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OSCILLATION OF DIFFERENCE EQUATIONS

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1. Introduction. Recently there has been a considerable interest in the qualitative behavior of the solutions of difference equations of the form

$$y_{n+1} - y_n + p_n y_{n-k} = 0\,, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer (see for example the work in [1]–[4] and the references cited therein).

In this paper we are concerned with the oscillation of the solutions of the delay difference equations of the form

$$y_{n+1} - y_n + \sum_{i=1}^{K} p_{in} y_{n-m_i} = 0\,, \tag{1.2}$$

where $m_i,\ i = 1, \ldots, K$, are positive integers, and $p_{in},\ i = 1, \ldots, K,\ n = 1, 2, \ldots,$ are real numbers.

As usual a solution $\{y_n\}$ of (1.2) is called *oscillatory* if the terms y_n of the sequence are neither eventually positive nor eventually negative. Otherwise the solution is called *nonoscillatory*.

In Section 2 we establish some lemmas. The main results are given in Section 3. We emphasize that the positivity of $\{p_{in}\}$ is not required.

2. Some lemmas. The following lemmas will be used to derive sufficient conditions for the oscillation of the solutions of (1.2).

LEMMA 2.1. *Let $m_1 > \ldots > m_K > 0$ and suppose there exists a sufficiently large integer N such that*

$$p_{1n} \ge 0\,,\ p_{1n} + p_{2n} \ge 0\,, \ldots, p_{1n} + \ldots p_{Kn} \ge 0 \qquad \textit{for } n \ge N\,.$$

Assume further that for any given positive integer N_1 there exists an integer $N_2 \ge N_1$ such that $p_{in} \ge 0,\ i = 1, \ldots, K$, for $n \in [N_2, N_2 + m_1]$. Let $\{y_n\}$ be

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a solution of (1.2) *such that* y_n *is eventually positive. Then* y_n *is eventually nonincreasing and*

$$\sum_{i=1}^{K} p_{in} y_{n-m_i} \ge y_{n-m_K} \sum_{i=1}^{K} p_{in} \tag{2.1}$$

holds eventually.

Proof. Let $y_{n-m_1} > 0$ for $n \ge N$. Then there exists $N_2 \ge N$ such that $p_{in} \ge 0,\ i = 1, \dots, K,\ n \in [N_2, N_2 + m_1]$. This implies that

$$y_{n+1} - y_n = -\sum_{i=1}^{K} p_{in} y_{n-m_i} \le 0 \quad \text{for } n \in [N_2, N_2 + m_1]\,.$$

We shall show that y_n is nonincreasing for $n \in [N_2 + m_1, N_2 + m_1 + m_K]$. In fact,

$$n - m_i \in [N_2, N_2 + m_1] \quad \text{for } n \in [N_2 + m_1, N_2 + m_1 + m_K]\,.$$

So $y_{n-m_1} \ge y_{n-m_2} \ge \dots \ge y_{n-m_K}$. Therefore

$$\begin{aligned} y_{n+1} - y_n &= -\sum_{i=1}^{K} p_{in} y_{n-m_i} \\ &\le -(p_{1n} + p_{2n}) y_{n-m_2} - \sum_{i=3}^{K} p_{in} y_{n-m_i} \\ &\le \dots \le -\Big(\sum_{i=1}^{K} p_{in}\Big) y_{n-m_K} \le 0\,, \end{aligned} \tag{2.2}$$

for $n \in [N_2 + m_1, N_2 + m_1 + m_K]$. Repeating the above procedure we can show that y_n is nonincreasing for $n \in [N_2 + m_1 + l m_K, N_2 + m_1 + (l+1) m_K]$, for $l = 0, 1, 2, \dots$ That is, y_n is nonincreasing for $n \ge N_2$. From (2.2) it follows that (2.1) holds eventually. This completes the proof.

