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FINITE UNION OF H-SETS AND COUNTABLE COMPACT SETS

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1. Introduction. In [2], D. E. Grow and M. Insall construct a countable compact set which is not the union of two H-sets. We make precise this result in two directions, proving such a set may be, but need not be, a finite union of H-sets. Descriptive set theory tools like Cantor–Bendixson ranks are used; they are developed in the book of A. S. Kechris and A. Louveau [6]. Two proofs are presented; the first one is elementary while the second one is more general and useful. Using the last one I prove in my thesis, directed by A. Louveau, the existence of a countable compact set which is not a finite union of Dirichlet sets. This result, quoted in [3], is weaker because all Dirichlet sets belong to H. Other new results about the class H and similar classes of thin sets can be found in [4], [1] and [5].

Let $\mathbb{T}$ be the torus $\mathbb{R}/\mathbb{Z}$ endowed with its structure of compact topological group.

A compact subset K of $\mathbb{T}$ is an *H-set* if there exist a nonempty interval I of $\mathbb{T}$ and a strictly increasing sequence n_k of integers such that $n_k K \cap I = \emptyset$ for each integer k. The class of all H-sets is denoted by H.

THEOREM 1.1. *There exists a countable compact subset of $\mathbb{T}$ which is not a finite union of H-sets.*

THEOREM 1.2. *For every integer n, there exists a countable compact subset of $\mathbb{T}$ which is the union of $n+1$ H-sets, but not of n.*

2. Cantor–Bendixson ranks. For each compact metrizable space E we denote by $\mathcal{K}(E)$ (resp. $\mathcal{K}_\omega(E)$) the space of all compact (resp. countable compact) subsets of E.

A subset B of $\mathcal{K}(E)$ is said to be *hereditary* if for every $K \in B$ all compact subsets of K are also in B. Let B be an hereditary subset of $\mathcal{K}(E)$. We denote by B_f (resp. B_σ) the set of all compact subsets of E which are finite (resp. countable) unions of elements of B.

For each $K \in \mathcal{K}(E)$ define the B-*derivate* by

$$\mathrm{d}_B(K) = \{x \in K : \forall \text{ open } V \ (x \in V \Rightarrow \overline{K \cap V} \notin B)\}$$

and then by induction, let

$$K_B^{(0)} = K, \quad K_B^{(<\alpha)} = \bigcap_{\beta<\alpha} K_B^{(\beta)} \quad \text{and} \quad K_B^{(\alpha)} = \mathrm{d}_B(K_B^{(<\alpha)})\,.$$

The sequence $K_B^{(\alpha)}$ is a decreasing sequence of compact sets in E, hence stabilizes at some countable ordinal. It is easy to verify that $K_B^{(\alpha)}$ stabilizes at $\emptyset$ iff $K \in B_\sigma$. Hence we define the *Cantor–Bendixson rank* $\mathrm{rk}_B(K)$ to be the least α such that $K^{(\alpha)} = \emptyset$, if such an α exists, and ω_1 (the first uncountable ordinal) otherwise.

For $S = \emptyset \cup \{\text{singletons}\}$, rk_S is the classical Cantor–Bendixson rank on $\mathcal{K}_\omega(E)$.

PROPOSITION 2.1. *Let n be an integer and $K \in \mathcal{K}(E)$. If K is the union of n B-sets*, *then* $\mathrm{rk}_B(K) \le n$.

The previous result can be easily deduced from the following lemma.

LEMMA 2.2. *Let* $(K_1, K_2) \in (B_\sigma)^2$. *We have*

$$\mathrm{rk}_B(K_1 \cup K_2) \le \sup\{\mathrm{rk}_B(K_1), \mathrm{rk}_B(K_2)\} + \mathrm{rk}_B(K_1 \cap K_2)\,.$$

Proof. If $x \in (K_1 \cup K_2) \setminus (K_1 \cap K_2) = (K_1 \setminus K_2) \cup (K_2 \setminus K_1)$, then there exist an open neighbourhood V of x and $i = 1$ or 2 such that $V \cap (K_1 \cup K_2) \subset K_i$, thus $x \in \mathrm{d}_B(K_1 \cup K_2)$ iff $x \in \mathrm{d}_B(K_i)$. It follows by induction that $(K_1 \cup K_2)_B^{(\alpha)} \subset K_1 \cap K_2$ with $\alpha = \sup\{\mathrm{rk}_B(K_1), \mathrm{rk}_B(K_2)\}$. ∎

The following result is a simpler form of Theorem 6 of [6], p. 202.

LEMMA 2.3. *Let A and B be two subsets of $\mathcal{K}(E)$ which are closed under translations. Suppose that $A \subset B$, B is hereditary and there exists $K_1 \in A$ with $\mathrm{d}_B(K_1) \neq \emptyset$. Then rk_B is unbounded on A_σ.*

Proof. We can assume, by translating K_1 if necessary, that $0 \in \mathrm{d}_B(K_1)$. We construct by induction on α a compact set $K_\alpha \in A_\sigma$ with $\mathrm{rk}_B(K_\alpha) \ge \alpha + 1$. Let D be a countable dense subset of $K_1 \setminus \{0\}$. If α is a limit ordinal, choose $\alpha_n \nearrow \alpha$, and if $\alpha = \beta + 1$, set $\alpha_n = \beta$ for each n. Now put

$$K_\alpha = \{0\} \cup \bigcup_n ([K_{\alpha_n} \cap \mathrm{B}(0, \varepsilon_n)] + t_n)$$

where the sequence t_n enumerates infinitely many times each element of D and the sequence ε_n decreases to 0. It can be easily verified that K_α is compact, $K_\alpha \in A_\sigma$ and $0 \in (K_\alpha)_B^{(\alpha)}$. Thus rk_B is unbounded on A_σ. ∎

3. Elementary proof

Fact 3.1. *There is a countable compact set L with $\mathrm{d}_H(L) \neq \emptyset$. In particular*, $L \notin H$.

