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SOME PROPERTIES OF THE PISIER–XU INTERPOLATION SPACES

BY

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For a closed subset I of the interval $[0,1]$ we let $A(I) = [v_1(I), C(I)]_{\frac{1}{2}2}$. We show that $A(I)$ is isometric to a 1-complemented subspace of $A(0,1)$, and that the Szlenk index of $A(I)$ is larger than the Cantor index of I. We also investigate, for ordinals $\eta < \omega_1$, the bases structures of $A(\eta), A^*(\eta)$, and $A_*(\eta)$ [the isometric predual of $A(\eta)$].

All the results of this paper extend, with obvious changes in the proofs, to the interpolation spaces $[v_1(I), C(I)]_{\theta q}$.

0. Preliminaries. In this section we will recall the definitions of the concepts we are going to work with, and state some of the needed properties. In what follows ω_0 denotes the first infinite ordinal, and ω_1 the first uncountable ordinal.

0.1. *Real interpolation*. We will give the definitions only in the case that interests us.

Let X_0 and X_1 be two Banach spaces, and let $j : X_0 \to X_1$ be an injective continuous linear operator. By abuse of notation we will identify X_0 with $j(X_0)$, hence considering X_0 as a (not necessarily closed) subspace of X_1.

For each $t > 0$ we define an equivalent norm K_t on X_1 by

$$K_t(x; X_0, X_1) = K_t(x) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\}$$

and we define a new Banach space $[X_0, X_1]_{\frac{1}{2}2}$ by

$$[X_0, X_1]_{\frac{1}{2}2} = \Big\{x \in X_1 : \|x\|_{\frac{1}{2}2} = \Big(\int_0^\infty (K_t(x)/t)^2\, dt\Big)^{1/2} < \infty\Big\}.$$

It is known that X_0 is $\|\cdot\|_{\frac{1}{2}2}$-dense in $[X_0, X_1]_{\frac{1}{2}2}$, and that for some constant $k < \infty$, $\|\cdot\|_{\frac{1}{2}2} \le k\|\cdot\|_{X_0}$. Moreover, if X_0 is $\|\cdot\|_{X_1}$-dense in X_1, then $[X_0, X_1]^*_{\frac{1}{2}2}$ may be canonically identified with $[X_0^*, X_1^*]_{\frac{1}{2}2}$ (the latter interpolation space being defined via the map $j^* : X_1^* \to X_0^*$ which is injective since j has dense range).

If (X_0, X_1) and (Y_0, Y_1) are two interpolation couples, and if $T : X_1 \to Y_1$ is a linear map such that $T(X_0) \subset Y_0$ and $\|T\| = \max(\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}) < \infty$, then T defines a bounded operator from $[X_0, X_1]_{\frac{1}{2}2}$ into $[Y_0, Y_1]_{\frac{1}{2}2}$ with norm at most $\|T\|$.

0.2. *The Cantor index.* Let K be a topological space. We define its *Cantor derived set* K' by

$$K' = \{x \in K : x \text{ is an accumulation point of } K\}$$

and its *Cantor index* $o(K)$ by

$$o(K) = \sup\{\alpha < \omega_1 : K^{(\alpha)} \neq \emptyset\}$$

where the sets $K^{(\alpha)}$ are defined inductively by

$$\begin{aligned} K^{(0)} &= K\,, \\ K^{(\alpha+1)} &= (K^{(\alpha)})'\,, \\ K^{(\alpha)} &= \bigcap_{\beta<\alpha} K^{(\beta)} \quad \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

It is well known that for each ordinal $\alpha < \omega_1$ one has $o([0, \omega_0^\alpha]) = \alpha$, where $[0, \eta]$ denotes the set $\{\varrho \text{ ordinal} : 0 \le \varrho \le \eta\}$ equipped with the order topology.

