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A THEOREM OF O'NAN FOR FINITE LINEAR SPACES

BY

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One of the most important and beautiful results on doubly transitive permutation groups is O'Nan's characterization of $\mathrm{PSL}(n,q)$, $3 \le n$, by the normal structure of its one-point stabilizer. In a first attack, O'Nan [9] proves that each doubly transitive permutation group G on a finite set X such that the one-point stabilizer G_x has an abelian normal subgroup which does not act semiregularly on $X \setminus \{x\}$ satisfies $\mathbf{F}^*(G) \cong \mathrm{PSL}(n,q)$ for some integer $n \ge 3$ and some prime power q. In a second paper [10], the same conclusion is obtained under the hypothesis that G_x has a normal subgroup which is a T.I. set in G and which does not act semiregularly on $X \setminus \{x\}$.

In [14], the first of these two theorems has been generalized in a natural way to flag transitive automorphism groups of finite linear spaces which satisfy the following condition.

($*$) Each stabilizer of a block induces a regular group or a Frobenius group on the set of points incident with that block.

The main result of that paper appears here as Theorem 2.

The purpose of the present paper is the proof of the following analogous generalization of O'Nan's second theorem.

THEOREM 1. *Let G be a flag transitive automorphism group of a finite linear space D. Assume that G satisfies ($*$). Let X denote the point set of D. Let $x \in X$, and assume that G_x has a normal subgroup which is a T.I. set in G and which does not act semiregularly on $X \setminus \{x\}$. Then, for some integer $n \ge 3$ and some prime power q, we have $\mathbf{F}^*(G) \cong \mathrm{PSL}(n,q)$.*

The proof of this theorem will follow from Theorem 2 and from Propositions 7(ii), 8, 15, and 16. We shall imitate the argumentation in O'Nan's paper. In particular, we take over his results on the structural analysis of (H,K,L) configurations.

Flag transitive automorphism groups of finite linear spaces satisfying ($*$) have been investigated repeatedly; see [2], [3], [13], and [14]. We also note that in [1] a general attack on the classification of all flag transitive automorphism groups of finite linear spaces is announced. Clearly, this classification depends on the classification of the finite simple groups which is not needed in the present paper.

The geometric terminology and notation used in this paper follows that of [4]. The group-theoretic notation is standard. In addition, we define

$$\mathbf{F}_X(H) := \{x \in X : H \leq G_x\}$$

for each subgroup H of a permutation group G on a set X.

By a *linear space* we mean an incidence structure $D := (X, B, I)$ which satisfies $[x, y] = 1$ and $[x] \geq 2 \leq [j]$ for all $x, y \in X$ with $x \neq y$ and for each $j \in B$. D is called *finite* if $|X|$ is finite. For convenience in notation, we set $j = (j)$ for each $j \in B$. In particular, we write $\in$ instead of I and (X, B) instead of (X, B, I).

The following above-mentioned result [14; Satz 2] will play a crucial role in this paper.

THEOREM 2. *Let G be a flag transitive automorphism group of a finite linear space D. Assume that G satisfies $(*)$. Let X denote the point set of D. Let $x \in X$, and assume that G_x has an abelian normal subgroup which does not act semiregularly on $X \setminus \{x\}$. Then, for some integer $n \geq 3$ and some prime power q, we have $\mathbf{F}^*(G) \cong \mathrm{PSL}(n, q)$.*

For the remainder of this paper, we assume that G, D and X satisfy the hypotheses of Theorem 1.

1. Preliminary results. For a proof of the following lemma see [14; Lemmas 3 and 4].

LEMMA 3. *Let $x \in X$. Then*:

(i) *G_x acts transitively on $\{G_{xy} : y \in X \setminus \{x\}\}$ via conjugation.*
(ii) *Each normal subgroup of G_x is weakly closed in G_x with respect to G.*

Let $x \in X$ and $N \trianglelefteq G_x$. Then, by Lemma 3(ii), each one-point stabilizer of G contains exactly one conjugate of N. Thus, for each $r \in X$, we denote by N^r the unique conjugate of N contained in G_r. In particular, we write N^x instead of N. Furthermore, we set

$$N_s^r := N^r \cap G_s$$

for arbitrary elements $r, s \in X$ with $r \neq s$.

LEMMA 4. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Then*:

(i) *G_x acts transitively on $\{N_y^x : y \in X \setminus \{x\}\}$ via conjugation.*
(ii) *For each $y \in X \setminus \{x\}$, we have $N_y^x \trianglelefteq G_{xy}$.*
(iii) *If N^x is a T.I. set in G, then $\mathbf{N}_G(A) \leq G_x$ for each non-trivial subgroup A of N^x.*

P r o o f. Note that $N^x_y = N^x \cap G_{xy}$ for each $y \in X \setminus \{x\}$. Therefore, (i) follows from Lemma 3(i), and (ii) is obvious.