Proof ([6], p. 38). Enumerate all the rational intervals of $\mathbb{T}$ in a sequence I_n. For each n and each $i = 1, \ldots, n$, choose a point $r_i^n \in [0, 1/n]$ with $nr_i^n \in I_i$. Let finally $x_1, x_2, \ldots$ enumerate the r_i^n's. Clearly $L = \{0\} \cup \{x_1, x_2, \ldots\}$ is compact and $0 \in \mathrm{d}_H(L)$. ∎

Fact 3.2. *H is closed under translations.*

Proof. Let $K \in H$; let n_k be a sequence and I an interval witnessing that. Let $x \in \mathbb{T}$; we prove that $K + x \in H$. By compactness of $\mathbb{T}$ we can assume that $n_k x \to y$ for some $y \in \mathbb{T}$. Let J be the interval with the same center as I and half its length. Then $n_k(K + x) \cap (J + y) = \emptyset$ for k large enough. ∎

Proof of Theorem 1.1. Using Lemma 2.2 and Lemma 2.3 with $A = S$, $B = H$ and $K_1 = L$ we deduce that the countable compact set K_ω (where ω is the first infinite ordinal) is not a finite union of H-sets. ∎

Remark. Salinger [8] proved that every countable compact set of finite classical Cantor–Bendixson rank n is the union of 2^{n-1} H-sets.

Proof of Theorem 1.2. Let us return to the proof of Lemma 2.3 with $A = S$, $B = H$ and $K_1 = L$. The ε_n's can be chosen such that $\mathrm{B}(t_n, \varepsilon_n) \cap \mathrm{B}(t_m, \varepsilon_m) = \emptyset$ if $t_n \neq t_m$, because all elements of $L \setminus \{0\}$ are isolated, so K_α has classical Cantor–Bendixson rank equal to $\alpha + 1$.

Let n be an integer. By Salinger's Theorem, K_n is the union of 2^n H-sets. By Proposition 2.1, K_n cannot be the union of n H-sets. So there is a compact subset of K_n which is the union of $n + 1$ H-sets, but not of n. ∎

4. Descriptive set theory proof. For each compact metrizable space E, the space $\mathcal{K}(E)$ with the Hausdorff topology generated by the sets $\{K \in \mathcal{K}(E) : K \subset V\}$ and $\{K \in \mathcal{K}(E) : K \cap V \neq \emptyset\}$, where V is open in E, is compact and metrizable.

Let B be a Borel hereditary subset of $\mathcal{K}(E)$.

Fact 4.1. *B_f is an analytic subset of $\mathcal{K}(E)$.*

Proof. The function $\Phi : \mathcal{K}(E) \times \mathcal{K}(E) \to \mathcal{K}(E)$, $(K, L) \mapsto K \cup L$, is continuous and $B_f = \bigcup B_n$ with $B_0 = B$ and $B_{n+1} = \Phi(B_n \times B_n)$. ∎

Fact 4.2 ([6], pp. 140, 194, 198). *B_σ is a coanalytic subset of $\mathcal{K}(E)$ and* rk_B *is a coanalytic rank on B_σ.*

Let us recall the Boundedness Theorem: if C is a coanalytic set with a coanalytic rank and A is an analytic subset of C, then the rank of elements of A is uniformly bounded by a countable ordinal.

The following result can be deduced immediately from the Boundedness Theorem and Lemma 2.3.

LEMMA 4.3. *Let A and B be two subsets of $\mathcal{K}(E)$ which are closed under translations. Suppose that $A \subset B$, B is Borel and hereditary and there exists $K \in A$ with $\mathrm{d}_B(K) \neq \emptyset$. Then there is no analytic set P with $A_\sigma \subset P \subset B_\sigma$.*

FACT 4.4 ([7]). *H is a $\mathcal{K}_{\sigma\delta}$ subset of $\mathcal{K}(\mathbb{T})$.*

Proof. A compact subset K of $\mathbb{T}$ belongs to H if there exists an open rational interval I of $\mathbb{T}$ such that for every integer k, there exists an integer $n \geq k$ such that $nK \cap I = \emptyset$. The last condition is clearly closed in $\mathcal{K}(\mathbb{T})$. ∎

Using both previous results we have:

THEOREM 4.5. *There is no analytic set P with $\mathcal{K}_\omega(\mathbb{T}) \subset P \subset H_\sigma$.*

Theorem 1.1 can now be easily deduced. Indeed, H_f is an analytic subset of H_σ, whence $\mathcal{K}_\omega(\mathbb{T}) \not\subset H_f$.

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