0.3. *The Szlenk index.* Let X be a Banach space, C a bounded subset of X, and K a weak* compact subset of X^*. For $\varepsilon > 0$ we define a weak* compact set by

$$\begin{aligned} \sigma_{C,\varepsilon}(K) = \{x^* \in K : \exists (x_n)_{n\ge1} \subset C, \exists (x_n^*)_{n\ge1} \subset K \text{ with} \\ 0 = w\text{-}\lim_{n\to\infty} x_n,\ x^* = w^*\text{-}\lim_{n\to\infty} x_n^*,\ \text{and } \inf_n |x_n^*(x_n)| \ge \varepsilon\}\,. \end{aligned}$$

The *Szlenk index* $\mathrm{Sz}(X)$ of X is given by

$$\mathrm{Sz}(X) = \sup_{\varepsilon>0} [\sup\{\alpha < \omega_1 : S_\alpha(\varepsilon) \neq \emptyset\}]$$

where the sets $S_\alpha(\varepsilon)$ are defined inductively by

$$\begin{aligned} S_0(\varepsilon) &= \mathrm{Ball}(X^*), \\ S_{\alpha+1}(\varepsilon) &= \sigma_{\mathrm{Ball}(X),\varepsilon}(S_\alpha(\varepsilon))\,, \\ S_\alpha(\varepsilon) &= \bigcap_{\beta<\alpha} S_\beta(\varepsilon) \quad \text{if } \alpha \text{ is a limit ordinal}\,. \end{aligned}$$

It is known that if X is separable, then X^* is nonseparable if $\mathrm{Sz}(X) = \omega_1$.

0.4. *Projectional resolution of the identity* (*P.R.I.*), *transfinite bases*. Let X be a Banach space and μ an ordinal number. A sequence of projections $(P_\alpha)_{0\le\alpha\le\mu}$ is called a *P.R.I.* of X if the following holds:

(i) $P_0 = 0$ and $P_\mu = \mathrm{Id}$.
(ii) $\sup_{0\le\alpha\le\mu}\|P_\alpha\| < \infty$.
(iii) $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$.
(iv) For every $x \in X$, the map $\varphi_x\colon [0,\mu] \to X$ defined by $\varphi_x(\alpha) = P_\alpha(x)$ is continuous.

Under conditions (ii) and (iii), it is not hard to prove that (iv) is equivalent to (see [JZ])

(iv)′ For every $\alpha \le \mu$, $P_\alpha(X) = \overline{\bigcup_{\beta<\alpha} P_{\beta+1}(X)}$.

A sequence of vectors $(x_\alpha) \subset X$ is called a *basis* of X if every $x \in X$ has a unique decomposition $x = \sum_{\alpha\le\mu} a_\alpha x_\alpha$ (with norm convergence).

It is well known and easy to check that basic sequences are (up to normalization) in 1-1 correspondence with P.R.I.'s that satisfy $\operatorname{rank}(P_{\alpha+1}-P_\alpha) = 1$ for every α.

1. The spaces $A(I)$. Let Γ denote either a closed subset I of $\mathbb{R}$, or the compact space $[1,\eta]$ for some ordinal number η. We denote by $C(\Gamma)$ the space of continuous functions on Γ, and we define the spaces $v_p(\Gamma), 1 \le p \le \infty$, by

$$v_p(\Gamma)=\Big\{ f \in C(\Gamma) : \|f\|_{v_p}=\sup\Big(|f(t_0)|^p + \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p\Big)^{1/p} < \infty\Big\}$$

where the sup runs over all ordered finite subsets $\{t_0 < t_1 < \ldots < t_n\}$ of Γ.

The spaces $A(\Gamma)$ are defined by

$$A(\Gamma) = [v_1(\Gamma), C(\Gamma)]_{\frac{1}{2}2}\,.$$

Let us show first that for every ordinal $\eta < \omega_1$, the space $A(\eta) = A([1,\eta])$ is isometric to $A(I_\eta)$ for some closed subset I_η of $[0,1]$. Indeed:

For every $\eta < \omega_1$, let $\phi_\eta : [0,\eta] \to [0,1]$ be a continuous map with the property that $\phi_\eta(\alpha) < \phi_\eta(\beta)$ whenever $\alpha < \beta \le \eta$. (The existence of such maps is well known, and can be easily proved by transfinite induction). From the definitions it is clear that the map Φ_η defined by $\Phi_\eta(f) = f\phi_\eta$ is an onto isometry from the interpolation couple $(v_1(I_\eta), C(I_\eta))$ into $(v_1(\eta), C(\eta))$ where $I_\eta = \phi_\eta([0,\eta])$. Hence Φ_η also defines an onto isometry between $A(I_\eta)$ and $A(\eta)$.