To prove (iii), let $g \in \mathbf{N}_G(A)$. Then

$$\mathbf{1} \neq A \leq N^x \cap (N^x)^g .$$

Since N^x is a T.I. set in G, this implies that $g \in \mathbf{N}_G(N^x) = G_x$. ∎

LEMMA 5. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Assume that N^x is a T.I. set in G. Let $y \in X \setminus \{x\}$, and assume that N^x_y is abelian. Let $p \in \pi(N^x_y)$, and denote by $W_p(x, y)$ the weak closure of $\mathbf{O}_p(N^x_y)$ in G_{xy} with respect to G. Set $j := \mathbf{F}_X(\mathbf{O}_p(N^x_y))$ and $B_p := j^G$. Then*:

(i) *(X, B_p) is a linear space on which G acts flag transitively.*
(ii) *$W_p(x, y)$ is an abelian p-group.*
(iii) *$j = \mathbf{F}_X(W_p(x, y))$.*
(iv) *$G_j = \mathbf{N}_G(W_p(x, y))$.*
(v) *$G_{xj} = \mathbf{N}_{G_x}(\mathbf{O}_p(N^x_y))$.*

P r o o f. (i) Assume that there exist $r, s \in X$ such that $r \neq s$ and

$$\mathbf{F}_X(\mathbf{O}_p(N^r_s)) \neq \mathbf{F}_X(\mathbf{O}_p(N^s_r)) .$$

Then, by Lemma 4(i), there exists $t \in \mathbf{F}_X(\mathbf{O}_p(N^r_s))$ such that $t \notin \mathbf{F}_X(\mathbf{O}_p(N^s_r))$.

Since $t \in \mathbf{F}_X(\mathbf{O}_p(N^r_s))$, $\mathbf{O}_p(N^r_s) \leq G_t$. Thus, by Lemma 4(ii), $\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t)$ is a group.

Since $t \notin \mathbf{F}_X(\mathbf{O}_p(N^s_r))$, $\mathbf{O}_p(N^s_r) \not\leq G_t$. Thus, by Lemma 4(i), $\mathbf{O}_p(N^s_t) \not\leq G_r$, whence

$$(\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t))_r < \mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t) . \tag{1}$$

As $\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t)$ is a p-group, we find an element $g \in \mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t) \backslash G_r$ which normalizes $(\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t))_r$.

We now have

$$\mathbf{O}_p(N^r_s) \leq (\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t))_r = ((\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t))_r)^g \leq G_{r^g}$$

and

$$\mathbf{O}_p(N^{r^g}_s) \leq ((\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t))_r)^g = (\mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t))_r \leq G_r .$$

Thus, we conclude that $\mathbf{O}_p(N^r_s) = \mathbf{O}_p(N^r_{r^g})$ and $\mathbf{O}_p(N^{r^g}_s) = \mathbf{O}_p(N^{r^g}_r)$, so, by (1),

$$\mathbf{O}_p(N^{r^g}_r)\,\mathbf{O}_p(N^r_{r^g}) < \mathbf{O}_p(N^r_s)\,\mathbf{O}_p(N^s_t) .$$

By Lemma 4(i), this leads to the contradiction

$$\mathbf{1} < \mathbf{O}_p(N^{r^g}_r) \cap \mathbf{O}_p(N^r_{r^g}) .$$

Thus, we have shown that

$$\mathbf{F}_X(\mathbf{O}_p(N_s^r)) = \mathbf{F}_X(\mathbf{O}_p(N_r^s)) \tag{2}$$

for all $r, s \in X$ with $r \neq s$.

Let $r, s, t, u \in X$ with $r \neq s$, $t \neq u$, and $t, u \in \mathbf{F}_X(\mathbf{O}_p(N_s^r))$. We shall show that $\mathbf{F}_X(\mathbf{O}_p(N_s^r)) = \mathbf{F}_X(\mathbf{O}_p(N_u^t))$.

Without loss of generality we may assume that $r \neq t$. Since $t \in \mathbf{F}_X(\mathbf{O}_p(N_s^r))$, $\mathbf{O}_p(N_s^r) \leq G_t$, whence $\mathbf{O}_p(N_s^r) = \mathbf{O}_p(N_t^r)$. But now (2) yields $\mathbf{F}_X(\mathbf{O}_p(N_s^r)) = \mathbf{F}_X(\mathbf{O}_p(N_r^t))$. Therefore, since $u \in \mathbf{F}_X(\mathbf{O}_p(N_s^r))$, $\mathbf{O}_p(N_r^t) \leq G_u$, which yields $\mathbf{O}_p(N_r^t) = \mathbf{O}_p(N_u^t)$. It follows that $\mathbf{F}_X(\mathbf{O}_p(N_s^r)) = \mathbf{F}_X(\mathbf{O}_p(N_u^t))$, as desired.

