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POLYHEDRAL SUMMABILITY OF MULTIPLE FOURIER SERIES
(AND EXPLICIT FORMULAS FOR DIRICHLET KERNELS ON $\mathbb{T}^n$ AND ON COMPACT LIE GROUPS)

BY

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We study polyhedral Dirichlet kernels on the n-dimensional torus and we write a fairly simple formula which extends the one-dimensional identity $\sum_{j=-N}^{N} e^{ijt} = \sin((N+\frac{1}{2})t)/\sin(\frac{1}{2}t)$. We prove sharp results for the Lebesgue constants and for the pointwise boundedness of polyhedral Dirichlet kernels; we apply our results and methods to approximation theory, to more general summability methods and to Fourier series on compact Lie groups, where we write an asymptotic formula for the Dirichlet kernels.

Introduction. Multiple Fourier series are usually summed either by disks or by squares (or rectangles). The latter is the simplest way since one can separate variables, thereby reducing several questions on the partial sum operator to one-dimensional problems. In this paper we study the polyhedral summability, which is naturally related to the square summability, but cannot be handled by separating variables. Polyhedral sums have already been considered for L^p problems (see [10], [12], [15]) and they are partly motivated by the appearance of several papers on general summability methods (see e.g. [2], [3], [5], [18], [19]). A specific interest comes also from the study of Fourier series on compact Lie groups, which often gives rise to problems on the maximal torus where one does not find square sums, but rather polyhedral sums (which obey the complicated symmetries of the Weyl group and no separation of variables is possible).

Another reason for studying Fourier summability through arbitrary polyhedra will appear at the end, since we shall see that polyhedral Dirichlet kernels on $\mathbb{T}^n$ are exactly as good as the square ones. Indeed, we shall see that

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(i) their Lebesgue constants grow as $\log^n(N)$,

(ii) they are uniformly bounded away from certain hyperplanes perpendicular to the edges and they are unbounded at any point of a dense subset of the union of these hyperplanes.

The last result depends on a rather explicit formula for the Dirichlet kernel, which in the one-dimensional case reduces to the elementary identity $\sum_{j=-N}^{N} e^{ijt} = \sin((N+\frac{1}{2})t)/\sin(\frac{1}{2}t)$.

Results as (i) and (ii) are useful for the L^1 theory and we shall give some applications. Then we shall consider more general summability methods and apply our results to the theory of Fourier series on compact Lie groups, where we shall write an asymptotic formula for the Dirichlet kernels.

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Polyhedral sums. Let $\mathbf{p}_1, \ldots, \mathbf{p}_r$ be points with integral coordinates in $\mathbb{R}^n$; we call their convex hull $\widetilde{P}$ in $\mathbb{R}^n$ a *convex polyhedron*. For any positive integer N we define $\widetilde{P}_N$ as the convex hull of the points $N\mathbf{p}_1, \ldots, N\mathbf{p}_r$. Let $P_N = \widetilde{P}_N \cap \mathbb{Z}^n$; we still call P_N a polyhedron. The tilde over a symbol will always indicate a subset of $\mathbb{R}^n$ and if $\widetilde{A}$ is such a set, we shall write A for its restriction to $\mathbb{Z}^n$. A general reference for n-dimensional polyhedra is [4].

An $(n-1)$-face of $\widetilde{P}$ is called a *facet*. We say that P is *simple* if $\widetilde{P}$ is simple, i.e. if any vertex of $\widetilde{P}$ is contained in exactly n facets. If $\widetilde{P}$ has $2n$ facets, each parallel to another one, we call $\widetilde{P}$ and P *parallelepipeds*. A 1-face of $\widetilde{P}$ is called an *edge*. A segment (or edge) with extremities $\mathbf{a}$, $\mathbf{b}$ will be denoted by $[\mathbf{a}, \mathbf{b}]$.

Convexity (and even connectedness) will not be used in the sequel, and we shall define a *summability polyhedral set* to be any finite union of nonoverlapping (i.e. with disjoint interiors) convex polyhedra (in $\mathbb{R}^n$) containing the origin strictly in its interior. Observe that the above union is not unique; however, if $\widetilde{P}_{(1)}, \ldots, \widetilde{P}_{(s)}$ are nonoverlapping polyhedra and $\widetilde{P} = \widetilde{P}_{(1)} \cup \ldots \cup \widetilde{P}_{(s)}$ is a summability polyhedral set, then $\widetilde{P}_N = (\widetilde{P}_{(1)})_N \cup \ldots \cup (\widetilde{P}_{(s)})_N$ depends on $\widetilde{P}$ but not on the $\widetilde{P}_{(j)}$'s. We write $P_N = \widetilde{P}_N \cap \mathbb{Z}^n$ and since $\bigcup_{N=1}^{\infty} P_N = \mathbb{Z}^n$ we can study the *polyhedral Dirichlet kernel*

$$D_N(\mathbf{t}) = \sum_{\mathbf{m} \in P_N} e^{i\mathbf{m}\cdot\mathbf{t}}$$

and the Nth *Fourier polyhedral partial sum*

$$S_N f(\mathbf{t}) = (D_N * f)(\mathbf{t}) .$$

We shall write c, $c_1, \ldots$ for positive constants independent of N, which may change from line to line.

Lebesgue constants. The following theorem has been proved in [14]. Here we propose a different (perhaps more elementary) proof.

