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ON THE SPECTRUM OF THE SUM OF GENERATORS OF A FINITELY GENERATED GROUP, II

BY

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Introduction. We consider a finitely generated group Γ, and the two usual C^*-algebras coming along with Γ: the *full* C^**-algebra* $C^*(\Gamma)$ associated with the universal representation π_{un} of Γ on $\mathcal{H}_{\mathrm{un}}$; and the *reduced* C^**-algebra* $C^*_{\mathrm{r}}(\Gamma)$ associated with the left regular representation λ of Γ on $\ell^2(\Gamma)$.

Given a finite set S of generators of Γ (here we mainly deal with the non-symmetric case $S \neq S^{-1}$), we set:

$$h = \frac{1}{|S|} \sum_{s \in S} s \in C^*(\Gamma)\,.$$

In [HRV], to which this paper is a sequel, we initiated a study of the spectral properties of h. For example, we proved that the intersection of the spectrum $\mathrm{Sp}\, h$ with the unit circle $\mathbb{T}$ equals either $\mathbb{T}$ or the group C_n of n-th roots of 1, for some $n \geq 1$. Concerning $\lambda(h)$, there is the archetypal result of [Day]: Γ is amenable if and only if 1 is in the spectrum of $\lambda(h)$, if and only if the spectral radius of $\lambda(h)$ is 1. In the present paper, we examine more closely how properties of Γ and its representation theory are reflected in properties of $\mathrm{Sp}\, h$ and $\mathrm{Sp}\, \lambda(h)$. Here is a summary of our results.

(1) Γ has Kazhdan's Property (T) if and only if the resolvent $R : \mathbb{C} - \mathrm{Sp}\, h \to \mathcal{L}(\mathcal{H}_{\mathrm{un}})$ has a pole at 1; this is equivalent to the compactness in the norm topology of $\mathcal{L}(\mathcal{H}_{\mathrm{un}})$ of the closed semigroup generated by h. It is also equivalent to a uniform ergodic property for h.

(2) Γ is finite if and only if the resolvent $R : \mathbb{C} - \mathrm{Sp}\, \lambda(h) \to \mathcal{L}(\ell^2(\Gamma))$ has a pole at 1; this is still equivalent to h being algebraic.

(3) The equality $\mathrm{Sp}\, h = C_n$ holds if and only if Γ is isomorphic to C_n and S is reduced to one generator; the equality $\mathrm{Sp}\, h = \mathbb{T}$ holds if and only if Γ is isomorphic to $\mathbb{Z}$ and S is reduced to one generator.

(4) Let $\varrho(\lambda(h)) = \lim_{k\to\infty} \|\lambda(h^k)\|^{1/k}$ be the spectral radius of $\lambda(h)$, and

set $\sigma(h) = \limsup_{k\to\infty} \|h^k\|_2^{1/k}$. Then

$$\frac{1}{\sqrt{|S|}} \le \sigma(h) \le \varrho(\lambda(h)) \le 1\,.$$

Moreover, for $|S| \ge 2$, the equality $1/\sqrt{|S|} = \sigma(h)$ holds if and only if S generates a free semigroup.

(5) For a class of groups Γ that includes hyperbolic groups in the sense of Gromov (but not in general), one has $\sigma(h) = \varrho(\lambda(h))$.

(6) If Γ is either the free group $\mathbb{F}_n$ on $S = \{s_1, \ldots, s_n\}$, where $n \ge 2$, or the surface group

$$\Gamma_g = \left\langle a_1, b_1, \ldots, a_g, b_g \;\Big|\; \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right\rangle$$

with $S = \{a_1, b_1, \ldots, a_g, b_g\}$, where $g \ge 2$, then

$$\operatorname{Sp} \lambda(h) = \{z \in \mathbb{C} : |z| \le 1/\sqrt{|S|}\}\,.$$

Properties of $\lambda(h)$ can be expressed in terms of *digraphs* (i.e. directed graphs). More precisely, let $\mathcal{G}(\Gamma, S)$ denote the Cayley digraph of (Γ, S), with set of vertices Γ and set of directed edges $S \times \Gamma$, an edge (s, γ) having origin γ and extremity $s^{-1}\gamma$. The matrix of $|S|\lambda(h)$ with respect to the canonical basis $(\delta_\gamma)_{\gamma\in\Gamma}$ of $\ell^2(\Gamma)$ is then precisely the adjacency matrix of $\mathcal{G}(\Gamma, S)$ (see e.g. Chapter 13 of [Har]). Thus, (3) above means that the oriented cycles and the oriented line are respectively characterized, as Cayley digraphs, by their spectra. (4) and (5) mean that, in case Γ is hyperbolic and $|S| \ge 2$, the spectral radius of $\mathcal{G}(\Gamma, S)$ is $1/\sqrt{|S|}$ if and only if $\mathcal{G}(\Gamma, S)$ contains a regular rooted tree of degree $|S|$.

Although this paper is a sequel to [HRV], the results are largely independent of those in [HRV].

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1. Characterizations of Property (T). Let us recall some facts from [DuS]. Let x be an operator of norm 1 on a Hilbert space $\mathcal{H}$. Suppose that $\operatorname{Sp} x$ has an isolated point z_0; the point z_0 is a *pole of order* p of x if the resolvent

$$R : \mathbb{C} - \operatorname{Sp} x \to \mathcal{L}(\mathcal{H}) : z \to (x - z)^{-1}$$

has a pole of order p at z_0. It is known that a pole z_0 of modulus 1 is necessarily simple, so that z_0 is an eigenvalue of h ([DuS], Lemma VII.3.18).

