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ON AN EXTENDED CONTACT BOCHNER CURVATURE TENSOR ON CONTACT METRIC MANIFOLDS

BY

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1. Introduction. On Sasakian manifolds, Matsumoto and Chūman [3] defined a contact Bochner curvature tensor (see also Yano [7]) which is invariant under D-homothetic deformations (for D-homothetic deformations, see Tanno [5]). On the other hand, Tricerri and Vanhecke [6] defined a general Bochner curvature tensor with conformal invariance on almost Hermitian manifolds.

In this paper we define an extended contact Bochner curvature tensor which is invariant under D-homothetic deformations of contact metric manifolds; we call it the EK-contact Bochner curvature tensor. We show that a contact metric manifold with vanishing EK-contact Bochner curvature tensor is a Sasakian manifold.

2. Preliminaries. Let M be a $(2n+1)$-dimensional contact metric manifold with structure tensors (ϕ, ξ, η, g). They satisfy

$$\phi\xi = 0\,, \quad \eta(\xi) = 1\,, \quad \phi^2 = -I + \eta\otimes\xi\,, \quad \eta(X) = g(\xi, X)\,,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \qquad g(\phi X, Y) = d\eta(X, Y)$$

for any vector fields X and Y on M. Define an operator h by $h = -\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes Lie differentiation. Then the vector field ξ is Killing if and only if h vanishes. It is well known that h and ϕh are symmetric operators, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\eta\circ h = 0$, $\operatorname{Tr} h = 0$ and $\operatorname{Tr}\phi h = 0$. Moreover, for every contact metric manifold M the following general formulas were obtained:

$$\nabla_X\xi = \phi X + \phi hX\,, \tag{2.1}$$

$$\tfrac{1}{2}(R(\xi, X)\xi - \phi R(\xi, \phi X)\xi) = h^2X + \phi^2X\,, \tag{2.2}$$

$$g(Q\xi, \xi) = 2n - \operatorname{Tr} h^2\,, \tag{2.3}$$

$$\sum_{i=1}^{2n+1}(\nabla_{E_i}\phi)E_i = -2n\xi \qquad (\{E_i\} \text{ is an orthonormal frame})\,, \tag{2.4}$$

$$(2.5)\quad (\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y = -2g(X,Y)\xi + \eta(Y)(X + hX + \eta(X)\xi),$$

$$(2.6)\quad \phi(\nabla_\xi h)X = X - \eta(X)\xi - h^2X - R(X,\xi)\xi\,,$$

where ∇ is the covariant differentiation with respect to g, Q is the Ricci operator of M, R is the curvature tensor field of M and $\operatorname{Tr} h$ denotes the trace of h (cf. [1], [2] and [4]). Moreover, using $\phi h\xi = 0$, we get

$$(2.7)\quad \phi(\nabla_Y h)\xi = -hY - h^2Y\,.$$

If ξ is Killing on a contact metric manifold M, then M is said to be a *K-contact Riemannian manifold*. If a contact metric manifold M is *normal* (i.e., $N + 2d\eta\otimes\xi = 0$, where N denotes the Nijenhuis tensor formed with ϕ), then M is called a *Sasakian manifold*. Every Sasakian manifold is a K-contact Riemannian manifold. On a Sasakian manifold with structure tensors (ϕ,ξ,η,g), we have

$$\nabla_X\xi = \phi X\,,\qquad (\nabla_X\phi)Y = R(X,\xi)Y = -g(X,Y)\xi + \eta(Y)X\,,$$
$$\phi Q = Q\phi\,,\qquad Q\xi = 2n\xi$$

(see, e.g., [8]).

3. ***D*-homothetic deformations.** Let M be an $(m+1)$-dimensional $(m=2n)$ contact metric manifold. Define the tensor field B^{es} on M by