THEOREM 1. *For any polyhedral set P there are two positive constants c_1 and c_2 (depending only on P) such that*

$$c_1 \log^n(N) \le \|D_N\|_{L^1(\mathbb{T}^n)} \le c_2 \log^n(N)\,.$$

P r o o f. We start with the right hand side inequality. Let $\widetilde{\gamma}$ be a hyperplane of dimension $\le n-1$, determined by some of the vertices of $\widetilde{P}$. Then $\gamma_N = \widetilde{\gamma} \cap P_N$ is a union of lower dimensional polyhedra and by an induction argument we can suppose that

$$\Big\| \sum_{\mathbf{m}\in\gamma_N} e^{i\mathbf{m}\cdot\mathbf{t}} \Big\|_{L^1(\mathbb{T}^n)} \le c \log^{n-1}(N)\,.$$

Now we write $\widetilde{P}$ as a union of nonoverlapping simple polyhedra. Because of the previous remark we do not worry about the contribution of their boundaries to the norm of the Dirichlet kernel. Thus we can assume P to be a simple polyhedron.

Let now $\mathbf{a}_1, \ldots, \mathbf{a}_s$ be the vertices of the simple polyhedron $\widetilde{P}$ and let $N\mathbf{a}_1, \ldots, N\mathbf{a}_s$ be the vertices of $\widetilde{P}_N$. Let $\Phi_1, \ldots, \Phi_s$ be nonnegative $C^\infty(\mathbb{R}^n)$ functions with compact support such that, for any j:

(i) $\Phi_j(\mathbf{a}_j) = 1$,

(ii) $\Phi(\mathbf{w}) = \sum_{j=1}^s \Phi_j(\mathbf{w}) = 1$ for any $\mathbf{w} \in \widetilde{P}$,

(iii) the support of Φ_j does not contain any point of the facets which do not contain $\mathbf{a}_j$.

Let $\phi_1, \ldots, \phi_s$ be Schwartz functions on $\mathbb{R}^n$ satisfying $\widehat{\phi}_j = \Phi_j$ for any j and let ${}_N\phi_j(\mathbf{t}) = N^n \phi_j(N\mathbf{t})$ for any positive integer N. Then $\|{}_N\phi_j\|_{L^1(\mathbb{R}^n)} = \|\phi_j\|_{L^1(\mathbb{R}^n)}$ and ${}_N\widehat{\phi}_j(\mathbf{w}) = \widehat{\phi}_j(N^{-1}\mathbf{w})$. Define

$${}_N\psi_j(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbb{Z}^n} {}_N\phi_j(\mathbf{t}+\mathbf{m})$$

on $\mathbb{T}^n$. Then $\|{}_N\psi_j\|_{L^1(\mathbb{T}^n)} \le \|\phi_j\|_{L^1(\mathbb{R}^n)}$ and by the Poisson summation formula we have ${}_N\widehat{\psi}_j(\mathbf{m}) = \widehat{\phi}_j(N^{-1}\mathbf{m})$. Then

$$\|D_N\|_{L^1(\mathbb{T}^n)} = \Big\| \sum_{j=1}^s {}_N\psi_j * D_N \Big\|_{L^1(\mathbb{T}^n)} \le \sum_{j=1}^s \|{}_N\psi_j * D_N\|_{L^1(\mathbb{T}^n)}\,.$$

Now let us fix j. By construction the support of $({}_N\psi_j * D_N)^\wedge$ contains $N\mathbf{a}_j$ and does not contain any point of the facets which do not contain $N\mathbf{a}_j$. Therefore we can find a set $Q_N \subset \mathbb{Z}^n$ which essentially coincides with a parallelepiped and such that

$$ {}_N\psi_j * D_N = {}_N\psi_j * \Big(\sum_{\mathbf{m}\in Q_N} e^{i\mathbf{m}\cdot\mathbf{t}} \Big) . \tag{1} $$

The construction of Q_N is as follows. Assuming N large we can find a parallelepiped $\widetilde{Q} \subset \widetilde{P}_N$ admitting $N\mathbf{a}_j$ as one of its vertices and such that if $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ are the facets of $\widetilde{Q}$ containing $N\mathbf{a}_j$ and $\widetilde{\varrho}_1, \ldots, \widetilde{\varrho}_n$ are the facets of $\widetilde{P}_N$ containing $N\mathbf{a}_j$, then $\widetilde{\sigma}_j \subseteq \widetilde{\varrho}_j$ for all j. By changing sign of some variables we can assume the points $\mathbf{q} - N\mathbf{a}_j$ to have nonnegative coordinates for any $\mathbf{q} \in \widetilde{P}_N$. Let $\widetilde{Q}_*$ be obtained by deleting from $\widetilde{Q}$ the n facets different from $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$.

Now observe that there are n distinct points $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{Z}^n$ so that, for any $j = 1, \ldots, n$, $\widetilde{\sigma}_j + \mathbf{v}_j$ is the facet of $\widetilde{Q}$ opposite to $\widetilde{\sigma}_j$. If $Q_* = \widetilde{Q}_* \cap \mathbb{Z}^n$ we can write the disjoint union

$$ Q_N = \bigcup_{h_1,\ldots,h_n=0}^{KN} \Big(Q_* + \sum_{j=1}^{n} h_j \mathbf{v}_j \Big) . $$

By a suitable choice of the integer K we have $Q_N \supset P_N$, hence (1) follows from the definition of Φ.