THEOREM 1. *Consider a group Γ, a finite generating set S of Γ and the corresponding contraction $h \in C^*(\Gamma)$. The following are equivalent*:

(i) *Γ has Kazhdan's Property* (T);

(ii) *there exists an integer n such that* $\mathrm{Sp}\, h \cap \mathbb{T}$ *is the group C_n of n-th roots of* 1, *and each of these is a simple pole of h*;

(iii) 1 *is a pole of h*;

(iv) *h has a pole z_0 of modulus* 1.

Moreover, *if these hold*, *the integer n in* (ii) *divides the exponent of the finite abelian group $\Gamma/[\Gamma, \Gamma]$*.

P r o o f. (i)$\Rightarrow$(iii). Assume that Γ has Property (T). Denote by $\widehat{\Gamma}$ the unitary dual of Γ.

The proof of Proposition I(4) in [HRV] shows that there exists a constant $\delta > 0$ such that any $z \in \mathbb{C}$ with $0 < |z-1| < \delta$ belongs to $\mathbb{C} - \mathrm{Sp}\, h$ and satisfies

$$\|\pi((z-h)^{-1})\| \le (\delta - |z-1|)^{-1}$$

for any $\pi \in \widehat{\Gamma}$ which is not the unit representation χ_1 of Γ in $\mathbb{C}$. On the other hand, $\chi_1((z-h)^{-1}) = (z-1)^{-1}$. It follows that

$$\|(z-h)^{-1}\| = \sup_{\pi \in \widehat{\Gamma}} \|\pi((z-h)^{-1})\| = O(|z-1|^{-1})$$

so that 1 is a simple pole of h.

(iii)$\Rightarrow$(i). Assume that (iii) holds; consider a unitary representation π of Γ on a Hilbert space $\mathcal{H}$, and suppose that there exists a sequence $(\xi_n)_{n\ge1}$ of vectors of norm 1 in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \max_{s\in S} \|\pi(s)\xi_n - \xi_n\| = 0\,.$$

We have to show that π has a non-zero fixed vector.

Observe that 1 is in $\mathrm{Sp}\, \pi(h)$ by assumption on $(\xi_n)_{n\ge1}$, that 1 is isolated in $\mathrm{Sp}\, \pi(h)$ because it is isolated in $\mathrm{Sp}\, h$, and that 1 is a simple pole of $\pi(h)$ because $\|\pi((z-h)^{-1})\| \le \|(z-h)^{-1}\|$ for z near 1 and $\ne 1$. As recalled at the beginning of this section, this implies that 1 is an eigenvalue of $\pi(h)$, so that π has non-zero fixed vectors by Proposition I(2) of [HRV].

The implications (ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are obvious. Let us show that (iv) implies (ii). As z_0 is isolated in $\mathrm{Sp}\, h$, the peripheral spectrum $\mathrm{Sp}\, h \cap \mathbb{T}$ is the set C_n for some $n \ge 1$, and $z_0 h$ is unitarily equivalent to h (see Proposition 3 and the proof of Lemma 7 of [HRV]). So (ii) holds.

Assume finally that the equivalent conditions (i) to (iv) hold. Then the abelian group $\Gamma/[\Gamma, \Gamma]$ is finite, by Proposition I.7 of [HaV]. By Proposition 3 of [HRV], $\Gamma/[\Gamma, \Gamma]$ maps homomorphically onto C_n, so that n divides the exponent of $\Gamma/[\Gamma, \Gamma]$.

R e m a r k s. (i) When $S = S^{-1}$ the operator h is self-adjoint, so that $z_0 \in \mathrm{Sp}\, h$ is isolated in $\mathrm{Sp}\, h$ if and only if z_0 is a simple pole of h. In this case, Theorem 1 is contained in [HRV], and says in particular that Γ has Property (T) if and only if 1 is isolated in $\mathrm{Sp}\, h$.

(ii) We do not know whether there exists a pair (Γ, S) such that the corresponding h has a spectrum where 1 is isolated but is not a pole. Note, however, that there exists a pair (Γ, S) and $\pi \in \widehat{\Gamma}$ such that 1 is isolated in $\mathrm{Sp}\, \pi(h)$ but is not a pole of $\pi(h)$ (see the example involving the Volterra operator, shortly before Lemma 5 of [HRV]).

Consider again an operator x of norm 1 on a Hilbert space $\mathcal{H}$; let p denote the orthogonal projection of $\mathcal{H}$ onto $\mathrm{Ker}(x-1)$. For each integer $m \geq 1$, form the average

$$A_m(x) = \frac{1}{m}(1 + x + \ldots + x^{m-1}).$$

By von Neumann's mean ergodic theorem, the sequence $(A_m(x))_{m\geq 1}$ always converges strongly to p (see e.g. n° 143 of [RiN]). We say that x is *uniformly ergodic* if this convergence is in norm, i.e. if

$$\lim_{m\to\infty} \|A_m(x) - p\| = 0.$$

Denote by $S(x)$ the norm closed semigroup generated by x in $\mathcal{L}(\mathcal{H})$. Kaashoek and West have shown that $S(x)$ is compact in the norm topology if and only if $\mathrm{Sp}\, x \cap \mathbb{T}$ is either empty or a finite set of simple poles (Theorem 3 of [KaW]).

If Γ is a group with Property (T), then there is a unique projection $p_\Gamma \neq 0$ in $C^*(\Gamma)$ such that $\pi(p_\Gamma) = 0$ for any $\pi \in \widehat{\Gamma} - \{\chi_1\}$ (see Lemma 3.1 of [Val]).