Now (i) follows from Lemma 4(i).

(ii) Let $g, h \in G$ such that $\mathbf{O}_p(N_y^x)^g \leq G_{xy}$ and $\mathbf{O}_p(N_y^x)^h \leq G_{xy}$. Then $x, y \in j^g \cap j^h$, whence, by (i), $j^g = j^h$. Now, by Lemma 4(ii), $\mathbf{O}_p(N_y^x)^g$ and $\mathbf{O}_p(N_y^x)^h$ normalize each other. Since, by hypothesis, N^x is a T.I. set in G, we even have $[\mathbf{O}_p(N_y^x)^g, \mathbf{O}_p(N_y^x)^h] = \mathbf{1}$.

(iii) follows from (i), and (iv) follows from (iii).

(v) We have

$$\mathbf{N}_{G_x}(W_p(x,y)) \leq \mathbf{N}_{G_x}(W_p(x,y) \cap N^x) = \mathbf{N}_{G_x}(\mathbf{O}_p(N_y^x))\,.$$

Therefore, the claim follows from (iv). ∎

LEMMA 6. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Assume that N^x is a T.I. set in G. Let $y \in X \setminus \{x\}$, and assume that N_y^x is abelian. Then $[N_x^y, \mathbf{N}_{N^x}(N_y^x)] \leq N_y^x$.*

Proof. The groups N^x and $N^x N_x^y$ act transitively on the set y^{N^x}. Furthermore, we have $(N^x)_y = N_y^x$ and $(N^x N_x^y)_y = N_y^x N_x^y$.

On the other hand, Lemma 5(i) yields $\mathbf{F}_X(N_y^x) = \mathbf{F}_X(N_y^x N_x^y)$. Thus, by [12; Theorem 3.5], $\mathbf{N}_{N^x}(N_y^x)$ and $\mathbf{N}_{N^x N_x^y}(N_y^x N_x^y)$ both act transitively on $y^{N^x} \cap \mathbf{F}_X(N_y^x)$, whence $|\,\mathbf{N}_{N^x}(N_y^x) : N_y^x| = |\,\mathbf{N}_{N^x}(N_y^x N_x^y) : N_y^x|$.

Thus, $\mathbf{N}_{N^x}(N_y^x N_x^y) = \mathbf{N}_{N^x}(N_y^x)$, which gives the desired conclusion. ∎

PROPOSITION 7. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Assume that N^x is a T.I. set in G. Let $y \in X \setminus \{x\}$, and assume that $N_y^x \neq \mathbf{1}$. Then*:

(i) *If N_y^x is abelian and the Sylow 2-subgroup of N_y^x is not cyclic, then either $N_y^x \trianglelefteq N^x$, or, for some integer $e \geq 2$, we have $N^x \cong \mathrm{SL}(2, 2^e)$ and $|N_y^x| = 2^e$.*

(ii) *One of the following holds.*
   - (a) *N^x is a Frobenius group with $\mathbf{Z}(\mathbf{F}(N^x))_y \neq \mathbf{1}$.*
   - (b) *N_y^x is a non-abelian Frobenius complement and a Hall subgroup of N^x. N_y^x has a normal complement in N^x.*
   - (c) *N_y^x is abelian.*

P r o o f. From Lemma 4(iii) we may conclude that N_x^y is isomorphic to a subgroup of $\mathrm{Aut}(N^x)$.

Let $g \in N_x^y \setminus \mathbf{1}$. Then, by Lemma 4(iii), $\mathbf{C}_{N^x}(g) \le N_y^x$. Since $[g, N_y^x] \le N^y \cap N^x = \mathbf{1}$, we thus have $\mathbf{C}_{N^x}(g) = N_y^x$. Clearly, Lemma 4(i) implies that $N_y^x \cong N_x^y$. Thus, we have a constrained (N_x^y, N^x, N_y^x) configuration in the sense of O'Nan [10].

Now (i) follows from Lemma 6 and [10; Proposition 4.26], and (ii) is a consequence of [10; Propositions 4.9 and 4.15]. ∎

Note that, if, for some $x \in X$, G_x has a normal subgroup which is a T.I. set in G and which satisfies condition (a) of Proposition 7(ii), then Theorem 1 follows from Theorem 2. In the following two sections, we shall treat the cases (b) and (c) of Proposition 7(ii).

2. Case (b) of Proposition 7(ii). The purpose of this section is the proof of the following proposition which shows that case (b) of Proposition 7(ii) leads to a contradiction.