It is now useful to decompose Q_N in a different way. Let $\mathbf{q}_1, \ldots, \mathbf{q}_M$ be all the elements in Q_*. Write

$$ Q_N = \bigcup_{i=1}^{M} \{\mathbf{q}_i + h_j \mathbf{v}_j\}_{h_1,\ldots,h_n=0}^{KN} = \bigcup_{i=1}^{M} B_i . \tag{2} $$

Then, for any $i = 1, \ldots, M$,

$$ \Big\| \sum_{\mathbf{m}\in B_i} e^{i\mathbf{m}\cdot\mathbf{t}} \Big\|_{L^1(\mathbb{T}^n)} = \Big\| \sum_{h_1,\ldots,h_n=0}^{KN} e^{i(h_1\mathbf{v}_1+\ldots+h_n\mathbf{v}_n)\cdot\mathbf{t}} \Big\|_{L^1(\mathbb{T}^n)} $$

$$ = \Big\| \prod_{j=1}^{n} \Big(\sum_{h_j=0}^{KN} e^{ih_j\mathbf{v}_j\cdot\mathbf{t}} \Big) \Big\|_{L^1(\mathbb{T}^n)} = \Big\| \prod_{j=1}^{n} \frac{\sin\big((\frac{KN+1}{2})\mathbf{v}_j\cdot\mathbf{t}\big)}{\sin(\frac{1}{2}\mathbf{v}_j\cdot\mathbf{t})} \Big\|_{L^1(\mathbb{T}^n)} . $$

In order to estimate the last norm, cover the fundamental domain $[-\pi, \pi)^n$ with a minimal union of (nonoverlapping) translates of the parallelepiped with edges $[\mathbf{0}, \frac{2\pi}{KN}\mathbf{v}_1], \ldots, [\mathbf{0}, \frac{2\pi}{KN}\mathbf{v}_n]$ and observe that the polynomial in the last norm has constant sign on any such parallelepiped. Then a computation shows that

$$ \Big\| \sum_{\mathbf{m}\in B_i} e^{i\mathbf{m}\cdot\mathbf{t}} \Big\|_{L^1(\mathbb{T}^n)} \le c \log^n(N) , $$

which, by (1) and (2), gives the right hand side inequality in the theorem.

When P is a convex polyhedron, the left hand side inequality is a consequence of a general result of Yudin [19]. When P is a polyhedral set, fix a vertex $\mathbf{a}$ in $\widetilde{P}$ and let Φ be a $C^\infty(\mathbb{R}^n)$ function with compact convex support, attaining value 1 in a neighbourhood of $\mathbf{a}$ and such that the support of Φ does not meet any facet of $\widetilde{P}$ other than the facets containing $\mathbf{a}$; let ϕ satisfy $\widehat{\phi} = \Phi$ and let $\psi_N(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \phi_N(\mathbf{t}+\mathbf{m})$. Then

$$\|D_N\|_{L^1(\mathbb{T}^n)} \ge c\|\psi_N * D_N\|_{L^1(\mathbb{T}^n)}\,.$$

Now Yudin's proof works for the polynomial $\psi_N * D_N$ with minor changes. ∎

A formula for polyhedral Dirichlet kernels on $\mathbb{T}^n$. Precise information on the pointwise boundedness of Dirichlet kernels is frequently used in studying the convergence of Fourier series (e.g. in localization theorems). In the one-dimensional case the elementary identity $\sum_{j=-N}^{N} e^{ijt} = \sin((N+\frac12)t)/\sin(\frac12 t)$ is very useful for this and many other problems. We now prove a similar identity for polyhedral sums (our Lemma is an extension of the Lemma in [11]).

LEMMA. *Let $\widetilde{P} = \widetilde{P}_{(1)} \cup \ldots \cup \widetilde{P}_{(s)}$ be a summability polyhedral set written as a union of convex polyhedra. Let $[\mathbf{a}_1,\mathbf{b}_1],\ldots,[\mathbf{a}_v,\mathbf{b}_v]$ be a maximal set of pairwise nonparallel edges of $\widetilde{P}_{(1)},\ldots,\widetilde{P}_{(s)}$. For any j, let $\mathbf{m}_j \in \mathbb{Z}^n$ such that $[\mathbf{0},\mathbf{m}_j]$ is a segment of minimal length parallel to $[\mathbf{a}_j,\mathbf{b}_j]$. Let*

$$E(\mathbf{t}) = \prod_{j=1}^{v}(1 - e^{i\mathbf{m}_j\cdot\mathbf{t}})\,.$$

Then

$$D_N(\mathbf{t}) = \frac{G_N(\mathbf{t})}{E(\mathbf{t})}\,,$$

where there exists $c > 0$, independent of N, such that the polynomial $G_N(\mathbf{t}) = \sum \widehat{G}_N(\mathbf{n})e^{i\mathbf{n}\cdot\mathbf{t}}$ has the following property: for any $\mathbf{n}$ appearing in the above sum (with nonzero coefficient) there exists a vertex $\mathbf{a}$ of one of the $(\widetilde{P}_{(j)})_N$'s satisfying $|\mathbf{a}-\mathbf{n}| \le c$ (in particular, the number of terms in G_N is uniformly bounded); moreover, the coefficients $\widehat{G}_N(\mathbf{n})$ are uniformly bounded integers: $|\widehat{G}_N(\mathbf{n})| \le c$.

Let us give an idea of the proof first. Consider $\mathbb{T}^2$ with variables (x,y) and let $\widetilde{P}_N$ be the triangle with vertices $(-N,-N)$, $(2N,-N)$, $(0,2N)$. Let $P_N = \widetilde{P}_N \cap \mathbb{Z}^2$ and let $D_N(\mathbf{t})$ be the associated Dirichlet kernel. Observe that $\widehat{D}_N$ is the characteristic function of P_N and look at the passage from D_N to $D_N \cdot (e^{ix}-1)$: after the product, $\widehat{D}_N$ disappears but for some segments parallel and close to one of the nonhorizontal edges of P_N. Two more products are necessary to get the polynomial G_N in the Lemma (once P_N

is given, squared paper is all we need in order to write out the coefficients of G_N in dimension two).

Proof of the Lemma. We can assume P to be a convex polyhedron. Let d be a fixed integer satisfying $d > |\mathbf{m}_1| + \ldots + |\mathbf{m}_v|$. Of course, for any N there are a bounded number of points $\mathbf{n} \in \mathbb{Z}^n$ satisfying $|\mathbf{a} - \mathbf{n}| \le 4d$ for some vertex $\mathbf{a}$ of P_N; let A_N be the complement of this set in $\mathbb{Z}^n$; we want to prove that $\widehat{G}_N(\mathbf{n}) = 0$ for all $\mathbf{n} \in A_N$.