THEOREM 2. *Notation being as in Theorem* 1, *the following are equivalent*:

(i) *Γ has Kazhdan's Property* (T);
(ii) *$S(h)$ is compact*;
(iii) *h is uniformly ergodic.*

Moreover, *if these hold*, *then* $\lim_{m\to\infty} A_m(h) = p_\Gamma$.

Proof. (i)⇒(ii) follows from our Theorem 1 and Theorem 3 of [KaW].

(ii)⇒(iii) is a particular case of Theorem 1 of [KaW].

(iii)⇒(i). Suppose that (iii) holds, and let $p \in C^*(\Gamma)$ be the norm limit of the sequence $(A_m(h))_{m\geq 1}$. Since, for any $m \geq 1$, we have $\|A_m(h)\| = 1$, we see that p is non-zero. We have recalled above that $h(\xi) = \xi$ for any $\xi \in \mathrm{Im}\, p \subseteq \mathcal{H}_{\mathrm{un}}$. For such a $\xi \in \mathrm{Im}\, p$, we have successively:

$s(\xi) = \xi$ for all $s \in S$ by Lemma 3 of [HRV];
$\gamma(\xi) = \xi$ for all $\gamma \in \Gamma$ because S generates Γ;
$x(\xi) = \chi_1(x)\xi$ for all $x \in C^*(\Gamma)$ because Γ generates $C^*(\Gamma)$.

Consequently, $pC^*(\Gamma)p = \mathbb{C}p$, so that p is a minimal projection such that $\chi_1(p) = 1$. Then Γ has Property (T) by Lemma 1.2 of [Val].

The last assertion follows from the proof of (iii)⇒(i) and the uniqueness of p_Γ.

COROLLARY 1. *Let Γ be a group with Property* (T) *and let S, h, p_Γ be as above. Assume that, for any $n \geq 2$, there does not exist any homomorphism of Γ onto C_n which is constant on S* (*this is for example the case if Γ is perfect*). *Then*

$$\lim_{m\to\infty} \|h^m - p_\Gamma\| = 0\,.$$

P r o o f. It follows from the assumption and Proposition 3 of [HRV] that $\operatorname{Sp} h \cap \mathbb{T} = \{1\}$. The spectrum of $h(1 - p_\Gamma)$ is then contained in the open unit disk of $\mathbb{C}$. Notice that, for any $m \geq 1$,

$$h^m - p_\Gamma = h^m(1 - p_\Gamma) = (h(1 - p_\Gamma))^m\,.$$

Let γ be a simple closed rectifiable curve in the open unit disk, surrounding $\operatorname{Sp} h(1-p_\Gamma)$. Then, by holomorphic functional calculus, we have for $m \geq 1$,

$$h^m - p_\Gamma = \frac{1}{2\pi i} \oint_\gamma \mathrm{z}^{\mathrm{m}}[\mathrm{z} - \mathrm{h}(1 - \mathrm{p}_\Gamma)]^{-1}\,\mathrm{dz}\,.$$

Hence

$$\|h^m - p_\Gamma\| \leq \frac{1}{2\pi}[\sup_{z\in\gamma} \|[z - h(1 - p_\Gamma)]^{-1}\| \oint_\gamma |\mathrm{z}|^{\mathrm{m}}\,|\mathrm{dz}|\,.$$

Since z^m converges to 0 uniformly on γ, we have the assertion.

2. Reconstructing Γ and S from $\operatorname{Sp} h$.

As before, Γ will be a group with a finite generating subset S, and h will denote the corresponding element in $C^*(\Gamma)$. The theme of this section is: how far does the spectrum of h or $\lambda(h)$ determine Γ and S? We shall consider the following cases: $\lambda(h)$ has a pole of modulus one; $\operatorname{Sp} h$ is finite; $\operatorname{Sp} h$ is a closed subgroup of $\mathbb{T}$.

2.1. *Poles of $\lambda(h)$.* We recall that $\operatorname{Sp}\lambda(h) \cap \mathbb{T}$ is non-empty if and only if Γ is amenable (see the end of Section B of [HRV]).

PROPOSITION 1. *If $\lambda(h)$ has a pole of modulus* 1, *then Γ is finite.*

P r o o f. By the preceding remark, Γ is amenable. It follows firstly that $C^*(\Gamma)$ and $C^*_{\mathrm{r}}(\Gamma)$ are isomorphic, and secondly (by Theorem 1) that Γ has Property (T). As Γ is amenable and has Property (T), it is a finite group (see [HaV], Proposition 1.7).

2.2. *Finiteness of* $\operatorname{Sp} h$. It is tempting to conjecture that $\operatorname{Sp} h$ is finite if and only if Γ is finite. Good examples of operators with finite spectrum are algebraic operators, i.e. operators T for which there exists a non-zero polynomial P with complex coefficients such that $P(T) = 0$. For such a T, any point in $\mathrm{Sp}\, T$ is a pole of T (see Exercise VII.5.17 of [DuS]). Here is what we can show.

PROPOSITION 2. *Γ is finite if and only if h is algebraic.*

Proof. We prove the non-trivial implication "h algebraic $\Rightarrow \Gamma$ finite". Denote by n the dimension of the unital subalgebra of the complex group algebra $\mathbb{C}\Gamma$ generated by h; let Γ^+ be the subsemigroup generated by S, endowed with the natural word length. Fix $g \in \Gamma^+$, say of length k; then g belongs to the support of h^k. Writing h^k as a linear combination of $1, h, h^2, \ldots, h^{n-1}$, we have

$$\mathrm{supp}\, h^k \subseteq \bigcup_{i=0}^{n-1} \mathrm{supp}\, h^i \, .$$

Hence

$$\Gamma^+ \subseteq \bigcup_{i=0}^{n-1} \mathrm{supp}\, h^i \, ,$$

so Γ^+ is finite. In particular, elements of S have finite order; as a consequence, $\Gamma = \Gamma^+$ and Γ is finite.