LEMMA 2.2. *Suppose that the assumptions of Lemma* 2.1 *hold. Further, assume that* $\sum_{j=N}^{\infty} \sum_{i=1}^{K} p_{ij} = \infty$. *Then every nonoscillatory solution* $\{y_n\}$ *of* (1.2) *satisfies*

$$\lim_{n\to\infty} y_n = 0\,. \tag{2.3}$$

Proof. Let $\{y_n\}$ be an eventually positive solution of (1.2). By Lemma 2.1, y_n is eventually nonincreasing and hence $\lim_{n\to\infty} y_n = l \ge 0$ exists. If $l > 0$, by summing (1.2) from N to n we have

$$0 = y_{n+1} - y_N + \sum_{j=N}^{n} \sum_{i=1}^{K} p_{ij} y_{j-m_i} \tag{2.4}$$

$$\geq y_{n+1} - y_N + \sum_{j=N}^{n} y_{j-m_K} \sum_{i=1}^{K} p_{ij}$$

$$\geq y_{n+1} - y_N + y_{n-m_K} \sum_{j=N}^{n} \sum_{i=1}^{K} p_{ij}\,.$$

Letting $n \to \infty$ we get a contradiction. Therefore $l = 0$. Thus the proof is complete.

LEMMA 2.3. *In addition to the assumptions of Lemma* 2.1, *suppose that there exists a positive number d such that*

$$\sum_{i=n-m_K}^{n} \sum_{j=1}^{K} p_{ij} \geq d > 0 \quad \textit{for all large } n\,. \tag{2.5}$$

Let $\{y_n\}$ be an eventually positive solution of (1.2). *Then y_{n-m_K}/y_n is eventually bounded above.*

Proof. From (2.5), for any large integer $\overline{N}$ there exists an integer n such that $\overline{N} \in [n - m_K, n]$ and

$$\sum_{j=n-m_K}^{\overline{N}} \sum_{i=1}^{K} p_{ij} \geq \frac{d}{2}\,, \qquad \sum_{j=\overline{N}}^{n} \sum_{i=1}^{K} p_{ij} \geq \frac{d}{2}\,. \tag{2.6}$$

Summing (1.2) form $n - m_K$ to $\overline{N}$ we have

$$y_{\overline{N}+1} - y_{n-m_K} + \sum_{j=n-m_K}^{\overline{N}} \sum_{i=1}^{K} p_{ij} y_{j-m_i} = 0\,.$$

Hence

$$y_{n-m_K} \geq y_{\overline{N}+1} + \sum_{j=n-m_K}^{\overline{N}} \sum_{i=1}^{K} p_{ij} y_{j-m_i} \geq y_{\overline{N}+1} + y_{\overline{N}-m_K} d/2\,. \tag{2.7}$$

Similarly, summing (1.2) from $\overline{N}$ to n we have

$$y_{n+1} - y_{\overline{N}} + \sum_{j=\overline{N}}^{n} \sum_{i=1}^{K} p_{ij} y_{j-m_i} = 0\,.$$

Hence

$$y_{\overline{N}} \geq y_{n+1} + y_{n-m_K} d/2\,. \tag{2.8}$$

Combining (2.7) and (2.8) we have

$$y_{\overline{N}-m_K}/y_{\overline{N}} \leq (2/d)^2\,. \tag{2.9}$$

Since $\overline{N}$ is arbitrary the proof is complete.

LEMMA 2.4. *Under the assumptions of Lemma* 2.3, *if* $\{y_n\}$ *is an eventually positive solution of* (1.2) *then eventually*

$$\frac{y_{n-m_K}}{y_{n+1}} \le \frac{8}{d^4}[1-2(d/2)^3+\sqrt{1-4(d/2)^3}]\,, \tag{2.10}$$

where $0<d<1$.

Proof. From (2.9), for all large n we have $y_{n+1} \ge (d/2)^2 y_n$. In view of (2.8),

$$\begin{aligned} y_{\overline{N}+1} &\ge (d/2)^2 y_{\overline{N}} \ge (d/2)^2\left[y_{n+1} - y_{n-m_K}\frac{d}{2}\right] \\ &\ge (d/2)^2 y_{n+1} + (d/2)^3\left[y_{\overline{N}+1} + y_{\overline{N}-m_K}\frac{d}{2}\right] \end{aligned}$$

or

$$y_{\overline{N}+1}[1-(d/2)^3] \ge y_{\overline{N}-m_K}(d/2)^4\,.$$

Hence

$$\frac{y_{\overline{N}-m_K}}{y_{\overline{N}+1}} \le \frac{1-(d/2)^3}{(d/2)^4} = l_1\,.$$