Let N be large and let $\mathbf{m} \in P_N \cap A_N$. Let $B(\mathbf{m}, 2d)$ be the ball with center $\mathbf{m}$ and radius $2d$; for any $\mathbf{r} \in P_N \cap B(\mathbf{m}, 2d)$ there exists at least one direction $[\mathbf{0}, \mathbf{m}_k]$ such that

$$\{\mathbf{r} + j\mathbf{m}_k\}_{j=-2d}^{2d} \subset P_N\,.$$

Let

$$H(\mathbf{t}) = D_N(\mathbf{t})(1 - e^{i\mathbf{m}_k\cdot\mathbf{t}})\,.$$

Then for any $\mathbf{n} \in B(\mathbf{m}, 2d)$ we have $\widehat{H}(\mathbf{n}) = 0$. Now we write

$$G_N(\mathbf{t}) = H(\mathbf{t}) \prod_{j\ne k} (1 - e^{i\mathbf{m}_j\cdot\mathbf{t}})\,.$$

By the choice of d we have $\widehat{G}_N(\mathbf{n}) = 0$ for all $\mathbf{n} \in B(\mathbf{m}, d)$. This proves that $\widehat{G}_N(\mathbf{n}) = 0$ when $\mathbf{n} \in A_N$. Thus the first part of the Lemma is proved, the second being obvious. ■

Remark. The above Lemma works under more general hypotheses. Indeed, the previous proof only needed the following two assumptions: (i) the faces of the polyhedra are parallel, (ii) the lengths of the edges of the polyhedra diverge (we could omit (ii) too, but in this case it would be complicated to give the statement of the Lemma). Anyhow, the Lemma can be restated as follows (we write out the statement for convex polyhedra only).

LEMMA*. *Let Y_N be a sequence of convex polyhedra in $\mathbb{Z}^n$ satisfying the following conditions*:

(i) *if $\sigma_1, \ldots, \sigma_v$ are the faces of Y_1, then any Y_N has* (*as only*) *faces $\sigma_1^N, \ldots, \sigma_v^N$ with σ_j^N parallel to σ_j for any j*;

(ii) *any edge of Y_N has length $\ge cN$.*

Let $[\mathbf{a}_1, \mathbf{b}_1], \ldots, [\mathbf{a}_v, \mathbf{b}_v]$ be a maximal set of pairwise nonparallel edges of Y_1. For any j, let $\mathbf{m}_j \in \mathbb{Z}^n$ such that $[\mathbf{0}, \mathbf{m}_j]$ is a segment of minimal length parallel to $[\mathbf{a}_j, \mathbf{b}_j]$. Let E and G_N be as in the Lemma. Then the conclusion of the Lemma holds.

Observe that the passage from the Lemma to Lemma* is similar to the passage from square sums to rectangular sums. Lemma* will be useful when dealing with compact Lie groups, where we shall work with a sequence of polyhedra which are not exactly dilates of each other.

Pointwise boundedness of Dirichlet kernels. The above Lemma implies the following

THEOREM 2. *Let P be a summability polyhedral set and let $[\mathbf{0},\mathbf{m}_1],\ldots$ $\ldots,[\mathbf{0},\mathbf{m}_v]$ be as in the statement of the Lemma. For $j=1,\ldots,v$ consider the hyperplane $\widetilde{E}_j=\{\mathbf{x}\in\mathbb{R}^n:\mathbf{x}\cdot\mathbf{m}_j=0\}$ orthogonal to $\mathbf{m}_j$ and let*

$$\widetilde{E}=\bigcup_{\mathbf{m}\in\mathbb{Z}^n}\Big(2\pi\mathbf{m}+\bigcup_{j=1}^{v}\widetilde{E}_j\Big).$$

Being periodic, $\widetilde{E}$ defines a set E in $\mathbb{T}^n$ (e.g. let E be the intersection of $\widetilde{E}$ with the fundamental domain $[-\pi,\pi)^n$; observe that E may be larger than $[-\pi,\pi)^n\cap\bigcup_{j=1}^{v}\widetilde{E}_j$). Then for any closed set $F\subset\mathbb{T}^n$ disjoint from E we have

$$|D_N(\mathbf{t})|\le c_F$$

for all $\mathbf{t}\in F$ (c_F being independent of N).

The above theorem is best possible in the following sense.

THEOREM 3. *Let P and E be as in Theorem 2. Then*

$$\sup_{N\in\mathbb{N}}|D_N(\mathbf{x})|=\infty$$

for $\mathbf{x}$ in a dense subset of E.

Proof. Let $\mathbf{x}\in E$ and let $D_N(\mathbf{x})=G_N(\mathbf{x})/E(\mathbf{x})$ as in the Lemma; clearly $G_N(\mathbf{x})=E(\mathbf{x})=0$; thus $\sup_{N\in\mathbb{N}}|D_N(\mathbf{x})|=\infty$ by the de l'Hopital theorem provided we find a direction $\mathbf{v}$ such that $\sup_{N\in\mathbb{N}}|(\partial/\partial\mathbf{v})G_N(\mathbf{x})|$ $=\infty$.