Remarks. (1) Using the bound $|\mathrm{supp}\, h^i| \le |S|^i$ in the above proof, we get a crude estimate on the order of Γ:

$$|\Gamma| \le \frac{|S|^n - 1}{|S| - 1} \, .$$

(2) If $S = S^{-1}$, then $h = h^*$; hence $\mathrm{Sp}\, h$ is finite if and only if h is algebraic, if and only if Γ is finite.

(3) We point out the relation between Proposition 2 and a paper of Formanek [For] where he considers the element

$$x = \frac{1}{|S|^2} \sum_{s \in S} s = \frac{h}{|S|} \in \mathbb{C}\Gamma$$

and shows two things:

(i) $1 - x$ is not a left divisor of zero in $\mathbb{C}\Gamma$;
(ii) if $1 - x$ is right invertible in $\mathbb{C}\Gamma$, then Γ is finite.

This provides an alternative proof of Proposition 2: indeed, if h is algebraic, then so is $1 - x$; by (i) the minimal polynomial of $1 - x$ over $\mathbb{C}$ has a non-zero constant term, meaning that $1 - x$ is invertible in $\mathbb{C}\Gamma$; then (ii) implies that Γ is finite.

(4) Assume that Γ is finite. Then $\mathrm{Sp}\, h$ is a subset of some finite extension $\mathbb{K}$ of the rational field $\mathbb{Q}$. Conversely, let Sp be a subset of $\mathbb{K}$, and let $n \ge 1$

be an integer. Let $(\Gamma_j)_{j\in J}$ be a family of pairwise non-isomorphic finite groups, each having a set S_j of n generators. Let h_j be the corresponding element in $\mathbb{C}\Gamma_j$. It was shown by Strunkov [Str] that, if $\mathrm{Sp}\, h_j = \mathrm{Sp}$ for every $j \in J$, then J is finite.

2.3. $\mathrm{Sp}\, h$ *is a closed subgroup of* $\mathbb{T}$. Since we know from [HRV] that we have either $\mathrm{Sp}\, h \cap \mathbb{T} = \mathbb{T}$ or $\mathrm{Sp}\, h \cap \mathbb{T} = C_n$ for some $n \geq 1$, it is natural to consider cases where $\mathrm{Sp}\, h$ equals either $\mathbb{T}$ or C_n. This is done in this subsection. It is here that we get our most precise results, since we completely characterize pairs (Γ, S) with $\mathrm{Sp}\, h \subseteq \mathbb{T}$. The next lemma has its own interest; it owes much to conversations with L. Brown and T. Fack.

LEMMA 1. *Let M be a finite von Neumann algebra. Let $x \in M$ be such that $\|x\| = 1$ and $\mathrm{Sp}\, x \subseteq \mathbb{T}$. Then x is unitary.*

Proof. We shall appeal to the theory of characteristic values, or s-numbers, initiated by Fack [Fac] and elaborated by Brown [Bro], of which we recall some basic facts.

Let τ be a positive faithful trace on M such that $\tau(1) = 1$. Fix an element $y \in M$ with $\|y\| = 1$. For $t \in [0, 1]$, define the t-th characteristic value (or s-number) $\mu_y(t)$ by

$$\mu_y(t) = \inf \|ye\|$$

where the infimum is taken over all projections e in M such that $\tau(e) \geq 1-t$. Clearly $\mu_y : [0, 1] \to \mathbb{R}$ is a non-negative function with values decreasing from 1 to 0. By Proposition 1.3 of [Fac], one also has

$$\mu_y(t) = \min\{\lambda \geq 0 : \tau(e_\lambda) \geq 1 - t\}$$

where e_λ is the unique resolution of the identity, continuous on the right, such that $|y| = \int_0^1 \lambda \, de_\lambda$.

Assume from now on that y is invertible in M. In §3 of [Bro], one defines a finite positive measure ν on $\mathrm{Sp}\, y$ with the property that, for any $t \in [0, 1]$,

$$\int_0^t \log(s(u))\, du \leq \int_0^t \log(\mu_y(u))\, du \leq 0$$

where $s(\cdot)$ is the decreasing rearrangement of $z \to |z|$ relative to ν (see Proposition 1.11 and Theorem 3.6 of [Bro]). If $\mathrm{Sp}\, y \subseteq \mathbb{T}$, then s is the constant function 1, so the above inequalities say that $\mu_y(t) = 1$ for almost all $t \in [0, 1]$. Because μ_y is decreasing, this means that

$$\mu_y(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1\,, \\ 0 & \text{for } t = 1\,. \end{cases}$$

Using the definition of μ_y in terms of spectral resolution, we see that this forces $|y| = 1$, i.e. that y is unitary.

Now, let Γ be a group with a finite generating subset S such that $\operatorname{Sp} h \subseteq \mathbb{T}$. Because $\operatorname{Sp} \lambda(h) \subseteq \operatorname{Sp} h$, we see that Γ is amenable.

PROPOSITION 3. *If* $\operatorname{Sp} h = C_n$ *for some* $n \geq 1$, *then* Γ *is isomorphic to* C_n *and* $|S| = 1$; *if* $\operatorname{Sp} h = \mathbb{T}$, *then* Γ *is isomorphic to* $\mathbb{Z}$ *and* $|S| = 1$.

