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AN ESTIMATE FOR THE NUMBER OF REDUCIBLE BESSEL POLYNOMIALS OF BOUNDED DEGREE

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1. Introduction. The nth Bessel polynomial is

$$y_n(x)=\sum_{j=0}^{n}\frac{(n+j)!}{2^j(n-j)!j!}x^j\,.$$

In [3], E. Grosswald conjectured that $y_n(x)$ is irreducible over the rationals for every positive integer n. In [1], the first author proved that almost all $y_n(x)$ are irreducible and later [2] sharpened this by showing that the number of $n\le t$ for which $y_n(x)$ is reducible is $\ll t/\log\log\log t$. The object of this paper is to give a further sharpening.

THEOREM. *The number of $n\le t$ for which $y_n(x)$ is reducible is $\ll t^{2/3}$.*

The first author's earlier work used the Chebotarev Density Theorem, but the proof given here uses only elementary estimates. Our starting point is the Corollary to Lemma 2 in [1], which states that if

$$\Big(\prod_{p\mid n(n+1)}p\Big)^2\Big(\prod_{\substack{p\mid(n-1)\\ p\text{ odd}}}p\Big)\Big(\prod_{\substack{p\mid(n+2)\\ p>3}}p\Big)>n^2(n+1)^2\,,\tag{1}$$

then $y_n(x)$ is irreducible. We shall show that (1) holds for most n by showing that the non-squarefree part of $(n-1)n(n+1)(n+2)$ is typically very small.

2. Preliminaries. For every positive integer n, we define

$$a_n=\prod_{\substack{p^\alpha\parallel n\\ \alpha\text{ odd}}}p\quad\text{and}\quad b_n=\prod_{p^\alpha\parallel n}p^{[\alpha/2]}\,,$$

where $p^\alpha\|n$ denotes, as usual, that p^α is the highest power of p dividing n.

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We then have that $n = a_n b_n^2$ and

$$a_n \le \prod_{p|n} p\,. \tag{2}$$

In the next lemma, we use (2) to state (1) in a more usable form.

LEMMA 1. *If $y_n(x)$ is reducible and $t < n \le 2t$ then*

$$b_{n-1} b_n^2 b_{n+1}^2 b_{n+2} > \tfrac{1}{3} t\,.$$

Proof. From (1) and (2), we see that if $y_n(x)$ is reducible, then

$$\frac{n-1}{b_{n-1}^2} \cdot \frac{n^2}{b_n^4} \cdot \frac{(n+1)^2}{b_{n+1}^4} \cdot \frac{n+2}{b_{n+2}^2} \le 6n^2(n+1)^2\,.$$

The result now follows.

LEMMA 2. *If y is a positive real number, then*

$$\#\{n \in (t, 2t] : b_n > y\} \ll \frac{t}{y} + t^{1/2}\,.$$

Proof. The left-hand side is at most

$$\sum_{t<n\le 2t} \sum_{\substack{b^2|n \\ b>y}} 1 \ll \sum_{y<b\le\sqrt{2t}} \left(\frac{t}{b^2} + 1\right) \ll \frac{t}{y} + t^{1/2}\,.$$

LEMMA 3. *If $z \ge 2$ and y are real numbers, then*

$$\#\{n \in (t, 2t] : b_n b_{n+1} > z,\ b_n \le y,\ \text{and}\ b_{n+1} \le y\} \ll \frac{t \log z}{z} + y^2\,.$$

Proof. The left-hand side is

$$\begin{aligned} &\le \sum_{t<n\le 2t} \sum_{\substack{b^2|n,\, c^2|(n+1) \\ bc>z,\, b\le y,\, c\le y}} 1 \ll \sum_{\substack{bc>z \\ b\le y,\, c\le y}} \left(\frac{t}{b^2c^2} + 1\right) \\ &\ll y^2 + \sum_{bc\ge z} \frac{t}{b^2c^2}\,. \end{aligned} \tag{3}$$

Now the last sum in (3) is at most

$$t \sum_{r\ge z} d(r) r^{-2}\,, \tag{4}$$

where $d(r)$ denotes the number of divisors of r. Using the elementary estimate $\sum_{r\le x} d(r) \ll x \log x$ and partial summation, we find that (4) is

$$\ll \frac{t \log z}{z}\,.$$

This completes the proof.

3. Proof of the theorem. We will bound

(5) $$\#\{n \in (t, 2t] : b_{n-1} b_n^2 b_{n+1}^2 b_{n+2} > \tfrac{1}{3} t\} .$$

By Lemma 2, those n with any of $b_{n-1}, b_n, b_{n+1}, b_{n+2}$ greater than $t^{1/3}$ contribute $\ll t^{2/3}$. The remaining n all have $b_{n+j} \le t^{1/3}$ for $-1 \le j \le 2$. By Lemma 3, those n with any of $b_{n-1}b_n, b_n b_{n+1}, b_{n+1} b_{n+2}$ greater than $t^{1/3} \log t$ contribute $\ll t^{2/3}$. The remaining n all have

$$b_{n-1}b_n, b_n b_{n+1}, b_{n+1} b_{n+2} \le t^{1/3} \log t .$$

Using the condition in (5), we see that

$$b_{n-1}b_n \cdot b_n b_{n+1} \cdot b_{n+1} b_{n+2} > \tfrac{1}{3} t ,$$

so in fact the remaining n satisfy the stronger conditions

(6) $$\tfrac{1}{3} t^{1/3} \log^{-2} t \le b_{n-1}b_n, b_n b_{n+1}, b_{n+1} b_{n+2} \le t^{1/3} \log t .$$

Now consider those n satisfying (6) with $b_n > t^{2/9}$. Then $b_{n-1}, b_{n+1} < t^{1/9} \log t$ and $b_{n+2} > \frac{1}{3} t^{2/9} \log^{-3} t$. In other words, these n have

$$b_n \le t^{1/3}, \quad b_{n+2} \le t^{1/3} \quad \text{and} \quad b_n b_{n+2} > \tfrac{1}{3} t^{4/9} \log^{-3} t .$$

By an easy variant of the argument giving Lemma 3, these n contribute

$$\ll t^{5/9} \log^4 t + t^{2/3} \ll t^{2/3} .$$

A similar argument can be used to get the same bound for those n with $b_{n+1} > t^{2/9}$.

The remaining n have $b_n, b_{n+1} \le t^{2/9}$. By (6), $b_{n-1} \ge \frac{1}{3} t^{1/9} \log^{-2} t$ and

$$\tfrac{1}{9} t^{4/9} \log^{-4} t \le b_{n-1} b_n b_{n+1} \le t^{5/9} \log t .$$

The number of such n is

(7) $$\ll \sum_{\frac{1}{9} t^{4/9} \log^{-4} t \le m \le t^{5/9} \log t} \left(\frac{t}{m^2} + 1 \right) d_3(m)$$

where $d_3(m)$ denotes the number of ways of writing m as a product of three factors. Using the trivial estimate $\sum_{m \le x} d_3(m) \ll x \log^2 x$ and partial summation, we see that (7) is

$$\ll t^{5/9} \log^6 t \ll t^{2/3} .$$

This concludes the proof.

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