PROPOSITION 8. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Assume that N^x is a T.I. set in G. Let $y \in X \setminus \{x\}$. Then one of the following conditions must be false*:

(a) *N_y^x is cyclic of prime order.*
(b) *N_y^x is a Sylow subgroup of N^x.*
(c) *N_y^x has a normal complement in N^x.*

For the sake of clarity we break up the proof of Proposition 8 into a sequence of lemmas.

For the remainder of this section, let $x \in X$, and let N^x be a normal subgroup of G_x which is a T.I. set in G. Let $y \in X \setminus \{x\}$, and assume that y satisfies (a), (b), and (c) of Proposition 8. By M^x we denote the normal complement of N_y^x in N^x.

At the end of this section, we shall obtain a contradiction from Lemma 13.

LEMMA 9. *N^x is a Frobenius group with kernel M^x and complement N_y^x.*

P r o o f. By Lemma 6, we have $[N_x^y, \mathbf{C}_{M^x}(N_y^x)] \le M^x \cap N_y^x = \mathbf{1}$. Thus, Lemma 4(iii) yields $\mathbf{C}_{M^x}(N_y^x) = \mathbf{1}$. ∎

Set $p := |N_y^x|$, and define $W_p(x,y)$, j, and B_p as in Lemma 5. For each $r \in X$, we set $(r) := \{h \in B_p : r \in h\}$. We temporarily define $W := W_p(x,y)$, $B := B_p$,

$$\mathcal{J} := \{N_s^r : r, s \in j,\ r \neq s\},$$

and

$$\mathcal{M} := \{V \le W : |W : V| = p,\ \mathbf{C}_{M^x}(V) \neq \mathbf{1}\}.$$

Lemma 10. (i) $\mathcal{M} \neq \emptyset$.

(ii) $\bigcap_{V \in \mathcal{M}} V = \mathbf{1}$.

(iii) *Let* $r, s \in X$ *with* $r \neq s$. *Suppose* $V \in \mathcal{M}$. *Then* $N_s^r \cap V = \mathbf{1}$.

Proof. From [8; Satz 7.22] we conclude that

$$M^x = \langle \mathbf{C}_{M^x}(V) : V \in \mathcal{M} \rangle \,. \tag{3}$$

(i) follows from (3).

(ii) Set $T := \bigcap_{V \in \mathcal{M}} V$. Then, by (3), M^x centralizes T. On the other hand, by Lemma 5(iii), $T \leq G_{(j)}$. Note also that, by (b) and (c), M^x acts transitively on (x). Thus, we conclude that $T = \mathbf{1}$, as desired.

(iii) If $r \neq x$, then $\mathbf{C}_{M^x}(N_s^r) = \mathbf{1}$, by Lemma 4(iii). If $r = x$, the same conclusion follows from Lemma 9. Thus, (iii) is a consequence of the definition of $\mathcal{M}$. ∎

Lemma 11. *Let* $V \in \mathcal{M}$, *and set* $B(V) := \{h \in B : h \subseteq \mathbf{F}_X(V)\}$. *Then*:

(i) $(\mathbf{F}_X(V), B(V))$ *is a linear space.*

(ii) $\mathbf{C}_G(V)$ *acts flag transitively on* $(\mathbf{F}_X(V), B(V))$.

(iii) $\mathbf{C}_{M^x}(V)$ *acts regularly on* $(x) \cap B(V)$.

Proof. Set $Y := \mathbf{F}_X(V)$, and let $r \in Y$. Assume that there exists $h \in (r)$ such that $h \subseteq Y$.

Since $r \in Y$, $V \leq G_r$. Therefore, $M^r V$ is a group acting on (r).

By Lemmas 5(v) and 9, M^r acts regularly on (r). Thus, $M^r V$ acts faithfully on (r), and we have $V = (M^r V)_h$. In particular, by [12; Theorem 3.5], $\mathbf{C}_{M^r}(V)$ acts transitively on $\{i \in (r) : V \leq G_i\}$. But, as $h \in (r)$ and $r \subseteq Y$, this implies that $i \subseteq Y$ for all $i \in (r)$ with $V \leq G_i$.

Since $j \subseteq Y$, the preceding discussion shows that $i \subseteq Y$ for every $i \in (x)$ with $2 \leq |i \cap Y|$. In particular, the opening assumption is satisfied for each $r \in Y$, and we conclude that (i) and (iii) hold.

(ii) follows from (iii) and [9; Lemma 4.9]. ∎

Lemma 12. *We have* $|W| = p^2$.

Proof. From Lemma 11(ii) it follows that $\mathbf{C}_{G_j}(V)$ acts transitively on $\mathcal{J}$. Since $\langle \mathcal{J} \rangle = W$, we may apply [10; Lemmas 3.15 and 3.16]. Thus, there exists a subgroup P of W such that $|W : P| = p^2$ and $P \leq V$ for each $V \in \mathcal{M}$. Now the claim follows from Lemma 10(ii). ∎

Lemma 13. *We have* $|j| = 2$.