$$\begin{aligned}
(3.1)\quad & B^{\mathrm{es}}(X,Y)\\
&= R(X,Y) + h\phi X\wedge h\phi Y + \frac{1}{2(m+4)}(QY\wedge X - (\phi Q\phi Y)\wedge X\\
&\quad + \tfrac12(\eta(Y)Q\xi\wedge X + \eta(QY)\xi\wedge X) - QX\wedge Y + (\phi Q\phi X)\wedge Y\\
&\quad - \tfrac12(\eta(X)Q\xi\wedge Y + \eta(QX)\xi\wedge Y) + (Q\phi Y)\wedge\phi X + (\phi QY)\wedge\phi X\\
&\quad - (Q\phi X)\wedge\phi Y - (\phi QX)\wedge\phi Y + 2g(Q\phi X,Y)\phi + 2g(\phi QX,Y)\phi\\
&\quad + 2g(\phi X,Y)\phi Q + 2g(\phi X,Y)Q\phi - \eta(X)QY\wedge\xi + \eta(X)(\phi Q\phi Y)\wedge\xi\\
&\quad + \eta(Y)QX\wedge\xi - \eta(Y)(\phi Q\phi X)\wedge\xi)\\
&\quad - \frac{k+m}{m+4}(\phi Y\wedge\phi X + 2g(\phi X,Y)\phi) - \frac{k-4}{m+4}Y\wedge X\\
&\quad + \frac{k}{m+4}(\eta(Y)\xi\wedge X + \eta(X)Y\wedge\xi)\\
&\quad - \frac{1}{(m+4)(m+2)}\operatorname{Tr} h^2(\phi Y\wedge\phi X\\
&\quad + 2g(\phi X,Y)\phi + Y\wedge X + \eta(X)\xi\wedge Y - \eta(Y)\xi\wedge X)\,,
\end{aligned}$$

where $k = \frac{S+m}{m+2}$ (S is the scalar curvature tensor of M) and $(X\wedge Y)Z =$

$g(Y,Z)X - g(X,Z)Y$ (cf. [3]). (3.1) implies the following identities:

$$
\begin{aligned}
&B^{\mathrm{es}}(X,Y)Z = -B^{\mathrm{es}}(Y,X)Z\,,\\
&B^{\mathrm{es}}(X,Y)Z + B^{\mathrm{es}}(Y,Z)X + B^{\mathrm{es}}(Z,X)Y = 0\,,\\
&g(B^{\mathrm{es}}(X,Y)Z,W) = -g(Z,B^{\mathrm{es}}(X,Y)W)\,,\\
&g(B^{\mathrm{es}}(X,Y)Z,W) = g(B^{\mathrm{es}}(Z,W)X,Y)\,.
\end{aligned}
\tag{3.2}
$$

If M is a Sasakian manifold, then B^{es} coincides with the contact Bochner curvature tensor of Matsumoto and Chūman [3] and

$$
B^{\mathrm{es}}(\xi,Y)Z = B^{\mathrm{es}}(X,Y)\xi = 0\,,\qquad B^{\mathrm{es}}(\phi X,\phi Y)Z = B^{\mathrm{es}}(X,Y)Z\,,
\tag{3.3}
$$

where we have used $R(\phi X,Y)Z - R(\phi Y,X)Z = g(\phi Z,X)Y - g(\phi Z,Y)X - g(Z,X)\phi Y + g(Z,Y)\phi X$ on a Sasakian manifold.

Consider a *D-homothetic deformation* $g^* = \alpha g + \alpha(\alpha-1)\eta\otimes\eta$, $\phi^* = \phi$, $\xi^* = \alpha^{-1}\xi$, $\eta^* = \alpha\eta$ on a contact metric manifold M, where α is a positive constant. We then say that $M(\phi,\xi,\eta,g)$ is *D-homothetic* to $M(\phi^*,\xi^*,\eta^*,g^*)$. It is well known that if a contact metric manifold $M(\phi,\xi,\eta,g)$ is D-homothetic to $M(\phi^*,\xi^*,\eta^*,g^*)$, then $M(\phi^*,\xi^*,\eta^*,g^*)$ is a contact metric manifold. Moreover, if $M(\phi,\xi,\eta,g)$ is a K-contact Riemannian manifold (resp. Sasakian manifold), then so is $M(\phi^*,\xi^*,\eta^*,g^*)$ (see [5]). Denoting by $W^i{}_{jk}$ the difference $\overset{*}{\Gamma}{}^i{}_{jk} - \Gamma^i{}_{jk}$ of Christoffel symbols, by (2.1) on a contact metric manifold M we have

$$
\begin{aligned}
W(X,Y) &= (\alpha-1)(\eta(Y)\phi X + \eta(X)\phi Y)\\
&\qquad + \frac{\alpha-1}{2\alpha}((\nabla_X\eta)(Y) + (\nabla_Y\eta)(X))\xi \qquad \text{(see [5])}\\
&= (\alpha-1)(\eta(Y)\phi X + \eta(X)\phi Y) + \frac{\alpha-1}{\alpha}g(\phi hX,Y)\xi\,.
\end{aligned}
$$