From (2.8) and (2.7) it follows that

$$\begin{aligned} y_{\overline{N}+1} &\ge \left(\frac{d}{2}\right)^2\left[y_{n+1} + y_{n-m_K}\frac{d}{2}\right] \ge y_{n-m_K}\left(\frac{d}{2}\right)^2\left(\frac{d}{2}+\frac{1}{l_1}\right) \\ &\ge \left(y_{\overline{N}+1} + y_{\overline{N}-m_K}\frac{d}{2}\right)\left(\frac{d}{2}\right)^2\left(\frac{d}{2}+\frac{1}{l_1}\right). \end{aligned}$$

Hence

$$\frac{y_{\overline{N}-m_K}}{y_{\overline{N}+1}} = \frac{1-\left(\frac{d}{2}\right)^2\left(\frac{d}{2}+\frac{1}{l_1}\right)}{\left(\frac{d}{2}\right)^3\left(\frac{d}{2}+\frac{1}{l_1}\right)} = l_2 < l_1\,.$$

By induction we can show that

$$\frac{y_{\overline{N}-m_K}}{y_{\overline{N}+1}} \le l_n\,, \qquad n=1,2,\ldots, \tag{2.11}$$

and $0<l_n<l_{n-1}<\ldots<l_1$, where

$$l_n = \frac{1-\left(\frac{d}{2}\right)^2\left(\frac{d}{2}+\frac{1}{l_{n-1}}\right)}{\left(\frac{d}{2}\right)^3\left(\frac{d}{2}+\frac{1}{l_{n-1}}\right)}\,.$$

Clearly $\lim_{n\to\infty} l_n = l$ exists and

$$l = \frac{1 - \left(\frac{d}{2}\right)^2 \left(\frac{d}{2} + \frac{1}{l}\right)}{\left(\frac{d}{2}\right)^3 \left(\frac{d}{2} + \frac{1}{l}\right)} . \tag{2.12}$$

From (2.12) we get

$$l = \frac{1 - 2\left(\frac{d}{2}\right)^3 \pm \sqrt{1 - 4\left(\frac{d}{2}\right)^3}}{2\left(\frac{d}{2}\right)^4} . \tag{2.13}$$

Combining (2.11) and (2.13) and noting that $\overline{N}$ is arbitrary we get (2.10). This completes the proof of the lemma.

3. Main results

THEOREM 3.1. *In addition to the hypotheses of Lemma* 2.1 *suppose that*

$$\liminf_{n\to\infty} \frac{1}{m_K} \sum_{j=n-m_K}^{n-1} \sum_{i=1}^{K} p_{ij} > \frac{m_K^{m_K}}{(m_K+1)^{m_K+1}} . \tag{3.1}$$

Then every solution of (1.2) *is oscillatory.*

Proof. If not, let $\{y_n\}$ be an eventually positive solution of (1.2). Then by Lemma 2.1,

$$y_{n+1} - y_n \le \sum_{i=1}^{K} p_{in} y_{n-m_K} \le -\sum_{i=1}^{K} p_{in} y_n . \tag{3.2}$$

Hence, eventually

$$1 - \frac{y_{n+1}}{y_n} \ge \sum_{i=1}^{K} p_{in}$$

and so

$$\begin{aligned} \frac{1}{m_K} \sum_{j=n-m_K}^{n-1} \sum_{i=1}^{K} p_{ij} &\le \frac{1}{m_K} \sum_{i=n-m_K}^{n-1} \left(1 - \frac{y_{i+1}}{y_i}\right) \\ &= 1 - \frac{1}{m_K} \sum_{i=n-m_K}^{n-1} \frac{y_{i+1}}{y_i} \le 1 - \left(\prod_{i=n-m_K}^{n-1} \frac{y_{i+1}}{y_i}\right)^{1/m_K} \\ &= 1 - \left(\frac{y_n}{y_{n-m_K}}\right)^{1/m_K} . \end{aligned} \tag{3.3}$$