Proof. We assume that $3 \leq |j|$ $(= |\mathcal{J}|)$. Define $A := G_j/G_{(j)}$. From Lemmas 12 and 5(iv) we conclude that A is isomorphic to a subgroup of $\mathrm{PGL}(2, p)$.

Assume that p divides $|A|$. Then, since $3 \leq |\mathcal{J}|$, $|\mathcal{J}| \in \{p, p+1\}$. Therefore, by Lemma 10(iii), $|\mathcal{M}| = 1$, contrary to Lemma 10(ii).

Thus, p does not divide $|A|$. In particular, A considered as a permutation group on j has cyclic one-point stabilizers. On the other hand, Lemma 3(i) implies that all two-point stabilizers of A have the same size. Thus, we have $G_{xy} = G_{(j)}$, which means that A acts regularly or as a Frobenius group on j.

Let K denote the regular normal subgroup of A. Suppose $V \in \mathcal{M}$. Then, by Lemma 11(ii), $\mathbf{C}_{G_j}(V)$ acts transitively on j. Thus, the image of $\mathbf{C}_{G_j}(V)$ in A contains K. Since $K \neq \mathbf{1}$ and V is arbitrary, we must have

$$|\mathcal{M}| = 2\,. \tag{4}$$

Define $v := |X|$, $k := |j|$, $a := |A_x|$, and $\{V_1, V_2\} := \mathcal{M}$. For each $i \in \{1,2\}$, we set $v_i := |\,\mathbf{F}_X(V_i)|$ and $r_i := |\,\mathbf{C}_{M^x}(V_i)|$.

First of all, by [7; Theorem 5.3.16],

$$M^x = \mathbf{C}_{M^x}(V_1)\,\mathbf{C}_{M^x}(V_2)\,.$$

Therefore, Lemma 9 yields $|M^x| = r_1 r_2$. On the other hand, by Lemmas 5(v) and 9, M^x acts regularly on (x). Thus, by Lemma 5(i),

$$v - 1 = r_1 r_2 (k-1) \tag{5}$$

and

$$|G| = v r_1 r_2 a |G_{xy}|\,. \tag{6}$$

From Lemma 11(i), (iii) we obtain

$$v_i - 1 = r_i (k-1) \tag{7}$$

for each $i \in \{1,2\}$.

From $3 \leq |\mathcal{J}|$ we conclude that, for each $i \in \{1,2\}$, $G_{xy} \leq \mathbf{N}_{G_{xj}}(V_i)$. Conversely, $\mathbf{N}_{G_{xj}}(V_i)$ normalizes N_y^x, V_1, and V_2; see Lemma 5(iv), (v) and (4). Thus, $\mathbf{N}_{G_{xj}}(V_i) = G_{xy}$, and so, by Lemma 11(ii),

$$|\,\mathbf{N}_G(V_i)| = v_i r_i |G_{xy}| \tag{8}$$

for each $i \in \{1,2\}$.

Assume without loss of generality that $r_1 \leq r_2$.

If $r_1 = r_2$, then, by (6) and (8), v_1 divides $v r_1 a$. Thus, since a divides $k-1$, (7) implies that v_1 divides v. Now, by (5) and (7), we have

$$1 + r_1(k-1) \,|\, 1 + r_1^2 (k-1)\,.$$

But clearly this is impossible. Consequently, we must have

$$r_1 < r_2\,. \tag{9}$$

From (7) and (9) we conclude that V_1 and V_2 cannot be conjugate in G. In particular, G_{xj} normalizes V_1, V_2, and N_y^x, whence $G_{xy} = G_{xj}$; equivalently,

$$a = 1\,. \tag{10}$$

Define $b := |B|$, $b_2 := |B(V_2)|$, and $c := |G : \mathbf{N}_G(V_2)|$. Then Lemma 5(i) implies that $bk = vr_1r_2$, and Lemma 11(i) yields $b_2k = v_2r_2$. In particular, by (6), (8), and (10), it follows that $b = b_2c$. Thus, V_2 is weakly closed in G_{xy} with respect to G; equivalently,

$$(X, \{\mathbf{F}_X(V_2)^g : g \in G\})$$

is a linear space (on which G acts block transitively).

Now (7), [4; 1.3.8], and (5) yield $r_2 \le r_2(k-1) = v_2 - 1 < r_1$, contrary to (9). ■

By Lemma 13, G acts doubly transitively on X, and G_x considered as a permutation group on $X \setminus \{x\}$ has a regular normal subgroup. Thus, [10; Lemma 3.7] yields a contradiction. This proves Proposition 8.