Proof. The preceding remark shows that Γ is amenable, so that λ provides an isomorphism $C^*(\Gamma) \simeq C^*_{\mathrm{r}}(\Gamma)$. As $\lambda(h)$ belongs to the finite von Neumann algebra $\lambda(\Gamma)''$ generated by $C^*_{\mathrm{r}}(\Gamma)$, the previous lemma shows that $\lambda(h)$ is unitary, so that h is unitary. Since unitaries are extreme points of unit balls in unital C^*-algebras (see 1.1.13 of [Ped]), the equality $h = (1/|S|)\sum_{s \in S} s$ shows that $|S| = 1$, and Proposition 3 follows.

3. On the spectrum and spectral radius of $\lambda(h)$. We recall that, if x is a bounded operator on a (non-zero) Hilbert space, the *spectral radius* of x, denoted by $\varrho(x)$, is

$$\varrho(x) = \lim_{k \to \infty} \|x^k\|^{1/k} = \max_{z \in \operatorname{Sp} x} |z| .$$

If Γ is a group with a given finite generating subset S, we shall deal in this section with the operator

$$\lambda(h) = \frac{1}{|S|} \sum_{s \in S} \lambda(s) .$$

(Sometimes we shall have to work with different generating subsets and we shall append a subscript to h to indicate the dependence on S). We first have the easy

PROPOSITION 4. (i) $\varrho(\lambda(h))$ *belongs to* $\operatorname{Sp} \lambda(h)$.

(ii) $\operatorname{Sp} \lambda(h)$ *is symmetric with respect to the real axis in* $\mathbb{C}$.

(iii) *Assume that* S *does not contain the unit* e *of* Γ, *and set* $T = S \cup \{e\}$; *then*

$$\operatorname{Sp} \lambda(h_T) = \frac{|S|}{|S|+1} \left[\frac{1}{|S|} + \operatorname{Sp} \lambda(h_S) \right]$$

and

$$\varrho(\lambda(h_T)) = \frac{|S|}{|S|+1} \left[\frac{1}{|S|} + \varrho(\lambda(h_S)) \right] .$$

(iv) *Suppose* $S = \{s_1, s_2\}$. *Then* $\|\lambda(h)\| = 1$; *moreover*, $\lambda(h)$ *is invertible if and only if* $s_1 s_2^{-1}$ *has finite*, *odd order.*

Proof. (i) In $\ell^2(\Gamma)$, the cone $C = \{\xi \in \ell^2(\Gamma) : \operatorname{Re} \xi(g) \geq 0$ and $\operatorname{Im} \xi(g) \geq 0$ for all $g \in \Gamma\}$ is normal in the sense of ordered topological vector spaces, and the operator $\lambda(h)$ is positive for the ordering defined by C; so the result follows from 2.2 in the Appendix of [Sch]. Note that this is an infinite-dimensional analogue of the Perron–Frobenius theory.

(ii) $\lambda(h)$ commutes with the (anti-linear) operator of complex conjugation on $\ell^2(\Gamma)$.

(iii) This follows from the identity

$$\lambda(h_T) = \frac{|S|}{|S|+1}\left[\frac{1}{|S|} + \lambda(h_S)\right]$$

and from (i).

(iv) From $\lambda(h) = \frac{1}{2}[\lambda(s_1 s_2^{-1}) + 1]\lambda(s_2)$, we have $\|\lambda(h)\| = \frac{1}{2}\|\lambda(s_1 s_2^{-1}) + 1\| = 1$ because $\lambda(s_1 s_2^{-1})$ is a unitary operator with 1 in its spectrum. Moreover, we see that $\lambda(h)$ is invertible if and only if $\lambda(s_1 s_2^{-1}) + 1$ is invertible, which happens exactly when $s_1 s_2^{-1}$ has finite order m and -1 is not an m-th root of 1, i.e. m is odd.

Recall that the canonical trace τ is defined on $C^*_{\mathrm{r}}(\Gamma)$ by

$$\tau : C^*_{\mathrm{r}}(\Gamma) \to \mathbb{C} : x \to \langle x\delta_e \mid \delta_e\rangle$$

where δ_e is the characteristic function of $\{e\}$. In the random walk defined on the group Γ by the Markov operator $\lambda(h)$, the trace $\tau(\lambda(h^n))$ is viewed as the probability of returning to e at the n-th step, given that the random walk starts at e. We wish to compare that probability with $\varrho(\lambda(h^n))$. We shall need the following lemma, which seems interesting for its own sake.

LEMMA 2. *Let M be a finite von Neumann algebra, and let t be a positive normalized trace on M. Then, for any $x \in M$,*

$$|t(x)| \le \varrho(x)\,.$$

Proof. Consider the centre Z of M and the centre-valued trace $T : M \to Z$. It follows from Proposition 8.3.10 of [KaR] that there exists a state f on Z such that $t = f \circ T$. Since Z is abelian,

$$t(x) = f(T(x)) \in \operatorname{conv} \operatorname{Sp} T(x)\,,$$

where conv means "convex hull". Now, it follows from [Ber] that one also has

$$\operatorname{conv} \operatorname{Sp} T(x) \subseteq \operatorname{conv} \operatorname{Sp}(x)\,.$$

The lemma immediately follows.

The next proposition extends Theorem 4.6(b) of [MoW] to the non-symmetric case.

PROPOSITION 5. *For any integer $n \ge 1$, one has*

$$\tau(\lambda(h^n)) \le \varrho(\lambda(h^n))\,.$$

Proof. Apply Lemma 2 to the positive normalized trace τ on the von Neumann algebra $\lambda(\Gamma)''$ generated by $C^*_{\mathrm{r}}(\Gamma)$.