Putting this into

$$
\begin{aligned}
R^*(X,Y)Z &= R(X,Y)Z + (\nabla_X W)(Z,Y) - (\nabla_Y W)(Z,X)\\
&\quad + W(W(Z,Y),X) - W(W(Z,X),Y)
\end{aligned}
$$

and using (2.1) we have

$$
\begin{aligned}
R^*(X,Y)Z = R(X,Y)Z &+ (\alpha-1)(2g(\phi X,Y)\phi Z + g(\phi Z,Y)\phi X\\
&- g(\phi Z,X)\phi Y + \eta(Y)(\nabla_X\phi)(Z) + \eta(Z)(\nabla_X\phi)(Y)\\
&- \eta(X)(\nabla_Y\phi)(Z) - \eta(Z)(\nabla_Y\phi)(X))\\
&- (\alpha-1)^2(\eta(Z)\eta(X)Y - \eta(Z)\eta(Y)X)\\
&- \frac{\alpha-1}{\alpha}(g(X,(\nabla_Y\phi)hZ)\xi - g(Y,(\nabla_X\phi)hZ)\xi\\
&+ g(X,\phi(\nabla_Y h)Z)\xi - g(Y,\phi(\nabla_X h)Z)\xi\\
&+ g(X,\phi hZ)\phi hY - g(Y,\phi hZ)\phi hX)
\end{aligned}
\tag{3.4}
$$

$$- \frac{(\alpha-1)^2}{\alpha}(\eta(X)g(hZ,Y)\xi - \eta(Y)g(hZ,X)\xi)\,.$$

Choosing a ϕ^*-basis with respect to g^* and using (2.4) and (2.6), we get

$$\begin{aligned}(3.5)\quad & \mathrm{Ric}^*(X,Y)\\ &= \mathrm{Ric}(X,Y) + (\alpha-1)(-2g(X,Y) + 2(2n+1)\eta(X)\eta(Y))\\ &\quad + 2n(\alpha-1)^2\eta(X)\eta(Y) - \frac{\alpha-1}{\alpha}(-g(X,Y) + \eta(X)\eta(Y)\\ &\quad - 2g(hX,Y) + g(hX,hY) + g(R(X,\xi)\xi,Y))\,,\end{aligned}$$

where Ric is the Ricci curvature of M.

From (3.5), we find

$$\begin{aligned}(3.6)\quad Q^*X &= \frac{1}{\alpha}QX + \frac{\alpha-1}{\alpha}(-2X + 2(2n+1)\eta(X)\xi)\\ &\quad - \frac{\alpha-1}{\alpha^2}g(X,Q\xi)\xi - 2n\left(\frac{\alpha-1}{\alpha}\right)^2\eta(X)\xi\\ &\quad - \frac{\alpha-1}{\alpha^2}(-X + \eta(X)\xi - 2hX + h^2X + R(X,\xi)\xi)\,,\end{aligned}$$

where we have used the fact that

$$Q^*\xi = \frac{1}{\alpha}Q\xi - \frac{\alpha-1}{\alpha^2}g(Q\xi,\xi)\xi + 2n\frac{\alpha^2-1}{\alpha^2}\xi\,.$$

By virtue of (2.3) we have

$$(3.7)\qquad S^* = \frac{1}{\alpha}S - 2n\frac{\alpha-1}{\alpha} + \frac{\alpha-1}{\alpha^2}\,\mathrm{Tr}\,h^2\,.$$

Moreover, using the definition of h, we have

$$(3.8)\qquad h^* = \frac{1}{\alpha}h\,,$$

from which we get

$$(3.9)\qquad \mathrm{Tr}\,h^{*2} = \frac{1}{\alpha^2}\,\mathrm{Tr}\,h^2\,.$$

By means of (2.2), (3.1) and (3.4)–(3.9), after some lengthy computation, we obtain

$$\begin{aligned}(3.10)\quad & \overset{*}{B}{}^{\mathrm{es}}(X,Y)Z\\ &= B^{\mathrm{es}}(X,Y)Z + (\alpha-1)(\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi\\ &\quad + 2\eta(X)\eta(Z)Y - 2\eta(Y)\eta(Z)X + \eta(Y)(\nabla_X\phi)Z + \eta(Z)(\nabla_X\phi)Y\\ &\quad - \eta(X)(\nabla_Y\phi)Z - \eta(Z)(\nabla_Y\phi)X)\\ &\quad - \frac{\alpha-1}{\alpha}(g(X,(\nabla_Y\phi)hZ)\xi - g(Y,(\nabla_X\phi)hZ)\xi\end{aligned}$$