3. Case (c) of Proposition 7(ii). In this section, we shall show that Theorem 1 holds if, for some $x \in X$, G_x has a normal subgroup N^x which is a T.I. set in G and which satisfies $\mathbf{1} \neq \mathbf{Z}(N^x_y) = N^x_y$ for each $y \in X \setminus \{x\}$.

LEMMA 14. *Let $x \in X$, and assume that G_x has a normal subgroup of odd order which is a T.I. set in G and which does not act semiregularly on $X \setminus \{x\}$. Then, for some integer $n \ge 3$ and some prime power q, we have $\mathbf{F}^*(G) \cong \mathrm{PSL}(n, q)$.*

P r o o f. Take N^x to be a normal subgroup of odd order of G_x that is minimal with respect to the property that N^x is a T.I. set in G which does not act semiregularly on $X \setminus \{x\}$.

By the Feit–Thompson Theorem [5], N^x is solvable. Thus, $(N^x)' < N^x$. Take $p \in \pi(N^x/(N^x)')$, and let M^x denote the (unique) smallest normal subgroup of N^x the factor group of which is an elementary abelian p-group. Then $M^x \trianglelefteq G_x$.

By Theorem 2, we are done if N^x is abelian. Clearly, if $M^x = \mathbf{1}$, then N^x must be abelian. We now consider the case that $M^x \neq \mathbf{1}$.

Let $y \in X \setminus \{x\}$. Then, by Lemma 4(iii), the (minimal) choice of N^x yields $\mathbf{C}_{M^x}(g) = \mathbf{1}$ for every $g \in N^y_x \setminus \mathbf{1}$. Therefore, M^x is a nilpotent p'-group, and $|N^x_y| = p$; see [7; Theorem 10.3.1(iv)]. In particular, by Proposition 8, $N^x_y \notin \mathrm{Syl}_p(N^x)$.

Let P be an N^y_x-invariant Sylow p-subgroup of N^x which contains N^x_y. Since $N^x_y < P$,

$$\mathbf{C}_{N^y_x P}(N^y_x N^x_y) = N^y_x N^x_y < \mathbf{N}_{N^y_x P}(N^y_x N^x_y)\,.$$

Thus, N^y_x is conjugate in G_x to each of the proper subgroups of $N^y_x N^x_y$ different from N^x_y. In particular, each element of $N^y_x N^x_y \setminus N^x_y$ induces a fixed-point-free automorphism of M^x, whence, by [7; Theorem 6.2.4], N^x_y

centralizes M^x. It follows that $\mathbf{1} \neq N_y^x \leq \mathbf{Z}(N^x)$. Thus, the minimal choice of N^x forces N^x to be abelian. ∎

PROPOSITION 15. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Assume that N^x is a T.I. set in G. Let $y \in X \setminus \{x\}$, and assume that N_y^x is a non-trivial abelian group of odd order. Then, for some integer $n \geq 3$ and some prime power q, we have $\mathbf{F}^*(G) \cong \mathrm{PSL}(n, q)$.*

Proof. Take N^x to be a normal subgroup of G_x that is minimal with respect to the properties that N^x is a T.I. set in G, and that N_y^x is a non-trivial abelian group of odd order.

If $|N^x|$ is odd, then we are done by Lemma 14. Therefore, we assume henceforth that N^x has even order. Let m be an involution in N^x. Then $\mathbf{F}_X(\langle m \rangle) = \{x\}$. Set $z := y^m$.

Take $p \in \pi(N_y^x)$, define B_p as in Lemma 5, and let $h \in B_p$ such that $y, z \in h$.

First of all, we shall prove that

$$(2, |h|) = 1 . \tag{11}$$

Assume first that there exists $g \in G_h$ such that $|m^g m|$ is even.

Since $\mathbf{F}_X(\langle m \rangle) = \{x\}$, $\langle m^g, m \rangle$ has a unique orbit Y of odd length in X.

Let a generate the subgroup of $\langle m^g, m \rangle$ which fixes all points of Y. Set $k := am$. Then k is an involution in $\langle m^g, m \rangle$. Thus, k is conjugate to m^g or to m. In particular, k is conjugate to some involution of N^x. On the other hand, since $a, m \in G_x$, $k \in G_x$. Consequently, we must have $k \in N^x$, whence $a = km \in N^x$.

Suppose $g \notin G_x$. Then $\{x\} \subset Y$. Thus, since $a \in (N^x)_Y$, $|a|$ is odd. It follows that $|m^g m| = |Y| \cdot |a|$ is odd, contrary to the choice of g.