THEOREM 1. *For every closed subset I of $[0,1]$, the space $A(I)$ is isometric to a 1-complemented subspace of $A(0,1)$.*

Proof. It is enough to construct operators $E : (v_1(I), C(I)) \to (v_1(0,1), C(0,1))$ and $R : (v_1(0,1), C(0,1)) \to (v_1(I), C(I))$, both of norm 1, and such that RE is the identity map. Indeed, this will imply that

$ER[A(0,1)]$ is a 1-complemented subspace of $A(0,1)$ which is isometric to $A(I)$.

For R we take the formal restriction map: $Rf = f_{|I}$. It is clear that R sends $C(0,1)$ into $C(I)$, and $v_1(0,1)$ into $v_1(I)$, and that $\|R\| = 1$.

Let us now define the operator E. In the next definition we will use the conventions $\min\emptyset = \max I$, and $\max\emptyset = \min I$. With these conventions we define, for $t \in [0,1]$,

$$\begin{aligned} t^+ = t_I^+ &= \min\{s \in I : s \geq t\}\,, \\ t^- = t_I^- &= \max\{s \in I : s \leq t\}\,. \end{aligned}$$

Observe that since I is closed, $t^\pm \in I$ for every $t \in [0,1]$, and $t^+ = t^-$ if and only if $t \in [0, \min I] \cup [\max I, 1] \cup I$.

If $f \in C(I)$ is given, we define its extension Ef to $[0,1]$ by

$$Ef(t) = \begin{cases} f(t^+) & \text{if } t^+ = t^-\,, \\ f(t^+) - \dfrac{t^+ - t}{t^+ - t^-}(f(t^+) - f(t^-)) & \text{if } t^+ \neq t^-\,. \end{cases}$$

Observe that Ef is linear on any interval of the form $[t^-, t^+]$.

It is clear from this definition that E sends $C(I)$ into $C(0,1)$, and that $\|Ef\|_{C(0,1)} = \|f\|_{C(I)}$. All what remains to check now is that $\|Ef\|_{v_1(0,1)} = \|f\|_{v_1(I)}$. For this we need only check that $\|Ef\|_{v_1(0,1)} \leq \|f\|_{v_1(I)}$ since the other inequality is trival.

Let $f \in v_1(I)$, fix $\{t_0 < t_1 < \ldots < t_k\} \subset [0,1]$, and let us show that

$$|Ef(t_0)| + \sum_{i=0}^{k-1} |Ef(t_{i+1}) - Ef(t_i)| \leq \|f\|_{v_1(I)}\,.$$

It is clear from the definition of Ef that we can suppose $t_0 \geq \min I$ and $t_k \leq \max I$, *so we will suppose that this is the case.*

Consider now the sets $P = \{t_i : 1 \leq i \leq k\} \cup \{t_i^\pm : 1 \leq i \leq k\}$ and $Q = P \cap I$, and order them, i.e. $P = \{\tilde{t}_0 < \tilde{t}_1 < \ldots < \tilde{t}_l\}$, $Q = \{s_0 < s_1 < \ldots < s_m\}$.

For each $j, 0 \leq j \leq m$, let $\pi(j)$ be such that $s_j = \tilde{t}_{\pi(j)}$. Observe that $\pi(j-1) \leq \pi(j) - 1$ for every $j \in [1,m]$. Moreover, if $\pi(j-1) \neq \pi(j) - 1$, then Ef is linear on $[s_{j-1}, s_j]$. (Indeed, if $i \in]\pi(j-1), \pi(j)[$, then $\tilde{t}_i^- = s_{j-1}$ and $\tilde{t}_i^+ = s_j$.)