Let $P=P_1\cup\ldots\cup P_s$. To prove the above we recall (see the Lemma) that the support of $\widehat{G}_N$ consists of points close to a vertex of one of the $(\widetilde{P}_{(j)})_N$'s. Then we fix a direction $\mathbf{v}$ not parallel to any face of the $\widetilde{P}_{(j)}$'s and such that $\mathbf{v}\cdot\mathbf{a}\ne0$ for any vertex $\mathbf{a}$ of one of the $\widetilde{P}_{(j)}$'s . Now, because of the properties of G_N in the Lemma, we have $|\mathbf{v}\cdot\mathbf{n}|\ge c|\mathbf{n}|\ge c_1N$ whenever $\widehat{G}_N(\mathbf{n})\ne0$. Let

$$(G_N)_{\mathbf{v}}(\mathbf{x})=\frac{\partial}{\partial\mathbf{v}}G_N(\mathbf{x})=\frac{\partial}{\partial\mathbf{v}}\sum\widehat{G}_N(\mathbf{n})e^{i\mathbf{n}\cdot\mathbf{x}}=i\sum\mathbf{v}\cdot\mathbf{n}\,\widehat{G}_N(\mathbf{n})e^{i\mathbf{n}\cdot\mathbf{x}}.$$

Then $|(G_N)^{\wedge}_{\mathbf{v}}(\mathbf{n})|\ge cN$ if $\widehat{G}_N(\mathbf{n})\ne0$ ($\widehat{G}_N(\mathbf{n})$ being an integer). Since $\mathbf{x}$ is (say) in the hyperplane E_j, the above expression is a trigonometric polynomial (in the $(n-1)$-dimensional variable $\mathbf{x}$) with a bounded number of coefficients, any of them with absolute value $\ge cN$. Indeed, observe that the restriction $(G_N)_{\mathbf{v}}|_{E_j}$ cannot be identically zero for any chosen $\mathbf{v}$, since this would imply $\nabla G_N|_{E_j} \equiv 0$ and, as a consequence (since $G_N|_{E_j} \equiv 0$) we would deduce $G_N/E|_{E_j} \equiv D_N|_{E_j} \equiv 0$, then $D_N(\mathbf{0}) = 0$, which is impossible. Then, if H is a fixed relatively open subset of E_j, an argument of Zygmund (see [20, p. 370], [13, Lemma 8.14]) shows that

$$\int_H |(G_N)_{\mathbf{v}}(\mathbf{x})|^2 \, d\mathbf{x} \geq c_H \sum |(G_N)^{\wedge}_{\mathbf{v}}(\mathbf{n})|^2 \geq c_H N^2$$

($d\mathbf{x}$ being the $(n-1)$-dimensional Lebesgue measure) and therefore $(G_N)_{\mathbf{v}}$ cannot be bounded on H. ■

An analogue of Theorem 3 will be considered at the end of the next section.

More general Dirichlet kernels. The arguments of the above sections apply to Dirichlet kernels defined through sets which coincide only locally with a polyhedron. We are able to deal with the case $n = 2$ and the precise definitions are as follows.

Let $\widetilde{V}$ be a compact set, the closure of an open set in $\mathbb{R}^2$, containing the origin strictly in its interior and such that $\partial\widetilde{V}$ has finite upper Minkowski measure. Suppose there exists an angle $\widetilde{W}$ with a vertex $\mathbf{a}$ and a disk $B(\mathbf{a}, \varepsilon)$ such that $\widetilde{W} \cap B(\mathbf{a}, \varepsilon) = \widetilde{V} \cap B(\mathbf{a}, \varepsilon)$. Then we call $\widetilde{V}$ a *partially polygonal set*. We define $\widetilde{V}_N$ through dilation and $V_N = \widetilde{V}_N \cap \mathbb{Z}^2$. In this section D_N denotes the (partially polygonal) Dirichlet kernel

$$D_N(\mathbf{t}) = \sum_{\mathbf{m} \in V_N} e^{i\mathbf{m}\cdot\mathbf{t}} .$$

The Lebesgue constants of D_N range from $\log^2(N)$ to $N^{1/2}$ (see the proof of Theorem 1 and [18]) and the "polygonal piece" inside $\widetilde{V}$ seems to play no role in the computation of the Lebesgue constants. The situation is perhaps more interesting if we study the pointwise boundedness of D_N.

On the one hand, we may have $\sup_N |D_N(\mathbf{t})| = \infty$ for almost every $\mathbf{t}$ (choose $\widetilde{V}$ to be the union of the upper closed unitary semidisk $\widetilde{K}$ centered at $(0, 0)$ and the closed triangle $\widetilde{T}$ with vertices $(-1, 0)$, $(0, -1)$, $(1, 0)$, then apply [16, p. 274] and Theorem 2).

On the other hand, we are going to show that D_N always keeps the "singularities" carried by the polygonal part of $\widetilde{V}$.

THEOREM 4. *Let $\widetilde{V}$ and $\widetilde{W}$ be as above and let $[\mathbf{0}, \mathbf{m}_1]$ and $[\mathbf{0}, \mathbf{m}_2]$ be segments parallel to the edges of $\widetilde{W}$. Let*

$$\widetilde{E} = \bigcup_{\mathbf{m} \in \mathbb{Z}^2} \Big(2\pi\mathbf{m} + \Big(\bigcup_{j=1}^{2} \{\mathbf{x} \in \mathbb{R}^2 : \ \mathbf{x} \cdot \mathbf{m}_j = 0\} \Big) \Big)$$

and $E = \widetilde{E} \cap [-\pi, \pi)^2$. *Then for any open set* $G \subset \mathbb{T}^2$ *intersecting* E *we have*

$$\sup_{N\in\mathbb{N},\mathbf{t}\in G} |D_N(\mathbf{t})| = \infty\,.$$

Proof. The proof is by contradiction. Suppose there exists a small disk $B = B(\mathbf{x}_0, \varrho)$ with $\mathbf{x}_0 \in E$ such that $\sup_{N\in\mathbb{N},\mathbf{t}\in B} |D_N(\mathbf{t})| \le c$. Following the notation in Theorem 2 we can assume $\mathbf{x}_0 \in E_1$ and $B \cap E_2 = \emptyset$. Since $\sup_{N\in\mathbb{N},\mathbf{t}\in B} |\mathrm{Re}(D_N(\mathbf{t}))| \le c$ and $\sup_{N\in\mathbb{N},\mathbf{t}\in B} |\mathrm{Im}(D_N(\mathbf{t}))| \le c$, it follows that $|D_N(\mathbf{t})|$ is also uniformly bounded on $-B = B(-\mathbf{x}_0, \varrho)$. Let $H = -B \cup B$. Then

