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SOME EIGENVALUE ESTIMATES FOR WAVELET RELATED TOEPLITZ OPERATORS

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By a straightforward computation we obtain eigenvalue estimates for Toeplitz operators related to the two standard reproducing formulas of the wavelet theory. Our result extends the estimates for Calderón–Toeplitz operators obtained by Rochberg in [R2].

In the first section we recall two standard reproducing formulas of the wavelet theory, we define Toeplitz operators and discuss some of their properties. The second section contains precise statements of our results and their proofs. At the end of the second section we include some comments about the range of applicability of our estimates.

1. Introduction and preliminaries. The first reproducing formula which we discuss is based on the Schrödinger representation of the Heisenberg group. We take any square integrable function ϕ with unit norm defined on the d-dimensional Euclidean space. This function provides the following resolution of the identity:

$$I = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_{pq} \otimes \phi_{pq} \, dp \, dq \,. \tag{1}$$

We denote by ϕ_{pq} the action of the Schrödinger representation of the Heisenberg group on the function ϕ, i.e.

$$\phi_{pq}(x) = e^{-\pi ipq+2\pi ipx}\phi(x-q) \,.$$

The symbol $\phi_{pq} \otimes \phi_{pq}$ stands for the orthogonal projection on the function ϕ_{pq}.

The second reproducing formula makes use of the standard action of the "$ax+b$"-group by translations and dilations. It is called the *Calderón reproducing formula*. Again we take a square integrable function ψ defined

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on the d-dimensional Euclidean space, but now we assume that it satisfies the *admissibility condition*, i.e. for almost every $\xi \in \mathbb{R}^d$,

$$\int_0^\infty |\widehat{\psi}(s\xi)|^2 \frac{ds}{s} = 1,$$

where $\widehat{\psi}$ is the Fourier transform of ψ, i.e.

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^d} \psi(x) e^{-2\pi i x\xi}\, dx.$$

Under the above assumption on ψ,

$$I = \int_G \psi_\zeta \otimes \psi_\zeta \, d\zeta. \tag{2}$$

The symbol G denotes the "$ax+b$"-group, i.e.

$$G = \{\zeta = (v,t) : v \in \mathbb{R}^d,\ t > 0\},$$

$d\zeta = t^{-d-1} dv\, dt$ is the left invariant measure on G, and for $\zeta = (v,t)$,

$$\psi_\zeta(x) = t^{-d/2} \psi\left(\frac{x-v}{t}\right).$$

Both these formulas are understood in a weak sense and both are easily checked by a direct Fourier transform computation. They are particular cases of reproducing formulas related to square integrable representations. The names "reproducing formulas" come from the interpretation of (1) and (2) as the identities

$$f = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle f, \phi_{pq} \rangle \phi_{pq}\, dp\, dq,$$

$$f = \int_G \langle f, \psi_\zeta \rangle \psi_\zeta \, d\zeta,$$

valid for all square integrable functions f.

The standard function ϕ which appears in the formula (1) is well localized near the origin and the same is true for its Fourier transform. In this case the function ϕ_{pq} is localized near q while its Fourier transform is localized near p.

A similar interpretation is valid for the function $\psi_{(u,s)}$ from the formula (2). It is natural to take ψ localized near the origin with Fourier transform concentrated in a neighborhood of the sphere $|\xi| = 1$. In this situation $\psi_{(u,s)}$ is localized near u while its Fourier transform is concentrated in a neighborhood of the sphere $|\xi| = s^{-1}$.