We conclude this section by looking at the behaviour of $\varrho(\lambda(h))$ under passage to a quotient group. If N is a normal subgroup of Γ, we denote by $\pi : \Gamma \to \Gamma/N$ the quotient homomorphism. We shall compare the spectral radii of the operators $\lambda(h_S)$ acting on $\ell^2(\Gamma)$ and $\lambda_{\Gamma/N}(h_{\pi(S)})$ acting on $\ell^2(\Gamma/N)$.

PROPOSITION 6. *Notation being as above, one has*:

(i) $\varrho(\lambda(h_S)) \le \varrho(\lambda_{\Gamma/N}(h_{\pi(S)}))$;
(ii) *if N is amenable*, *then* $\varrho(\lambda(h_S)) = \varrho(\lambda_{\Gamma/N}(h_{\pi(S)}))$.

Proof. (i) For any integer $k \ge 1$, let H_k be the subgroup of Γ generated by the symmetric subset

$$\operatorname{supp}(h_S^k)^* h_S^k = (S^k)^{-1} S^k = \{x^{-1}y : x, y \in S^k\}.$$

Then

$$\begin{aligned}\|\lambda(h_S^k)\| &= \|\lambda[(h_S^k)^* h_S^k]\|^{1/2} = \varrho(\lambda[(h_S^k)^* h_S^k])^{1/2}\\ &\le \varrho(\lambda_{\Gamma/N}[(h_{\pi(S)}^k)^* h_{\pi(S)}^k])^{1/2} = \|\lambda_{\Gamma/N}[(h_{\pi(S)}^k)^* h_{\pi(S)}^k]\|^{1/2}\\ &= \|\lambda_{\Gamma/N}(h_{\pi(S)}^k)\|\end{aligned}$$

where the inequality follows from Lemma 3.1 of [Ke1]; (i) is now clear.

(ii) Let $\lambda_{\Gamma/N} \circ \pi$ be the left regular representation of Γ/N, viewed as a representation of Γ. Since N is amenable, $\lambda_{\Gamma/N} \circ \pi$ is weakly contained in the left regular representation λ of Γ. So π induces a *-homomorphism $C_{\mathrm{r}}^*(\Gamma) \to C_{\mathrm{r}}^*(\Gamma/N)$ which is onto. Consequently, $\operatorname{Sp} \lambda_{\Gamma/N}(h_{\pi(S)}) \subseteq \operatorname{Sp} \lambda(h_S)$, so that $\varrho(\lambda_{\Gamma/N}(h_{\pi(S)})) \le \varrho(\lambda(h_S))$.

Remarks. In the symmetric case $S = S^{-1}$, Proposition 6 appears in [Ke1], Lemma 3.1 and Corollary 2. In that case, the converse of (ii) is true as well, by Theorem 1 of [Ke1]. Our Proposition 9 below shows that the converse of (ii) does not hold in general; indeed, the quotient homomorphism $\pi : \mathbb{F}_{2g} \to \Gamma_g$ has non-amenable kernel but, with S the usual set of generators of $\mathbb{F}_{2g}$, one has

$$\operatorname{Sp} \lambda_{\Gamma_g}(h_{\pi(S)}) = \operatorname{Sp} \lambda(h_S).$$

4. Free subsemigroups and a result of Kesten. We first quote without proof a remarkable result of Kesten in the symmetric case $S = S^{-1}$ (see Theorem 3 of [Ke1] and the Theorem of [Ke2]).

PROPOSITION 7. *Let S^+ be a generating subset of Γ with $|S^+| = n$. Form $S = S^+ \cup (S^+)^{-1}$. Then*, *for the corresponding operator* $\lambda(h_S)$,

$$\frac{\sqrt{2n-1}}{n} \le \varrho(\lambda(h_S)) = \|\lambda(h_S)\| \le 1$$

with equality on the right if and only if Γ is amenable and, provided $n \geq 2$, equality on the left if and only if Γ is freely generated by S^+. In that case $\mathrm{Sp}\,\lambda(h_S) = [-\sqrt{2n-1}/n, \sqrt{2n-1}/n]$.

We wish to indicate what Proposition 7 becomes in the non-symmetric case. So let S be a finite, non-symmetric generating subset of Γ. Proposition 4(iv) above shows that, at least for $|S| = 2$, the norm $\|\lambda(h)\|$ contains no information (1). So we shall focus on the spectral radius $\varrho(\lambda(h))$. However, we shall see that, besides $\varrho(\lambda(h))$, a new number $\sigma(h)$ comes in, defined by

$$\sigma(h) = \limsup_{k\to\infty} \|h^k\|_2^{1/k}\,.$$

If $S = S^{-1}$, then $\sigma(h) = \varrho(\lambda(h))$ by Lemma 2.2 of [Ke1]. In general, $\sigma(h)$ only gives information on the semigroup Γ^+ generated by S, because its definition does not use inverses. In contrast, the definition of $\varrho(\lambda(h))$ makes use of inverses, since we view $\lambda(h)$ as acting on $\ell^2(\Gamma)$, not on $\ell^2(\Gamma^+)$.

Before stating our results, we recall some definitions.

Let F be a finite, symmetric, generating subset of Γ such that $e \in F$. We say that Γ has *subexponential growth* if $\lim_{k\to\infty} |F^k|^{1/k} = 1$; and that Γ has *property* (RD) if there are constants $C > 0$, $r \geq 0$ such that

$$\|\lambda(f)\| \leq C\|f(1+L)^r\|_2 \quad \text{for any } f \in \mathbb{C}\Gamma$$

where L is the length function on Γ associated with F. These definitions do not depend on the generating subset F (up to a change in the constant C for property (RD)).

Examples of groups with property (RD) are on the one hand groups with polynomial growth ([Jol], Theorem 3.1.17), on the other hand free groups ([Haa], Lemma 1.5) and more generally hyperbolic groups à la Gromov ([dHa], [JoV], Théorème 2).