$$\sup_{N\in\mathbb{N},\mathbf{t}\in H} |D_N(\mathbf{t})| \le c\,. \tag{3}$$

Let N_0 be a fixed large number and write $\widetilde{V} = \widetilde{V}_{N_0}$. Now let $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{a}, \mathbf{c}]$ be the edges of $\widetilde{W}$, $[\mathbf{a}, \mathbf{b}]$ being parallel to $[\mathbf{0}, \mathbf{m}_1]$, and $\mathbf{b}, \mathbf{c} \in B(\mathbf{a}, \varepsilon)$ (see the definition above). We want to modify the angle W so that it turns into a right angle. This can be done (thanks to the fixed dilation N_0) by adding or subtracting a suitable triangle $\widetilde{R}$ (with vertices $\mathbf{a}$, $\mathbf{c}$, $\mathbf{d}$ such that $\mathbf{d} \in B(\mathbf{a}, \varepsilon)$, while $[\mathbf{a}, \mathbf{d}]$ and $[\mathbf{a}, \mathbf{b}]$ are perpendicular and $[\mathbf{c}, \mathbf{d}]$ is not parallel to $[\mathbf{a}, \mathbf{b}]$). Then we get a set of the form $\widetilde{I} = \overline{\widetilde{V} \setminus \widetilde{R}}$ or $\widetilde{I} = \widetilde{V} \cup \widetilde{R}$ and there exists a disk $B(\mathbf{a}, \delta)$ and a square $\widetilde{S}$ such that $\widetilde{I} \cap B(\mathbf{a}, \delta) = \widetilde{S} \cap B(\mathbf{a}, \delta)$.

We shall use this fact to split a neighbourhood of the vertex $\mathbf{a}$ from $\widetilde{V}$ via a smooth function, but first we must turn to the Dirichlet kernel and show that changing $\widetilde{V}$ into $\widetilde{I}$ does not affect the property (3).

Indeed, define the dilated set $\widetilde{I}_N$ and $I_N = \widetilde{I}_N \cap \mathbb{Z}^2$. Passing from V_N to I_N we have added or deleted a triangle (less an edge) which does not contain any edge parallel to $[\mathbf{0}, \mathbf{m}_1]$. Thus, by Theorem 2, the polynomial

$$\sum_{\mathbf{m}\in V_N} e^{i\mathbf{m}\cdot\mathbf{t}} - \sum_{\mathbf{m}\in I_N} e^{i\mathbf{m}\cdot\mathbf{t}}$$

is bounded on H, independently of N and then $D_N^{\#}(\mathbf{t}) = \sum_{\mathbf{m}\in I_N} e^{i\mathbf{m}\cdot\mathbf{t}}$ satisfies (3).

Now let Φ be a $C^\infty(\mathbb{R}^2)$ function with compact support, equal to 1 in a neighbourhood of the origin. We shall fix more conditions on Φ later on. Let ϕ be the Schwartz function on $\mathbb{R}^2$ satisfying $\widehat{\phi} = \Phi$ and let, for any positive integer N, $\phi_N(\mathbf{t}) = N^2\phi(N\mathbf{t})$. Again, let ψ_N be the function on $\mathbb{T}^2$ defined by $\psi_N(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} \phi(\mathbf{t} + \mathbf{m})$. Then $\widehat{\psi}_N(\mathbf{m}) = \Phi(N^{-1}\mathbf{m})$ and $\|\psi_N\|_{L^1(\mathbb{T}^2)} \le c$. Let $g_N(\mathbf{t}) = \psi_N(\mathbf{t})e^{-i\mathbf{t}\cdot\mathbf{a}}$; by a suitable choice of Φ we can suppose that the support of $\widehat{g}_N$ is contained in $B(N\mathbf{a}, N\delta)$. Write $g_N = h_N + k_N$, with $h_N = g_N\chi_{B(\mathbf{0},\varrho/2)}$ (characteristic function), and observe that $\|k_N\|_{L^\infty(\mathbb{T}^2)} \le c$. Then split $D_N^{\#} = r_N + s_N$, with $r_N = D_N^{\#}\chi_H$. Write

$$F_N = D_N^{\#} * g_N = r_N * g_N + s_N * h_N + s_N * k_N\,.$$

Let $H_1 = B(\mathbf{x}_0, \varrho/2) \cup B(-\mathbf{x}_0, \varrho/2)$. The previous construction and [18] yield

$$\|r_N * g_N\|_{L^\infty(\mathbb{T}^2)} \le \|r_N\|_{L^\infty(\mathbb{T}^2)}\|g_N\|_{L^1(\mathbb{T}^2)} \le c\,,$$

$$(\operatorname{supp}(s_N * h_N)) \cap H_1 = \emptyset\,,$$

$$\|s_N * k_N\|_{L^\infty(\mathbb{T}^2)} \le \|s_N\|_{L^1(\mathbb{T}^2)}\|k_N\|_{L^\infty(\mathbb{T}^2)} \le cN^{1/2}\,.$$

Therefore

$$\sup_{\mathbf{t}\in H_1} |F_N(\mathbf{t})| \le cN^{1/2}\,. \tag{4}$$

Now we consider the support of $\widehat{F}_N$ as part of a square (let us call it U_N) and we want to use copies of $\widehat{F}_N$ to recover the characteristic function of U_N (let $\mathbf{a}$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ be the vertices of U_N); we shall choose later the length of the edges of this square. Let $\mathbf{O}_j$ ($j = 2, 3, 4$) be a transformation of $\mathbb{T}^2$ sending the edges containing $\mathbf{a}$ into the edges containing $\mathbf{a}_j$. Let ${}_1F_N = F_N$ and ${}_jF_N = F_N \circ \mathbf{O}_j$ ($j = 2, 3, 4$). Observe that the $\mathbf{O}_j$'s consist of reflections and—on the Fourier transform side—of translations. Therefore, they do not modify the symmetric set H_1 and all the polynomials ${}_jF_N$ satisfy (4).