For functions $b(p,q)$ and $\mathrm{b}(\zeta)$ (called *symbol functions*) defined on $\mathbb{R}^d \times \mathbb{R}^d$ and G respectively we define two corresponding Toeplitz operators

$$T_b = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(p,q)\, \phi_{pq} \otimes \phi_{pq}\, dp\, dq\,,$$

$$\mathrm{T_b} = \int_G \mathrm{b}(\zeta)\, \psi_\zeta \otimes \psi_\zeta\, d\zeta\,.$$

These are integral operators acting on $L^2(\mathbb{R}^d)$ and their kernels are

$$K_b(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(p,q)\, \phi(x-q) e^{2\pi ipx} \overline{\phi(y-q)} e^{-2\pi ipy}\, dp\, dq\,,$$

and

$$\mathrm{K_b}(x,y) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{b}(u,s) \psi_s(x-u) \overline{\psi_s(y-u)}\, du\, \frac{ds}{s}\,,$$

where $\psi_s(x) = s^{-d}\psi(x/s)$. These Toeplitz operators may be thought of as perturbations of the corresponding reproducing formulas where a weight has been attributed to each projection $\phi_{pq} \otimes \phi_{pq}$, $\psi_\zeta \otimes \psi_\zeta$. The operator $\mathrm{T_b}$ is called a *Calderón–Toeplitz operator*. For more details and further motivation we refer the reader to [R1], [D2], [F]. In our considerations we restrict attention to compactly supported functions ϕ, ψ.

The purpose of this paper is to show the relation between the eigenvalues of Toeplitz operators with nonnegative compactly supported symbols and the squares of the absolute values of the Fourier coefficients of the function ϕ or ψ. In the time-frequency representation Toeplitz operators with nonnegative compactly supported symbols localize functions on which they act to the region which is essentially the support of the symbol. For this reason Toeplitz operators with pairwise disjoint supports of their symbols may be thought of as pairwise orthogonal. On the other hand, Toeplitz operators with nonnegative compactly supported symbols form building blocks for Toeplitz operators with general symbols.

The eigenvalue estimates for Calderón–Toeplitz operators based on the Haar function obtained by Rochberg in [R2] provide the motivation for our study. The result of Rochberg asserts that the eigenvalues of Calderón–Toeplitz operators with nonnegative, bounded, compactly supported symbols decay no faster than $1/n^2$ and that this estimate determines the cut-off at $p = 1/2$ in the scale of the Schatten ideals S^p. In the same paper Rochberg asks the question about the relationship between the properties of the wavelet function and the cut-off for the operator. Our results provide a partial solution of that problem and the analogous problem for Toeplitz operators based on the Schrödinger representation. Although our approach is a straightforward computation it provides a precise description of the behavior of the eigenvalues for some classes of functions ϕ, ψ. In particular, we get two-sided estimates by comparable sequences for convolution powers of the Haar function taken as ψ.

The problem of studying the cut-off in the behavior of certain classes of operators depending on symbol functions has recently attracted some attention. Some references dealing with this and related problems are [AFP], [BS], [D1], [DP], [JW], [PRW], [R2], [RS], [RT], [S]. Very often the cut-off is described in terms of Schatten or Schatten–Lorentz ideals. In some cases it is possible to get the estimates of the eigenvalues themselves. In this note we follow the second direction.

2. The main results. We start this section by recalling a standard proposition dealing with eigenvalue estimates.

PROPOSITION. *Let T be a compact, positive operator defined on a Hilbert space H and let s_n, $n = 0, 1, 2, \ldots$, denote its eigenvalues written in nonincreasing order. Suppose that V_N and V_{N+1} are, respectively, N-dimensional and $(N+1)$-dimensional subspaces of H.*

(i) *If*

$$\langle Tw, w\rangle \le b_N ||w||^2 \quad \text{for } w \in V_N^{\perp}\,, \tag{3}$$

then

$$s_N \le b_N\,.$$

(ii) *If*

$$\langle Tw, w\rangle \ge a_N ||w||^2 \quad \text{for } w \in V_{N+1}\,, \tag{4}$$

then

$$s_N \ge a_N\,.$$

Although our results are stated for symbols having the form of a product they also apply to general nonnegative, continuous, compactly supported symbols. We justify this statement for Toeplitz operators based on the Schrödinger representation. For Calderón–Toeplitz operators the argument goes along the same lines.