$$
\begin{aligned}
&+ g(X, \phi(\nabla_Y h)Z)\xi - g(Y, \phi(\nabla_X h)Z)\xi) \\
&- \frac{(\alpha-1)^2}{\alpha}(\eta(X)g(hZ,Y)\xi - \eta(Y)g(hZ,X)\xi) \\
&+ \frac{1}{2(2n+4)}\left\{\frac{3(\alpha-1)}{2\alpha}(\eta(X)\eta(Z)g(Y,Q\xi)\xi - \eta(Y)\eta(Z)g(X,Q\xi)\xi\right. \\
&+ g(Y,Z)g(X,Q\xi)\xi - g(X,Z)g(Y,Q\xi)\xi) \\
&+ \frac{1}{2}\frac{\alpha-1}{\alpha}(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))g(Q\xi,\xi)\xi \\
&+ \frac{\alpha-1}{\alpha}(-g(\phi X,Z)g(\phi Y,Q\xi)\xi + g(\phi Y,Z)g(\phi X,Q\xi)\xi \\
&\left. - 2g(\phi X,Y)g(\phi Z,Q\xi)\xi + 4ng(X,Z)\eta(Y)\xi - 4ng(Y,Z)\eta(X)\xi)\right\}.
\end{aligned}
$$

Now we introduce the *EK-contact Bochner curvature tensor* B^{ek} on M by

$$
\begin{aligned}
(3.11)\quad & B^{\mathrm{ek}}(X,Y)Z \\
&= B^{\mathrm{es}}(X,Y)Z - \eta(X)B^{\mathrm{es}}(\xi,Y)Z - \eta(Y)B^{\mathrm{es}}(X,\xi)Z \\
&\quad - \eta(Z)B^{\mathrm{es}}(X,Y)\xi - \eta(B^{\mathrm{es}}(X,Y)Z)\xi + \eta(X)\eta(B^{\mathrm{es}}(\xi,Y)Z)\xi \\
&\quad + \eta(Y)\eta(B^{\mathrm{es}}(X,\xi)Z)\xi + \eta(Y)\eta(Z)\phi B^{\mathrm{es}}(\phi X,\xi)\xi \\
&\quad + \eta(X)\eta(Z)\phi B^{\mathrm{es}}(\xi,\phi Y)\xi\,.
\end{aligned}
$$

In particular, if M is a Sasakian manifold, then $B^{\mathrm{ek}} = B^{\mathrm{es}}$ from (3.2) and (3.3). That is, B^{ek} coincides with the contact Bochner curvature tensor defined by Matsumoto and Chūman [3].

THEOREM 3.1. *The EK-contact Bochner curvature tensor is invariant under D-homothetic deformations* $M(\phi,\xi,\eta,g) \to M(\phi^*,\xi^*,\eta^*,g^*)$ *on a contact metric manifold* M.

Proof. Using (2.5)–(2.7) and (3.10), we find

$$
\begin{aligned}
(3.12)\quad & -\eta^*(X)\overset{*}{B}{}^{\mathrm{es}}(\xi^*,Y)Z = -\eta(X)\overset{*}{B}{}^{\mathrm{es}}(\xi,Y)Z \\
&= -\eta(X)B^{\mathrm{es}}(\xi,Y)Z \\
&\quad + (\alpha-1)(-\eta(X)\eta(Z)Y + \eta(X)\eta(Z)hY + \eta(X)(\nabla_Y\phi)Z) \\
&\quad + \frac{\alpha-1}{\alpha}(\eta(X)\eta(Y)\eta(Z)\xi + \eta(X)g(Y,R(Z,\xi)\xi)\xi) \\
&\quad + \frac{(\alpha-1)^2}{\alpha}\eta(X)g(Y,Z)\xi
\end{aligned}
$$

$$+\frac{(\alpha-1)(\alpha-2)}{\alpha}\eta(X)g(Y,hZ)\xi$$
$$+\frac{1}{2(2n+4)}\left\{\frac{3(\alpha-1)}{2\alpha}(-g(Y,Z)\eta(X)g(Q\xi,\xi)\xi\right.$$
$$+\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi)$$
$$+\frac{1}{2}\frac{\alpha-1}{\alpha}(-\eta(X)g(Y,Z)g(Q\xi,\xi)\xi$$
$$+\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi)$$
$$\left.+\frac{\alpha-1}{\alpha}(4ng(Y,Z)\eta(X)\xi-4n\eta(X)\eta(Y)\eta(Z)\xi)\right\},$$