From the above observation one can easily deduce that for every $j \in [1,m]$,

$$\sum_{i=\pi(j-1)}^{\pi(j)-1} |Ef(\tilde{t}_{i+1}) - Ef(\tilde{t}_i)| = |f(s_j) - f(s_{j-1})|\,.$$

We are now ready to show that $\|Ef\|_{v_1(0,1)} \le \|f\|_{v_1(I)}$. We distinguish two cases for the set $\{t_i : 0 \le i \le k\}$.

Case 1: $t_0 \in I$. In this case we have $t_0 = \tilde{t}_0 = s_0$, i.e. $\pi(0) = 0$. We also have $\pi(m) = l$. In what follows the first inequality comes from the triangular inequality.

$$\begin{aligned}
|Ef(t_0)| + \sum_{i=0}^{k-1} |Ef(t_{i+1}) - Ef(t_i)| \\
&\le |Ef(\tilde{t}_0)| + \sum_{i=0}^{l-1} |Ef(\tilde{t}_{i+1}) - Ef(\tilde{t}_i)| \\
&= |Ef(\tilde{t}_0)| + \sum_{j=1}^{m} \sum_{i=\pi(j-1)}^{\pi(j)-1} |Ef(\tilde{t}_{i+1}) - Ef(\tilde{t}_i)| \\
&= |f(s_0)| + \sum_{j=1}^{m} |f(s_j) - f(s_{j-1})| \le \|f\|_{v_1(I)} .
\end{aligned}$$

Case 2: $t_0 \notin I$. In this case we have $\tilde{t}_0 = s_0 < \tilde{t}_1 = t_0 < s_1$, which implies $s_0 = t_0^-$ and $s_1 = t_0^+$ and so Ef is linear on $[s_0, s_1]$. Let $\lambda = (s_1 - t_0)/(s_1 - s_0)$, i.e. $t_0 = \lambda s_0 + (1-\lambda)s_1$. Then

$$\begin{aligned}
|Ef(t_0)| + \sum_{i=0}^{k-1} |Ef(t_{i+1}) - Ef(t_i)| \\
&\le |Ef(\tilde{t}_1)| + \sum_{i=0}^{\pi(1)-1} |Ef(\tilde{t}_{i+1}) - Ef(\tilde{t}_i)| \\
&\quad + \sum_{j=2}^{m} \sum_{i=\pi(j-1)}^{\pi(j)-1} |Ef(\tilde{t}_{i+1}) - Ef(\tilde{t}_i)| \\
&= |Ef(\tilde{t}_1)| + |Ef(s_1) - Ef(\tilde{t}_1)| + \sum_{j=2}^{m} |f(s_j) - f(s_{j-1})| \\
&\le \lambda \left(|f(s_0)| + |f(s_1) - f(s_0)| \right) \\
&\quad + (1-\lambda)|f(s_1)| + \sum_{j=2}^{m} |f(s_j) - f(s_{j-1})| \\
&\le \|f\|_{v_1(I)} .
\end{aligned}$$

This concludes the proof of the theorem. ∎

R e m a r k. With the same proof, Theorem 1 can be extended as follows: if I and J are two closed subsets of $\mathbb{R}$ with $I \subset J$ and if B is a Banach space, then $A(I;B)$ is isometric to a 1-complemented subspace of $A(J;B)$.

THEOREM 2. *$\mathrm{Sz}(A(I)) \geq o(I)$ for every closed subset I of $[0,1]$.*

P r o o f. Observe first that Weierstrass' theorem implies that $v_1(I)$ is norm dense in $C(I)$. Therefore (§0.1), $A^*(I) = [\mathcal{M}(I), v_1^*(I)]_{\frac{1}{2}2}$ (where $\mathcal{M}(I)$ stands for the space of random measures on I). In particular, $\mathcal{M}(I)$ is norm dense in $A^*(I)$.