For a general nonnegative, continuous, compactly supported symbol b we take smooth, nonnegative, compactly supported functions b_1^0, b_2^0, b_1^1, b_2^1 which satisfy

$$b_1^0(p)b_2^0(q) \le b(p,q) \le b_1^1(p)b_2^1(q)\,.$$

The map $b \to T_b$ is positivity preserving, so

$$T_{b_1^0\otimes b_2^0} \le T_b \le T_{b_1^1\otimes b_2^1}\,.$$

The estimate from below for $T_{b_1^0\otimes b_2^0}$ and the estimate from above for $T_{b_1^1\otimes b_2^1}$ apply to T_b.

First we present estimates for Toeplitz operators based on the Schrödinger representation. Next we state their analogues for Calderón–Toeplitz operators.

THEOREM 1. *Let the square integrable function ϕ have compact support and let the symbol function b have the form*

$$b(p,q)=b_1(p)b_2(q)\,.$$

Assume that both b_1 and b_2 are nonnegative and compactly supported, and that b_1 is integrable and b_2 is smooth (it is enough to assume that the Fourier transform of b_2 is integrable).

Take any box $B=[x_1^1,x_2^1]\times\ldots\times[x_1^d,x_2^d]$ containing the algebraic sum $\operatorname{supp} b_2+\operatorname{supp}\phi$ *in its interior. For $m=(m^1,\ldots,m^d)$, where all m^j are integers, and $\gamma_j=(x_2^j-x_1^j)^{-1}$, let*

$$m_B=(\gamma_1 m^1,\ldots,\gamma_d m^d)\,.$$

Let s_n denote the nonincreasing rearrangement of the eigenvalues of the operator T_b and for any positive integer M let a_n^M be the nonincreasing rearrangement of the sequence

$$\int_{\mathbb{R}^d} b_1(p)\,|\widehat{\phi}(Mm_B-p)|^2\,dp\,. \tag{5}$$

Under the above assumptions there are positive constants c, C and a natural number M such that for all natural N,

$$ca_N^M\le s_N\le Ca_N^1\,.$$

Proof. The operator T_b may be viewed as acting on $L^2(B)$. This is because $K_b(x,y)=0$ if either x or y is outside B. The functions $e^{2\pi i m_B x}$ form an orthogonal basis of $L^2(B)$. We expand $K_b(x,y)$ in a Fourier series and we get

$$\begin{aligned}K_b(x,y)=c_B\sum_{m,n}&\widehat{b}_2(m_B-n_B)\\&\times\int_{\mathbb{R}^d} b_1(p)\widehat{\phi}(m_B-p)\overline{\widehat{\phi}(n_B-p)}\,dp\,e^{2\pi i m_B x}e^{-2\pi i n_B y}\,.\end{aligned}$$

Let

$$W_N^M=\Big\{\sum_{m\in V_N^M}\lambda_m e^{2\pi i m_B x}\Big\},\quad\text{where}$$

$$V_N^M=\{Mm_0,\ldots,Mm_{N-1}\},\qquad \int_{\mathbb{R}^d} b_1(p)|\widehat{\phi}(M(m_j)_B-p)|^2\,dp=a_j^M\,.$$

Take $w \in (W_N^1)^\perp$; then

$$\langle T_b w, w\rangle = c_B \sum_{m,n\notin V_N^1} \widehat{b}_2(m_B - n_B) \times \int_{\mathbb{R}^d} b_1(p)\widehat{\phi}(m_B - p)\overline{\widehat{\phi}(n_B - p)}\, dp\, \lambda_n \overline{\lambda}_m \le C \int_{\mathbb{R}^d} b_1(p) \sum_{m\notin V_N^1} |\widehat{\phi}(m_B - p)|^2 |\lambda_m|^2 dp \le C a_N^1 ||w||^2 .$$

The above shows that the estimate (3) is satisfied, thus

$$s_N \le C a_N^1 .$$

To get the estimate from below we take a natural number M so large that

$$\widehat{b}_2(0) > \sum_{m\neq 0} |\widehat{b}_2(M m_B)| .$$

The above condition guarantees that the operator of convolution by the sequence $\widehat{b}_2(M m_B)$ is invertible.