Thus, we have $g \in G_x$, whence

$$\langle m^g, m \rangle \leq N^x . \tag{12}$$

Set $W := W_p(y, z)$, where $W_p(y, z)$ is defined as in Lemma 5. Since $g \in G_h$, Lemma 5(iv) implies that $\langle m^g, m \rangle \leq \mathbf{N}_G(W)$. Let i denote the central involution of $\langle m^g, m \rangle$. Then, by [7; Theorem 6.2.4], (12), and Lemma 4(iii),

$$W = \langle \mathbf{C}_W(m), \mathbf{C}_W(i), \mathbf{C}_W(mi) \rangle \leq G_x .$$

In particular, by Lemma 5(iii), $x \in h$. Thus, (11) follows from $\mathbf{F}_X(\langle m \rangle) = \{x\}$.

Assume next that $|m^g m|$ is odd for every $g \in G_h$. Define $A := G_h / G_{(h)}$. Let m^* denote the image of m in A. Then, by a theorem of Glauberman [6, Theorem 1],

$$A = \mathbf{O}(A)\, \mathbf{C}_A(m^*) . \tag{13}$$

Assume first that A acts regularly on h. Then, by Lemma 5(v),

$$\mathbf{N}_{G_y}(\mathbf{O}_p(N^y_z)) = G_{yh} = G_{yz}\,,$$

whence $\mathbf{N}_{N^y}(\mathbf{O}_p(N^y_z)) = N^y_z$. By hypothesis, N^y_z is abelian. Thus, by a theorem of Burnside [7; Theorem 7.4.3], N^y has a normal p-complement.

The minimal choice of N^x yields $\mathbf{O}_{p'}(N^y)_z = \mathbf{1}$. Thus, by Lemma 4(iii), $\mathbf{C}_{\mathbf{O}_{p'}(N^y)}(g) = \mathbf{1}$ for each $g \in N^z_y \setminus \mathbf{1}$. In particular, by [7; Theorem 10.3.1(v)], $|\mathbf{\Omega}_1(\mathbf{O}_p(N^z_y))| = p$. But then the minimal choice of N^x forces $N^x = \mathbf{O}_{p'}(N^x)\mathbf{\Omega}_1(\mathbf{O}_p(N^x_y))$, contrary to Proposition 8.

Assume next that A acts as a Frobenius group on h. Let K denote the kernel of A. If $m^* \in K$, then $[\mathbf{O}(A), m^*] = \mathbf{1}$, whence $m^* = 1$, contrary to the choice of m. Thus, m^* has a fixed point in h. Again, (11) follows from $\mathbf{F}_X(\langle m \rangle) = \{x\}$.

Assume finally that A acts neither regularly nor as a Frobenius group on h. Then

$$(h, \{\mathbf{F}_X(G_{rs}) : r, s \in h\,,\ r \neq s\})$$

is a linear space on which A acts as a flag transitive automorphism group. Thus, by [4; 2.3.7(a)], A acts primitively on h. In particular, by (13), $\mathbf{O}(A) \neq \mathbf{1}$, which yields (11).

As $\mathbf{F}_X(\langle m \rangle) = \{x\}$, (11) yields

$$m \in \bigcap_{i \in (x)} G_i\,, \tag{14}$$

where, as usual, $(x) := \{i \in B_p : x \in i\}$.

Define $M^x := \langle m^g : g \in G_x \rangle$. Then, by (14) and Lemma 5(iv),

$$M^x \leq \mathbf{N}_G(W)\,, \tag{15}$$

where, as above, W denotes the weak closure of $\mathbf{O}_p(N^y_z)$ in G_{yz} with respect to G. Moreover, by the minimal choice of N^x, we must have $M^x_y = \mathbf{1}$ or $M^x = N^x$.

In the first case, (15) implies that

$$[m, W] \leq [M^x, W] \leq M^x \cap W \leq M^x_y = \mathbf{1}\ .$$

Hence, by Lemma 4(iii), $m \in G_y$, contrary to the choice of m.

In the second case, (15) implies that

$$N^x = \mathbf{N}_{N^x}(W) \leq \mathbf{N}_G(N^x_y)\,.$$

Thus, $\mathbf{1} \neq \mathbf{O}_p(N^x_y) \leq \mathbf{O}_p(N^x)_y$, contrary to the choice of N^x. ∎

Proposition 16. *Let $x \in X$ and $N^x \trianglelefteq G_x$. Assume that N^x is a T.I. set in G. Let $y \in X \setminus \{x\}$, and assume that N^x_y is an abelian group of even order. Then, for some integer $n \geq 3$ and some prime power q, we have $\mathbf{F}^*(G) \cong \mathrm{PSL}(n, q)$.*

P r o o f. Assume first that the Sylow 2-subgroup of N_y^x is not cyclic. Then, by Proposition 7(i), either $N_y^x \trianglelefteq N^x$, or, for some integer $e \geq 2$, we have $N^x \cong \mathrm{SL}(2, 2^e)$ and $|N_y^x| = 2^e$.