Let $k > 0$ be such that $\|x\|_{A(I)} \leq k\|x\|_{v_1(I)}$ for every $x \in v_1(I)$, and $\|x^*\|_{A^*(I)} \leq k\|x^*\|_{\mathcal{M}(I)}$ for every $x^* \in \mathcal{M}(I)$.

The result of the theorem will be an immediate consequence of the following:

LEMMA 3. *If $x \in I$ and $(x_n)_{n\geq 1} \in I \setminus \{x\}$ are such that $x = \lim_{n\to\infty} x_n$, then*:

(i) *$\delta_x = \lim_{n\to\infty} \delta_{x_n}$ in the weak* topology of $A^*(I)$, where δ_y denotes the Dirac measure at y.*

(ii) *There exist functions $f_n \in v_1(I), n \geq 1$, with $\|f_n\|_{v_1(I)} = 2$, such that*

$$\langle \delta_{x_n}, f_n \rangle = 1 \quad \text{for every } n \geq 1, \quad \text{and}$$
$$0 = \lim_{n\to\infty} f_n \quad \text{in the weak topology of } A(I)\,.$$

Indeed, this lemma implies—with the notation of §0.2, §0.3—that $S_\alpha(1/(2k^2)) \supset \{(1/k)\delta_x : x \in I^{(\alpha)}\}$, which clearly implies the assertion of Theorem 2.

It remains to prove Lemma 3.

(i) is clear as $\langle \delta_x, f\rangle = \lim_{n\to\infty}\langle \delta_{x_n}, f\rangle$ for every $f \in C(I)$.

(ii) Let $F_n \in C(0,1)$ be defined by

$$F_n(t) = \left(1 - \frac{2|t - x_n|}{|x - x_n|}\right)^+,$$

and let $f_n = F_{n|I}$. It is clear that $\|f_n\|_{v_1(I)} = 2$, for every $n \geq 1$, and that $\lim_{n\to\infty} f_n(t) = 0$ for every $t \in I$.

If $\mu \in \mathcal{M}(I)$, then Lebesgue's dominated convergence theorem (applied to $|\mu|$) implies that $\lim_{n\to\infty}\langle \mu, f_n\rangle = 0$. This implies that $0 = \lim_{n\to\infty} f_n$ in the weak topology of $A(I)$, as $(f_n)_{n\geq 1}$ is bounded in $A(I)$, and $\mathcal{M}(I)$ is norm dense in $A^*(I)$.

This concludes the proof of the lemma and thus of the theorem. ■

R e m a r k. Xu proved that the spaces $A(I)$ have nontrivial types [X], which implies in particular that they do not contain the l_n^1's uniformly [P], and therefore that $i(A(I)) = \omega_0$, where i denotes the l^1-Bourgain index [B].

We then have a transfinite family of Banach spaces with separable duals, namely $(A(\eta))_{\eta<\omega_1}$, such that $\omega_1 > \sup_{\eta<\omega_1} i(A(\eta))$, and $\omega_1 = \sup_{\eta<\omega_1} \mathrm{Sz}(A(\eta))$ [as $o([1,\omega_0^\alpha]) = \alpha$ for every ordinal $\alpha < \omega_1$]. This result can be looked at as a quantitative version of the—by now—well known result on the existence of separable Banach spaces not containing l^1, and with nonseparable duals.

2. The spaces $A(\eta)$. For the next result we need the following notation: If A is a set, χ_A will denote the characteristic function of A. Clearly $\chi_{]\alpha,\eta]} \in v_1(\eta)$ for every $0 \le \alpha < \eta$. We also define for $1 \le \alpha \le \eta$ the element $e_\alpha \in C^*(\eta) = l^1(\eta)$ by $\langle e_\alpha, f\rangle = f(\alpha)$.

THEOREM 4. *$(\chi_{]\alpha,\eta]})_{0\le\alpha<\eta}$ and $(e_\alpha)_{1\le\alpha\le\eta}$ are transfinite bases of $A(\eta)$ and $A^*(\eta)$ respectively.*

Proof. (i) Let us show that $(\chi_{]\alpha,\eta]})_{0\le\alpha<\eta}$ is a basis of $A(\eta)$.