Now we fix the length of U_N and the $C^\infty(\mathbb{R}^2)$ function Φ in such a way that

$$\sum_j {}_j\widehat{F}_N(\mathbf{t}) = \sum_{\mathbf{n}\in U_N} e^{i\mathbf{n}\cdot\mathbf{t}} \qquad (\,= D_N^U(\mathbf{t}))$$

(i.e. we construct a partition of unity through Φ). So far we have got a dilating sequence of cubes such that the associated Dirichlet kernel D_N^U satisfies

$$\sup_{\mathbf{t}\in B(\mathbf{x}_0,\varrho/2)} |D_N^U(\mathbf{t})| \le cN^{1/2} \tag{5}$$

where $\mathbf{x}_0$ belongs to the line E perpendicular to an edge of the square. Now we can either appeal to the proof of Theorem 3, which shows that the left hand side in (5) grows as N, or we can decompose U_N as we did at the end of the proof of Theorem 1 and then separate variables. ■

Remark. The previous theorem is obviously a variant of Theorem 3 for the case $n = 2$. However, if we apply Theorem 1 in place of [18] in the proof of Theorem 4, we can easily state and prove an analogue of Theorem 3 for any dimension n.

Applications to approximation theory and localization. Sharp estimates of Lebesgue constants are a main ingredient for results in approximation theory (see [1], [6], [15]). Applying Theorem 1 and arguing as in [6] one proves the following

THEOREM 5. *Let* $\mathfrak{b}$ *be either the space* $C(\mathbb{T}^n)$ *or* $L^1(\mathbb{T}^n)$. *Suppose that for some positive* k *the* k*-th modulus of continuity of a function* $f \in \mathfrak{b}$ *satisfies* $\omega_k(s, f) = o(|\log(s)|^{-n})$ *as* $s \to 0$. *Then the polyhedral partial sums of* f *converge to* f *in the norm of* $\mathfrak{b}$. *The result is false if we replace the small* o *with the big* O.

The following result is a consequence of Theorem 2. The proof runs as for the Riemann localization theorem.

THEOREM 6. *Let* $f \in L^1(\mathbb{T}^n)$ *be a function whose support does not intersect the set* E *defined in Theorem* 2. *Then, if* S_N *denotes the* N*-th polyhedral partial sum,* $S_N f(\mathbf{0})$ *vanishes as* N *goes to infinity.*

On the other hand, a duality argument and Theorem 3 prove

THEOREM 7. *For any open set* F *intersecting* E *there exists a function* $f \in L^1(\mathbb{T}^n)$ *such that* $S_N f(\mathbf{0})$ *does not converge to zero.*

If we apply Theorem 4 in place of Theorem 3 we can extend the previous result to partially polygonal sets in $\mathbb{Z}^2$.

Applications to Fourier series on compact Lie groups. Some of the previous results may be applied to Fourier Analysis on compact Lie groups. We need to fix the notation first.

With every integrable function f on a compact, simply connected, simple Lie group G it is possible to associate its Fourier series

$$f \approx \sum_{\lambda} d_\lambda \chi_\lambda * f ,$$

where d_λ and χ_λ are the dimension and the character of the irreducible unitary representation λ respectively. No analogue of square Fourier partial sums is possible and the best convergence results are obtained through polyhedral sums. To define them we must identify the irreducible unitary representations with the dominant weights.

Let $\mathbf{T}$ be a maximal torus of G , and $\mathfrak{t}$ and $\mathfrak{g}$ be the Lie algebras of $\mathbf{T}$ and G respectively. We choose a positive system Φ^+ in the set of roots of G, and let $\{\alpha_1, \ldots, \alpha_l\}$ be the associated system of simple roots. We denote by W the Weyl group generated by the reflections σ_j in the hyperplanes $\alpha_j(H) = 0$ $(j = 1, \ldots, R)$, and we consider W acting both on $\mathfrak{t}$ and on the dual $\mathfrak{t}^*$. The Killing form B defines a positive definite inner product $(\cdot,\cdot) = -B(\cdot,\cdot)$ in $\mathfrak{t}$. For every $\lambda \in i\mathfrak{t}^*$ there exists a unique $H_\lambda \in \mathfrak{t}$ such that $\lambda(H) = i(H_\lambda, H)$ for every $H \in \mathfrak{t}$. The vectors $H_j = 4\pi i H_{\alpha_j}/\alpha_j(H_{\alpha_j})$ generate the lattice Ker(exp). The elements of the set $\Lambda = \{\lambda \in i\mathfrak{t}^* :\ \lambda(H) \in 2\pi i\mathbb{Z},\ \forall H \in$

Ker(exp)} are called the *weights* of G, and the *fundamental weights* are defined by the relations $\lambda_j(H_k) = 2\pi i\delta_{jk}$, $j,k = 1,\ldots,l$.