Take $w \in W_{N+1}^M$; then

$$\langle T_b w, w\rangle = c_B \sum_{m,n\in V_{N+1}^M} \widehat{b}_2(m_B - n_B) \times \int_{\mathbb{R}^d} b_1(p)\widehat{\phi}(m_B - p)\overline{\widehat{\phi}(n_B - p)}\, dp\, \lambda_n \overline{\lambda}_m \ge c \int_{\mathbb{R}^d} b_1(p) \sum_{m\in V_{N+1}^M} |\widehat{\phi}(m_B - p)|^2 |\lambda_m|^2 dp \ge c a_N^M ||w||^2 .$$

This shows that the estimate (4) holds, therefore also

$$s_N \ge c a_N^M .$$

THEOREM 2. *Let ψ be a compactly supported, square integrable function satisfying the admissibility condition. Let the symbol function* b *have the form*

$$\mathrm{b}(u,s) = \mathrm{b}_1(u)\mathrm{b}_2(s) ,$$

where b_1 *is smooth, nonnegative, and compactly supported, and* b_2 *is integrable, nonnegative, and compactly supported in* $(0,\infty)$.

Take any box $B = [x_1^1, x_2^1] \times \ldots \times [x_1^d, x_2^d]$ such that the support of K_b *is contained in the interior of $B \times B$.*

Let s_n *denote the nonincreasing rearrangement of the eigenvalues of the operator* $\mathrm{T_b}$ *and let* a_n^M *be the nonincreasing rearrangement of the sequence*

$$\int_0^\infty \mathrm{b}_2(s)\,|\widehat{\psi}(sMm_B)|^2\,\frac{ds}{s}\,. \tag{6}$$

Under the above assumptions there are positive constants c, C *and a natural number* M *such that for all natural* N,

$$c\mathrm{a}_N^M \le \mathrm{s}_N \le C\mathrm{a}_N^1\,.$$

Proof. The proof follows the same pattern as the proof of Theorem 1 and we omit it.

COMMENTS. (1) We are primarily interested in two-sided estimates by comparable sequences. The estimates obtained in Theorems 1 and 2 are of that sort if for some constants c_1, c_2,

$$a_N^1 \le c_1 a_N^M \quad \text{and} \quad \mathrm{a}_N^1 \le c_2 \mathrm{a}_N^M$$

for all natural N.

If we take the function

$$\phi = c\chi_{[0,1]}^{*k_1} \otimes \ldots \otimes \chi_{[0,1]}^{*k_d}\,, \quad \text{i.e.}$$

$$\widehat{\phi}(\xi) = c\bigg(\frac{1-e^{-2\pi i\xi^1}}{\xi^1}\bigg)^{k_1} \cdots \bigg(\frac{1-e^{-2\pi i\xi^d}}{\xi^d}\bigg)^{k_d}\,,$$

$k_1,\ldots,k_d \in \{1,2,\ldots\}$, $\xi = (\xi^1,\ldots,\xi^d)$, then the sequences a_N^1, a_N^M obtained from the formula (5) are comparable. The same happens for the sequences a_N^1, a_N^M defined in (6) if $\psi = ch^{*k}$, where h is the Haar function, i.e.

$$h(x) = c\begin{cases} 1 & \text{for } 0 < x < 1,\\ -1 & \text{for } -1 < x < 0,\\ 0 & \text{elsewhere.}\end{cases}$$

In this case

$$c_M\,N^{-2k} \le \mathrm{a}_N^M \le C_M\,N^{-2k}$$

and the estimates in Theorem 2 extend the result of Rochberg in [R2].

There are some more examples of ϕ and ψ for which the explicit formulas for the Fourier transforms make it easy to show that a_N^1, a_N^M and a_N^1, a_N^M are comparable. We do not know a general description of the classes of functions ϕ, ψ for which such properties hold.

(2) The assumption that ϕ and ψ have compact supports is essential for our proofs. In the case of Toeplitz operators based on the Schrödinger representation it is enough to assume that $\widehat{\phi}$ is compactly supported. We simply observe that taking the Fourier transform of the kernel function K_b does not change its form.

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