The following proposition owes much to conversations with W. Woess.

PROPOSITION 8. *With Γ, S and h as usual, one has*:

(i) $1/\sqrt{|S|} \leq \sigma(h) \leq \varrho(\lambda(h)) \leq 1$;

(ii) *for $|S| \geq 2$, the equality $1/\sqrt{|S|} = \sigma(h)$ holds if and only if S generates a free semigroup*;

(iii) *if either S is symmetric or Γ has property* (RD), *then $\sigma(h) = \varrho(\lambda(h))$*;

(iv) *the equality $\varrho(\lambda(h)) = 1$ holds if and only if Γ is amenable*;

(v) *if either Γ is amenable and S is symmetric, or if Γ has subexponential growth, then $\sigma(h) = 1$.*

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(1) This was already observed in [BeC] and [DeG]. Note, however, that provided $e \in S$, the group Γ is amenable if and only if $\|\lambda(h)\| = 1$ (see Theorem 1 in [Day]).

Proof. (i) For any integer $k \geq 1$, set

$$S^k = \operatorname{supp} h^k = \{s_1 s_2 \ldots s_k : s_i \in S \text{ for } i = 1, 2, \ldots, k\}.$$

For $g \in S^k$, let $w_k(g)$ be the number of expressions of g of the form $g = s_1 s_2 \ldots s_k$; clearly $\sum_{g \in S^k} w_k(g) = |S|^k$ and

$$h^k(g) = \begin{cases} w_k(g)/|S|^k & \text{if } g \in S^k, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the Cauchy–Schwarz inequality,

$$(*) \qquad \|h^k\|_2 \geq \frac{1}{|S|^k |S^k|^{1/2}} \sum_{g \in S^k} w_k(g) = \frac{1}{|S^k|^{1/2}} \geq \frac{1}{|S|^{k/2}},$$

hence the first inequality. The second inequality follows from

$$\|h^k\|_2 = \|\lambda(h^k)\delta_e\|_2 \leq \|\lambda(h^k)\|;$$

the third from $\varrho(\lambda(h)) \leq \|\lambda(h)\| \leq 1$.

(ii) If S generates a free semigroup, then all inequalities in $(*)$ are equalities.

Now, assume that $|S| \geq 2$ and that S does *not* generate a free semigroup. Then, for some integer $m \geq 1$, one has $|S^m| \leq |S|^m - 1$. Then, for all integers $k \geq 1$, $|S^{mk}| \leq (|S|^m - 1)^k$, so that, using $(*)$,

$$\|h^{km}\|_2 \geq \frac{1}{|S^{mk}|^{1/2}} \geq \frac{1}{(|S|^m - 1)^{k/2}}.$$

Hence

$$\sigma(h) \geq \limsup_{k \to \infty} \|h^{km}\|_2^{1/km} \geq \frac{1}{(|S|^m - 1)^{1/2m}} > \frac{1}{\sqrt{|S|}}.$$

(iii) If $S = S^{-1}$, then $\sigma(h) = \varrho(\lambda(h))$ by Lemma 2.2 of [Ke1]. Suppose now that Γ has property (RD); set $F = S \cup S^{-1} \cup \{e\}$, and let C, r be the constants associated with F in the above definition. Since $\operatorname{supp} h^k \subseteq \{x \in \Gamma : L(x) \leq k\}$, one has

$$\|\lambda(h^k)\| \leq C\|h^k(1 + L)^r\|_2 \leq C(1 + k)^r \|h^k\|_2.$$

Then $\varrho(\lambda(h)) \leq \limsup_{k \to \infty} [C^{1/k}(1 + k)^{r/k} \|h^k\|_2^{1/k}] = \sigma(h)$.

(iv) This follows from Theorem 1 of [Day], or Théorème 1 of [Far], or Proposition 3 of [HRV].

(v) If Γ is amenable and $S = S^{-1}$, combine (iii) and (iv) above. If Γ has subexponential growth, then $\lim_{k \to \infty} 1/|S^k|^{1/k} = 1$. Together with $(*)$, this gives $\sigma(h) \geq 1$.

Remark. Assume that $|S| \geq 2$ and that S does *not* generate a free semigroup. Then, in the Cayley digraph $\mathcal{G}(\Gamma, S)$, we can find two vertices x, y joined by two directed paths without common vertices except x and

y. Let m be the length of the longest of the two paths. The proof of Proposition 8(ii) then gives the explicit lower bound:

$$\sigma(h) \geq \frac{1}{(|S|^m - 1)^{1/2m}} .$$

Observe that this bound decreases to $|S|^{-1/2}$ as m increases to ∞.

From Proposition 8, one has immediately:

COROLLARY 2. *Assume that Γ has property* (RD), *and that S has at least 2 elements. The following are equivalent*:

(i) $\varrho(\lambda(h)) = 1/\sqrt{|S|}$;
(ii) *S generates a free semigroup.*

EXAMPLE. Let G be a finitely generated *solvable* group, which is *not* almost nilpotent. Fix some integer $n \geq 2$. By Corollary 4.14 of [Ros], G contains a subset S with n elements that generates a free semigroup. Let Γ be the subgroup of G generated by S; because Γ is solvable, we have by Proposition 8,

$$\sigma(h) = \frac{1}{\sqrt{n}} \quad \text{and} \quad \varrho(\lambda(h)) = 1 .$$

So here $\sigma(h)$ and $\varrho(\lambda(h))$ are as remote as they can possibly be. This example shows that the assumption (RD) in Corollary 2 cannot be significantly weakened; it also shows that replacing S by $S \cup S^{-1}$ may drastically change $\sigma(h)$.