The set $\Sigma = \{\lambda \in \Lambda : \ \lambda = \sum_{j=1}^{l} m_j\lambda_j, \ m_j \in \mathbb{N}\}$ of *dominant weights* can be naturally identified with the set of equivalence classes of unitary irreducible representations of G. A dominant weight λ is *non-singular* if $m_j > 0$ for every $j = 1,\ldots,l$. If ξ is a character of $\mathbf{T}$, there exists a unique $\lambda \in i\mathfrak{t}^*$ such that

$$\xi \circ \exp H = e^{\lambda(H)} = e^{i(H_\lambda, H)}\,, \qquad H \in \mathfrak{t}\,.$$

For $\lambda \in \Sigma$ and $t = \exp H$ in $\mathbf{T}$ we define the alternating sum and the symmetric sum

$$A(\lambda)(t) = \sum_{\sigma\in W} \det(\sigma) e^{\sigma(\lambda)(H)}\,, \qquad S(\lambda)(t) = \sum e^{\sigma(\lambda)(H)}$$

where the last sum is over the orbit of λ under the action of the Weyl group.

For the character χ_λ and the dimension d_λ of the representation corresponding to the dominant weight λ we have the Weyl formulas:

$$\chi_\lambda(t) = (\Delta(t))^{-1} A(\lambda+\beta)(t)\,, \qquad d_\lambda = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\beta,\alpha)}{(\beta,\alpha)}$$

where $\beta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and

$$\Delta(t) = A(\beta)(t) = (-2i)^{|\Phi^+|} \prod_{\alpha\in\Phi^+} \sin(i\alpha(H)/2) \tag{6}$$

($|\Phi^+|$ denotes the cardinality of Φ). A reference for the theory is [17].

Let ω be a dominant weight, and let $P'(\omega)$ be the set of all the dominant λ's such that $(\lambda_j,\lambda) \le (\lambda_j,\omega)$ for every $j = 1,\ldots,l$. The polyhedron $P(\omega)$ is the union of the saturated hulls of the dominant weights $\lambda \in P'(\omega)$: $P(\omega) = \bigcup_{\sigma\in W}\sigma(P'(\omega))$. Let N be a positive integer. We denote by D_N the polyhedral Dirichlet kernel

$$D_N = \sum_{\lambda\in P'(N\omega)} d_\lambda \chi_\lambda\,.$$

In [7], [9], [11] several results have been proved for polyhedral Dirichlet kernels defined through reflections of a non-singular dominant weight. The Lemma in this paper allows us to drop the non-singularity hypothesis and therefore extend these results. Here we are not interested in this, but rather in extending an idea from [7] and writing an asymptotic formula for polyhedral Dirichlet kernels on compact Lie groups. Since D_N is central we only consider $D_N(t)$, $t = \exp H \in \mathbf{T}$; then (see [8, p. 154])

$$\begin{aligned} D_N(t) &= \sum_{\lambda\in P'(N\omega)} d_\lambda (\Delta(t))^{-1} A(\lambda+\beta)(t) \\ &= (\Delta(t))^{-1} \prod_{\alpha\in\Phi^+} \mathcal{D}_\alpha\Big(\sum_{\lambda\in P'(N\omega+\beta)} S(\lambda)(t)\Big) \end{aligned}$$

where $\mathcal{D}_\alpha$ denotes the partial derivative with respect to the tangent vector H_α. Observe that $\sum_{\lambda\in P'(N\omega+\beta)} S(\lambda)(t)$ is not a polyhedral Dirichlet kernel on $\mathbb{T}^l$ according to the definition in this paper, since the translation $+\beta$ damages the dilation. We then apply Lemma* in place of the Lemma; now the definition of the Weyl group implies that any edge of $P(N\omega+\beta)$ is parallel to a root, hence

$$\sum_{\lambda\in P'(N\omega+\beta)} S(\lambda) = G_N/\Delta$$

where G_N satisfies the properties in the Lemma. Then

$$\begin{aligned} D_N(t) &= (\Delta(t))^{-1} \prod_{\alpha\in\Phi^+} \mathcal{D}_\alpha(G_N(t)/\Delta(t)) \\ &= (\Delta(t))^{-1} \sum_{R\subseteq\Phi^+} \prod_{\alpha\in R} \mathcal{D}_\alpha(G_N(t)) \prod_{\alpha\in\Phi^+\setminus R} \mathcal{D}_\alpha(\Delta^{-1}(t)). \end{aligned}$$

Now, by the Weyl dimension formula $d_{N\omega} \approx N^{|Z|}$ where $Z = \{\alpha\in\Phi^+ : (\alpha,\omega)\neq 0\}$ (observe that $Z=\Phi^+$ when ω is non-singular). Then by applying the argument we used in the proof of Theorem 3 we have (almost everywhere)

$$\begin{aligned} D_N(t) &= (\Delta(t))^{-1} \sum_{R\subseteq Z} \prod_{\alpha\in R} \mathcal{D}_\alpha(G_N(t)) \prod_{\alpha\in\Phi^+\setminus R} \mathcal{D}_\alpha(\Delta^{-1}(t)) \\ &\quad + (\Delta(t))^{-1} \sum_{R\setminus Z\neq\emptyset} \prod_{\alpha\in R} \mathcal{D}_\alpha(G_N(t)) \prod_{\alpha\in\Phi^+\setminus R} \mathcal{D}_\alpha(\Delta^{-1}(t)) \\ &= (\Delta(t))^{-1} \sum_{R\subseteq Z} d_{N\omega} G_N^*(t) \prod_{\alpha\in\Phi^+\setminus R} \mathcal{D}_\alpha(\Delta^{-1}(t)) + o(N^{|Z|}) \end{aligned}$$

where $G_N^*(t) = d_{N\omega}^{-1} \prod_{\alpha\in R} \mathcal{D}_\alpha(G_N(t)) = \sum a_\lambda e^{\lambda(H)}$ satisfies the following conditions: either $a_\lambda = 0$ or $c_1 \le |a_\lambda| \le c_2$ and for any λ with $a_\lambda \neq 0$ there exists $\sigma\in W$ such that $|\lambda - \sigma(N\omega)| \le c$.

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