For each α, define a projection $P_\alpha : (v_1(\eta), C(\eta)) \to (v_1(\eta), C(\eta))$ by $P_\alpha f(\beta) = f(\min(\alpha,\beta))$ and observe that the projections so defined are increasing, i.e. $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$, and are of norm 1. Hence $(P_\alpha)_{0\le\alpha\le\eta}$ are increasing, norm 1 projections of $A(\eta)$. Let us show that they satisfy the continuity property (§0.4(iv)) on $A(\eta)$.

It is well known and easy to check that $(P_\alpha)_{0\le\alpha\le\eta}$ form a P.R.I. of $v_1(\eta)$, therefore

$$P_\alpha(v_1(\eta)) = \overline{\bigcup_{\beta<\alpha} P_{\beta+1}(v_1(\eta))}^{\|\cdot\|_{v_1}} \qquad \text{for every } 0 \le \alpha \le \eta\,.$$

On the other hand, $v_1(\eta)$ is $\|\cdot\|_A$-dense in $A(\eta)$, so

$$P_\alpha(A(\eta))=\overline{P_\alpha(v_1(\eta))}^{\|\cdot\|_A}\,.$$

This implies that

$$P_\alpha(A(\eta)) = \overline{\bigcup_{\beta<\alpha} P_{\beta+1}(A(\eta))}^{\|\cdot\|_A}$$

since $\|\cdot\|_A \le k\|\cdot\|_{v_1}$ for some constant k.

This finishes the proof of the first part as

$$(P_{\alpha+1} - P_\alpha)(f) = (f(\alpha+1) - f(\alpha))\chi_{]\alpha,\eta]}$$

for every f and every $\alpha < \eta$.

(ii) We show now that $(e_\alpha)_{1\le\alpha\le\eta}$ is a basis of $A^*(\eta)$. Using the facts that $A(\eta) = [v_{4/3}(\eta), v_4(\eta)]_{\frac12 2}$ (see [X]), and that $(\chi_{]\alpha,\eta]})_{0\le\alpha<\eta}$ is a basis for $v_p(\eta)$ if $1 \le p < \infty$ (see [E]), and therefore that $v_{4/3}(\eta)$ is $\|\cdot\|_{v_4}$-dense in $v_4(\eta)$, we deduce that $A^*(\eta) = [v_4^*(\eta), v_{4/3}^*(\eta)]_{\frac12 2}$ (§0.1).

It is also proved in [E] that $(e_\alpha)_{1\le\alpha\le\eta}$ is a basis of $v_p^*(\eta)$ if $1<p<\infty$, therefore the operators $(Q_\alpha)_{0\le\alpha\le\eta+1}$ defined by $Q_\alpha(e_\beta)=\chi_{]0,\alpha[}(\beta)e_\beta$ define a P.R.I. of the spaces $v_p^*(\eta)$.

Using the same proof as in part (i) we deduce that $(Q_\alpha)_{0\le\alpha\le\eta+1}$ defines a P.R.I. of $A(\eta)$. This concludes the proof since

$$(Q_{\alpha+1}-Q_\alpha)[A^*(\eta)]=\mathrm{sp}[e_\alpha]\,. \ \blacksquare$$

Remarks. (i) Using the same proof as for (ii) of Theorem 4, and the fact (see [E]) that $v_p(\eta)=Y_p^*(\eta)$ if $1<p<\infty$, where

$$Y_p(\eta)=\overline{\mathrm{sp}[e_\alpha:\alpha\le\eta,\ \alpha\ \text{nonlimit}]}^{\|\cdot\|_{v_p^*}}\,,$$

we can prove that $A(\eta)=B^*(\eta)$, where

$$B(\eta)=\overline{\mathrm{sp}[e_\alpha:\alpha\le\eta,\ \alpha\ \text{nonlimit}]}^{\|\cdot\|_{A^*}}\,.$$

(ii) Theorem 4 and the previous remark imply that $A(\eta)$ and $J(\eta)$ have the same measure theory properties. The proofs are the same as Edgar's proofs for $J(\eta)$.

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