It follows from (3.1) that there exist constants α and β such that for n sufficiently large,

$$\frac{m_K^{m_K}}{(m_K+1)^{m_K+1}} = \alpha < \beta \le \frac{1}{m_K} \sum_{j=n-m_K}^{n-1} \sum_{i=1}^{K} p_{ij} \,. \tag{3.4}$$

Combining (3.3) and (3.4) one gets

$$\left(\frac{y_n}{y_{n-m_K}}\right)^{1/m_K} \le 1-\beta \quad \text{for all large } n\,. \tag{3.5}$$

In particular, $\beta \in (0,1)$. From the fact that

$$\max_{0\le\beta\le1}[(1-\beta)\beta^{1/m_K}] = \frac{m_K}{(m_K+1)^{1+1/m_K}} = \alpha^{1/m_K}$$

it follows that (3.5) implies

$$\frac{\beta}{\alpha} y_n \le y_{n-m_K} \quad \text{for all large } n\,. \tag{3.6}$$

Substituting (3.6) into (3.2) we have

$$y_{n+1} - y_n \le -\frac{\beta}{\alpha} \sum_{i=1}^{K} p_{in} y_n \,.$$

Hence

$$1 - \frac{y_{n+1}}{y_n} \ge \frac{\beta}{\alpha} \sum_{i=1}^{K} p_{in} \,,$$

and so

$$\frac{\beta}{\alpha} \cdot \frac{1}{m_K} \sum_{j=n-m_K}^{n-1} \sum_{i=1}^{K} p_{ij} \le 1 - \left(\frac{y_n}{y_{n-m_K}}\right)^{1/m_K} .$$

Thus

$$\left(\frac{y_n}{y_{n-m_K}}\right)^{1/m_K} \le 1 - \frac{\beta^2}{\alpha}\,,$$

and eventually $(\beta/\alpha)^2 y_n \le y_{n-m_K}$. By induction, for every $m = 1, 2, \dots$ there exists an integer n_m such that

$$\left(\frac{\beta}{\alpha}\right)^m y_n \le y_{n-m_K}\,, \qquad n \ge n_m\,, \tag{3.7}$$

which implies that y_{n-m_K}/y_n is eventually unbounded. But this, in view of Lemma 2.3, is impossible. The proof is now complete.

If (3.1) is not satisfied then we have the following result:

THEOREM 3.2. *Assume that the hypotheses of Lemma* 2.1 *are satisfied. Further*, *suppose that*

$$\limsup_{n\to\infty} \sum_{j=n-m_K}^{n} \sum_{i=1}^{K} p_{ij} > 1 - \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1}, \tag{3.8}$$

where d *is defined by* (2.5). *Then every solution of* (1.2) *is oscillatory.*

Proof. If not, let $\{y_n\}$ be an eventually positive solution of (1.2). From (1.2) we have

$$y_{n+1} - y_n \le -y_{n-m_K} \sum_{i=1}^{K} p_{in}\,. \tag{3.9}$$

Summing (3.9) from $n - m_K$ to n we have

$$\begin{aligned} y_{n+1} - y_{n-m_K} &\le -\sum_{j=n-m_K}^{n} y_{j-m_K} \sum_{i=1}^{K} p_{ij} \\ &\le -y_{n-m_K} \sum_{j=n-m_K}^{n} \sum_{i=1}^{K} p_{ij}\,. \end{aligned} \tag{3.10}$$

Using Lemma 2.4 we have

$$y_{n+1} \ge \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1} y_{n-m_K}\,. \tag{3.11}$$

Now we combine (3.10) and (3.11) to get

$$y_{n-m_K}\left(\sum_{j=n-m_K}^{n} \sum_{i=1}^{K} p_{ij} - 1 + \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1}\right) \le 0\,.$$

This contradicts (3.8) and hence the proof is complete.

Remarks. Theorem 3.1 improves Theorem 4.1 and Theorem 3.1 of [2] and Theorem 4.3 of [1]. It is easy to check that Theorem 1 of [3] and Theorem 3 of [4] are special cases of Theorem 3.1. In the linear case Theorem 3.2 improves Theorem 2.5 of [1]. Erbe and Zhang take $p_{in} \ge 0$, $i = 1, \ldots, K$, $n = N, N+1, \ldots$ We have removed this restriction by the technique of Lemma 2.1.

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