(3.13)
$$-\eta^*(Y)\overset{*}{B}{}^{\mathrm{es}}(X,\xi^*)Z$$
$$=-\eta(Y)B^{\mathrm{es}}(X,\xi)Z+(\alpha-1)(\eta(Y)\eta(Z)X$$
$$-\eta(Y)\eta(Z)hX-\eta(Y)(\nabla_X\phi)Z)$$
$$-\frac{\alpha-1}{\alpha}(\eta(X)\eta(Y)\eta(Z)\xi+\eta(Y)g(X,R(Z,\xi)\xi)\xi)$$
$$-\frac{(\alpha-1)^2}{\alpha}g(X,Z)\eta(Y)\xi$$
$$-\frac{(\alpha-1)(\alpha-2)}{\alpha}g(hX,Z)\eta(Y)\xi$$
$$+\frac{1}{2(2n+4)}\left\{\frac{3(\alpha-1)}{2\alpha}(g(X,Z)\eta(Y)g(Q\xi,\xi)\xi\right.$$
$$-\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi)$$
$$+\frac{1}{2}\frac{\alpha-1}{\alpha}(g(X,Z)\eta(Y)g(Q\xi,\xi)\xi-\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi)$$
$$\left.+\frac{\alpha-1}{\alpha}(-4ng(X,Z)\eta(Y)\xi+4n\eta(X)\eta(Y)\eta(Z)\xi)\right\},$$

(3.14)
$$-\eta^*(Z)\overset{*}{B}{}^{\mathrm{es}}(X,Y)\xi^*$$
$$=-\eta(Z)B^{\mathrm{es}}(X,Y)\xi+(\alpha-1)(-\eta(X)\eta(Z)Y+\eta(Y)\eta(Z)X$$
$$-\eta(Y)\eta(Z)hX+\eta(X)\eta(Z)hY-\eta(Z)(\nabla_X\phi)Y$$
$$+\eta(Z)(\nabla_Y\phi)X),$$

(3.15)
$$-\eta^*(\overset{*}{B}{}^{\mathrm{es}}(X,Y)Z)\xi^*=-\eta(B^{\mathrm{es}}(X,Y)Z)\xi-\frac{\alpha-1}{\alpha}(\eta(X)g(hY,Z)\xi$$
$$-\eta(Y)g(hX,Z)\xi-g(X,(\nabla_Y\phi)hZ)\xi+g(Y,(\nabla_X\phi)hZ)\xi$$

$$
\begin{aligned}
&- g(X, \phi(\nabla_Y h)Z)\xi + g(Y, \phi(\nabla_X h)Z)\xi) \\
&+ \frac{1}{2(2n+4)}\bigg\{\frac{3(\alpha-1)}{2\alpha}(-g(Y,Z)g(X,Q\xi)\xi \\
&+ g(X,Z)g(Y,Q\xi)\xi - \eta(X)\eta(Z)g(Y,Q\xi)\xi \\
&+ \eta(Y)\eta(Z)g(X,Q\xi)\xi) + \frac{1}{2}\,\frac{\alpha-1}{\alpha}(-\eta(X)g(Y,Z)g(Q\xi,\xi)\xi \\
&+ \eta(Y)g(X,Z)g(Q\xi,\xi)\xi) + \frac{\alpha-1}{\alpha}(g(\phi X,Z)g(\phi Y,Q\xi)\xi \\
&- g(\phi Y,Z)g(\phi X,Q\xi)\xi + 2g(\phi X,Y)g(\phi Z,Q\xi)\xi \\
&- 4ng(X,Z)\eta(Y)\xi + 4ng(Y,Z)\eta(X)\xi)\bigg\}.
\end{aligned}
$$

Using (2.3), (2.5), (2.6) and (3.15), we get

$$
\begin{aligned}
(3.16)\quad & \eta^*(X)\eta^*(\overset{*}{B}{}^{\mathrm{es}}(\xi^*,Y)Z)\xi^* + \eta^*(Y)\eta^*(\overset{*}{B}{}^{\mathrm{es}}(X,\xi^*)Z)\xi^* \\
&= \frac{\alpha-1}{\alpha}(2g(Y,hZ)\eta(X)\xi - 2g(X,hZ)\eta(Y)\xi + g(Y,Z)\eta(X)\xi \\
&\quad - g(X,Z)\eta(Y)\xi - g(Y,R(Z,\xi)\xi)\eta(X)\xi + g(X,R(Z,\xi)\xi)\eta(Y)\xi) \\
&\quad + \frac{1}{2n+4}\,\frac{\alpha-1}{\alpha}(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi)\operatorname{Tr}h^2\,.
\end{aligned}
$$