In the first case, we conclude that $\mathbf{1} \neq \mathbf{O}_2(N_y^x) \trianglelefteq N^x$. This forces $\mathbf{Z}(\mathbf{O}_2(N^x))_y \neq \mathbf{1}$, and we are done by Theorem 2.

In the second case, we have

$$|N^x : \mathbf{N}_{N^x}(N_y^x)| = 2^e + 1 \tag{16}$$

and

$$|\,\mathbf{N}_{N^x}(N_y^x) : N_y^x| = 2^e - 1\,. \tag{17}$$

Define B_2 and j $(\in B_2)$ as in Lemma 5. Set $(x) := \{h \in B_2 : x \in h\}$. Since $N_y^x \in \mathrm{Syl}_2(N^x)$, N^x acts transitively on (x). Thus, by Lemma 5(v) and (16), $|(x)| = 2^e + 1$.

On the other hand, by Lemma 5(v) and (17), $2^e - 1$ divides $|j| - 1$. Thus, $|j| = 2^e$; see [4; 1.3.8]. In particular, (X, B_2) is an affine plane on which G acts doubly transitively. Thus, by [11; Theorem 1], G has a normal subgroup R acting regularly on X. It follows that $RN^x = RN^y$, whence

$$N_y^x = (RN^x)_y = (RN^y)_x = N_x^y\,,$$

contradiction.

Assume next that the Sylow 2-subgroup of N_y^x is cyclic. For all $r, s \in X$ with $r \neq s$ we denote by n_s^r the unique involution in N_s^r. Let $t \in \mathbf{F}_X(\langle n_s^r \rangle) \setminus \{r, s\}$. Then, by Lemma 4(ii), $[n_s^r, n_t^s] = 1$. Therefore, $n_t^s \in G_r$; see Lemma 4(iii). It follows that $n_r^s = n_t^s \in G_t$, whence $t \in \mathbf{F}_X(\langle n_r^s \rangle)$. Thus, we have shown that

$$\mathbf{F}_X(\langle n_s^r \rangle) = \mathbf{F}_X(\langle n_r^s \rangle) \tag{18}$$

for all $r, s \in X$ with $r \neq s$.

Take $z \in X \setminus \mathbf{F}_X(\langle n_y^x \rangle)$. Define $w := z^{n_y^x}$, and let Q be an n_z^w-invariant Sylow 2-subgroup of N^z such that $\mathbf{O}_2(N_w^z) \leq Q$.

Suppose $\mathbf{O}_2(N_w^z) = Q$. Then Q is cyclic. Thus, by [7; Theorem 7.4.3], N^z has a normal 2-complement. Set $M^z := \mathbf{O}(N^z)\langle n_w^z \rangle$. Then, by Proposition 8, we must have $\langle n_w^z \rangle < M_w^z$, whence $\mathbf{O}(N^z)_w \neq \mathbf{1}$. Thus, the desired assertion follows from Lemma 14.

Suppose $\mathbf{O}_2(N_w^z) < Q$. Then $\mathbf{O}_2(N_w^z)\langle n_z^w \rangle < Q\langle n_z^w \rangle$. Take $g \in \mathbf{N}_{Q\langle n_z^w \rangle}(\mathbf{O}_2(N_w^z)\langle n_z^w \rangle)$ such that $g \notin \mathbf{O}_2(N_w^z)\langle n_z^w \rangle = \mathbf{C}_{Q\langle n_z^w \rangle}(\mathbf{O}_2(N_w^z)\langle n_z^w \rangle)$. Then g fixes $\mathbf{\Omega}_1(\mathbf{O}_2(N_w^z)\langle n_z^w \rangle) = \langle n_w^z, n_z^w \rangle$ and $Q \cap \mathbf{\Omega}_1(\mathbf{O}_2(N_w^z)\langle n_z^w \rangle) = \langle n_w^z \rangle$. Thus, n_z^w and $n_w^z n_z^w$ are conjugate in G. In particular, by (18),

$$\mathbf{F}_X(\langle n_w^z \rangle) = \mathbf{F}_X(\langle n_w^z, n_z^w \rangle)\,. \tag{19}$$

Since $w = z^{n_y^x}$, $n_y^x \in \mathbf{C}_G(n_w^z n_z^w)$. Therefore, by Lemma 4(iii), $n_w^z n_z^w \in G_x$. Now (19) implies that $n_w^z \in G_x$, whence $n_w^z = n_x^z$. Thus, by (18), we must have $n_z^x \in G_w$, which yields $n_z^x = n_w^x$. It follows that $n_y^x \in \mathbf{C}_G(n_z^x)$.

Since $z \in X \setminus \mathbf{F}_X(\langle n_y^x \rangle)$ is arbitrary, $\langle n_z^x : z \in X \setminus \{x\} \rangle$ is an abelian normal subgroup of G_x. Thus, the desired assertion follows from Theorem 2. ∎

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