We now give some explicit computations of spectra.

PROPOSITION 9. *Let Γ be either the free group $\mathbb{F}_n$ on $S = \{s_1, \ldots, s_n\}$, where $n \geq 2$, or the surface group*

$$\Gamma_g = \Big\langle a_1, b_1, \ldots, a_g, b_g \,\Big|\, \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \Big\rangle$$

with $S = \{a_1, b_1, \ldots, a_g, b_g\}$ and $g \geq 2$. Then

$$\operatorname{Sp} \lambda(h_S) = \{z \in \mathbb{C} : |z| \leq 1/\sqrt{|S|}\} .$$

Proof. We begin with a few remarks, valid in both cases.

(1) Any element in the reduced C^*-algebra $C^*_{\mathrm{r}}(\Gamma)$ has a connected spectrum. Otherwise, holomorphic functional calculus would enable us to construct non-trivial idempotents in $C^*_{\mathrm{r}}(\Gamma)$, contradicting a result of Pimsner and Voiculescu [PiV] for free groups, and of Kasparov [Kas] for surface groups.

(2) There exists a homomorphism $\Gamma \to \mathbb{Z}$ mapping S to $\{1\}$. From that, one deduces that the spectrum of $\lambda(h_S)$ is invariant under multiplication

by any complex number of modulus 1 (see Lemma 7 of [HRV]). Together with (1), this means that $\operatorname{Sp} \lambda(h_S)$ is either a disk or an annulus, centred at 0 in any case.

(3) Since Γ has property (RD) (see [JoV], Théorème 2) and S generates a free semigroup, we have $\varrho(\lambda(h_S)) = 1/\sqrt{|S|}$ by Corollary 2.

Now, let Γ be the free group $\mathbb{F}_n$. We prove Proposition 9 by induction on n. In view of remarks (2) and (3), it is enough to check that $\lambda(h_S)$ is not invertible. For $n = 2$, this follows from Proposition 4(iv). For $n > 2$, set $S' = \{s_1 s_n^{-1}, s_2 s_n^{-1}, \ldots, s_{n-1} s_n^{-1}\}$ and $T = S' \cup \{e\}$. Then

$$\lambda(h_S) = \lambda(h_T)\lambda(s_n)\,.$$

We shall use the facts that the subgroup H of $\mathbb{F}_n$ generated by S' is free on $n-1$ generators, and that the restriction to H of the left regular representation of $\mathbb{F}_n$ is unitarily equivalent to a multiple of the left regular representation of $\mathbb{F}_{n-1}$. So our induction hypothesis means

$$\operatorname{Sp} \lambda(h_{S'}) = \{z \in \mathbb{C} : |z| \le 1/\sqrt{n-1}\}\,.$$

In particular, by Proposition 4(iii), the operator $\lambda(h_T)$ is not invertible, so that $\lambda(h_S)$ is not invertible either, and the proof is complete for $\Gamma = \mathbb{F}_n$.

For $\Gamma = \Gamma_g$, we appeal to the fact that $S' = \{a_1 b_g^{-1}, b_1 b_g^{-1}, a_2 b_g^{-1}, b_2 b_g^{-1}, \ldots, a_g b_g^{-1}\}$ freely generates a free group on $2g-1$ generators, and use the same argument as above to see that $\lambda(h_S)$ is not invertible.

Concluding remarks. (1) Consider $\mathbb{F}_n$ with $n \ge 2$ and S as in Proposition 9. Then $\|\lambda(h_S)\| = 2\sqrt{n-1}/n$, as computed in [AkO] (see also [Woe]).

(2) Bearing in mind Kesten's spectral characterization of free groups (Proposition 7), and comparing it with our Corollary 2, one might wonder whether, analogously, there is a spectral characterization of free groups among groups with property (RD) endowed with a finite, non-symmetric, generating system. This question was our motivation for Proposition 9, giving a negative answer.

(3) Let $\mathbb{E}_n^+$ be the free semigroup generated by $s_1, \ldots, s_n$ ($n \ge 2$). Consider the operator $\lambda(h^+) = (1/n)\sum_{i=1}^n \lambda(s_i)$ acting on $\mathcal{H} = \ell^2(\mathbb{E}_n^+)$. Since $\lambda(s_i)\mathcal{H}$ is orthogonal to $\lambda(s_j)\mathcal{H}$ for $i \ne j$, we have $\lambda(s_i)^*\lambda(s_j) = \delta_{ij}$, hence $\lambda(h^+)^*\lambda(h^+) = 1/n$, so that $\sqrt{n}\lambda(h^+)$ is an isometry. It is clearly not invertible, because $\lambda(h^+)$ is not onto. Hence $\operatorname{Sp}\sqrt{n}\lambda(h^+)$ is the closed unit disk in $\mathbb{C}$, i.e.

$$\operatorname{Sp} \lambda(h^+) = \{z \in \mathbb{C} : |z| \le 1/\sqrt{n}\}\,.$$

Added in proof (March 1993). Lemma 1 was proved in July 1992, after a conversation with L. Brown. Some time before, we had mentioned this lemma as a conjecture to G. Cassier & T. Fack, and they had proved it under the additional assumption that $\operatorname{Sp} x$ is distinct from the circle. Their proof, which is direct, makes a clever use of an operator kernel introduced by G. Cassier (*Ensembles K-spectraux et algèbres duales d'opérateurs*, Prépublications de l'Université de Lyon 1, no. 2 (1991), p. 10). Together with applications to the invariant subspace problem, their proof can be found in the preprint *Structure of contractions in von Neumann algebras*.

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