From (2.5) and (3.14) we have

$$
(3.17)\qquad
\begin{aligned}
\eta^*(Y)\eta^*(Z)\phi^*\overset{*}{B}{}^{\mathrm{es}}(\phi^*X,\xi^*)\xi^* &= \eta(Y)\eta(Z)\phi B^{\mathrm{es}}(\phi X,\xi)\xi \\
&\quad + 2(\alpha-1)\eta(Y)\eta(Z)hX\,, \\
\eta^*(X)\eta^*(Z)\phi^*\overset{*}{B}{}^{\mathrm{es}}(\xi^*,\phi^*Y)\xi^* &= -\eta(X)\eta(Z)\phi B^{\mathrm{es}}(\xi,\phi Y)\xi \\
&\quad - 2(\alpha-1)\eta(X)\eta(Z)hY\,.
\end{aligned}
$$

Therefore, using (2.3) and (3.10)–(3.17), we get our result.

4. Contact metric manifolds with vanishing EK-contact Bochner curvature tensor. We define

$$
(4.1)\qquad s^{\#} = \sum_{i,j=1}^{2n+1} g(R(E_i,E_j)\phi E_j, \phi E_i)\,,
$$

where $\{E_i\}$ is an orthonormal frame.

LEMMA 4.1 ([4]). *For any $(2n+1)$-dimensional contact metric manifold M we have*

$$
s^{\#} - S + 4n^2 = \operatorname{Tr}h^2 + \tfrac{1}{2}\{\|\nabla\phi\|^2 - 4n\} \ge 0\,.
$$

Moreover, M is Sasakian if and only if $\|\nabla\phi\|^2 - 4n = 0$ or equivalently

$$s^{\#} - S + 4n^2 = 0 .$$

THEOREM 4.1. *Let M be a contact metric manifold with vanishing EK-contact Bochner curvature tensor. Then M is a Sasakian manifold.*

Proof. Since the EK-contact Bochner curvature tensor of M vanishes we have

$$\begin{aligned} (4.2)\qquad & g(B^{\mathrm{es}}(X,Y)Z,W) \\ &= \eta(X)g(B^{\mathrm{es}}(\xi,Y)Z,W) + \eta(Y)g(B^{\mathrm{es}}(X,\xi)Z,W) \\ &\quad + \eta(Z)g(B^{\mathrm{es}}(X,Y)\xi,W) + \eta(W)\eta(B^{\mathrm{es}}(X,Y)Z) \\ &\quad - \eta(X)\eta(W)\eta(B^{\mathrm{es}}(\xi,Y)Z) - \eta(Y)\eta(W)\eta(B^{\mathrm{es}}(X,\xi)Z) \\ &\quad - \eta(Y)\eta(Z)g(\phi B^{\mathrm{es}}(\phi X,\xi)\xi,W) \\ &\quad - \eta(X)\eta(Z)g(\phi B^{\mathrm{es}}(\xi,\phi Y)\xi,W) . \end{aligned}$$

Setting $X = E_i$, $Y = E_j$, $Z = \phi E_j$, $W = \phi E_i$ ($\{E_i\}$ is a ϕ-basis) in each member of (4.2) and summing over i and j, we have

$$g(B^{\mathrm{es}}(E_i,E_j)\phi E_j,\phi E_i) = s^{\#} - S + 4n^2 - 2\,\mathrm{Tr}\,h^2 + (\mathrm{Tr}\,h^2)^2 = 0 .$$

Using Lemma 4.1, we obtain

$$(\mathrm{Tr}\,h^2)^2 - \mathrm{Tr}\,h^2 + \tfrac{1}{2}\{\|\nabla\phi\|^2 - 4n\} = 0 .$$

On the other hand, from (4.2) we find

$$\sum_{i=1}^{2n+1} g(B^{\mathrm{es}}(E_i,\xi)\xi,E_i) = -\sum_{i=1}^{2n+1} g(B^{\mathrm{es}}(\phi E_i,\xi)\xi,\phi E_i) .$$

Hence $g(Q\xi,\xi) = 2n$. Thus we get our result.

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