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THE SPACE OF WHITNEY LEVELS IS HOMEOMORPHIC TO l_2

BY

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If (X,d) is a metric continuum, $C(X)$ stands for the hyperspace of all nonempty subcontinua of X, endowed with the Hausdorff metric H. A *map* is a continuous function.

A *Whitney map* is a map $\mu : C(X) \to I$ such that $\mu(\{x\}) = 0$ for each $x \in X$, $\mu(X) = 1$ and if $A, B \in C(X)$, $A \subsetneq B$ then $\mu(A) < \mu(B)$. The space of Whitney maps $W(X)$ is endowed with the sup metric. Throughout this paper μ denotes a fixed Whitney map. A *Whitney level* is a subset of $C(X)$ of the form $\mu^{-1}(t)$ where μ is a Whitney map. By [5, p. 1032], Whitney levels are in $C(C(X)) = C^2(X)$. The space of Whitney levels, denoted by $N(X)$, is a subspace of $C^2(X)$.

Given $\mathcal{A}, \mathcal{B} \in N(X)$ we write $\mathcal{A} \le \mathcal{B}$ if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$, and we write $\mathcal{A} \ll \mathcal{B}$ if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subsetneq B$. The space of Whitney decompositions is $WD(X) = \{\{\omega^{-1}(t) \in C^2(X) \mid 0 \le t \le 1\} \in C(C(C(X))) \mid \omega \in W(X)\}$. Other conventions that we use: I denotes the interval $[0,1]$, the metric for $C^2(X)$ is denoted by H^2, $F_1(X)$ is the set of all one-element subsets of X.

The space $N(X)$ was introduced in [6]; it was useful to prove that $W(X)$ and $WD(X)$ are homeomorphic to the Hilbert space l_2 for all X (see [7] and [8]).

The aim of this paper is to prove

MAIN THEOREM. *The space $N(X)$ of Whitney levels is homeomorphic to the Hilbert space l_2 for all X.*

For that we use Toruńczyk's characterization of Hilbert space. Theorems 1 and 2 are intermediate results.

THEOREM 1. *$N(X)$ is topologically complete.*

DEFINITION 1.1. A *large ordered arc* (l.o.a.) in $C(X)$ is a subcontinuum γ of $C(X)$ such that $\bigcap \gamma \in F_1(X)$, $\bigcup \gamma = X$ and $A, B \in \gamma$ implies that $A \subset B$ or $B \subset A$.

An *antichain* in $C(X)$ is a subset $\mathcal{A}$ of $C(X)$ such that if $A, B \in \mathcal{A}$ and $A \subset B$ then $A = B$.

By [9, Lemma 1.3], every l.o.a. in $C(X)$ is homeomorphic to I and by [9, Thm. 2.8], if $A, B \in C(X)$ and $A \subset B$, then there exists a l.o.a. γ in $C(X)$ such that $A, B \in \gamma$. In [7] it was proved that if $\mathcal{A} \subset C(X) - (\{X\} \cup F_1(X))$, then $\mathcal{A}$ is a Whitney level if and only if $\mathcal{A}$ is a compact antichain which intersects every l.o.a. in $C(X)$.

Proof of Theorem 1. Let $\mathfrak{A} = \{D \in C^2(X) : D \cap \gamma \neq \emptyset$ for every l.o.a. γ in $C(X)\}$. Then $\mathfrak{A}$ is closed in $C^2(X)$, thus $\mathfrak{A}$ is topologically complete. For each $n \in \mathbb{N}$ define $\mathfrak{A}_n = \{D \in \mathfrak{A}$: there exist $A, B \in D$ such that $A \subset B$ and $H(A, B) \geq 1/n\}$ and $\mathfrak{B}_n = \{D \in \mathfrak{A} : D \cap F_1(X) \neq \emptyset$ and $D \cap \mu^{-1}[1/n, 1] \neq \emptyset\}$. It is easy to prove that $\mathfrak{A}_n$ and $\mathfrak{B}_n$ are closed subsets of $\mathfrak{A}$.

Clearly $\bigcup \mathfrak{A}_n \cup \bigcup \mathfrak{B}_n \subset \mathfrak{A} - N(X)$. Let $D \in \mathfrak{A} - N(X)$. If $X \in D$, then there exists $A \in D$ such that $A \neq X$. Thus there exists $n \in \mathbb{N}$ such that $D \in \mathfrak{A}_n$. If $D \cap F_1(X) \neq \emptyset$, since D intersects every l.o.a. in $C(X)$ and $D \neq F_1(X)$, we see that D is not contained in $F_1(X)$. Thus there exists $n \in \mathbb{N}$ such that $D \in \mathfrak{B}_n$. Finally, if $D \subset C(X) - (\{X\} \cup F_1(X))$, then since $D \notin N(X)$, D is not an antichain. Therefore $D \in \mathfrak{A}_n$ for some n.

Hence $\mathfrak{A} - N(X) = \bigcup \mathfrak{A}_n \cup \bigcup \mathfrak{B}_n$. Thus $N(X)$ is a G_δ subset of $\mathfrak{A}$. Therefore [12, Thm. 24.12], $N(X)$ is topologically complete.

THEOREM 2. *$N(X)$ is a metric AR.*

In [7] it was proved that for every $\mathcal{A}, \mathcal{B} \in N(X)$, the infimum and supremum of the set $\{\mathcal{A}, \mathcal{B}\}$ with respect to the order $\leq$ both exist. They were constructed in the following way: For each l.o.a. γ in $C(X)$, let A_γ (resp. B_γ) be the unique element in $A \cap \gamma$ (resp. $B \cap \gamma$) (notice that $A_\gamma \subset B_\gamma$ or $A_\gamma \supset B_\gamma$). The infimum of $\mathcal{A}$ and $\mathcal{B}$ is defined to be $\mathcal{A} \wedge \mathcal{B} = \{A_\gamma \cap B_\gamma : \gamma$ is a l.o.a. in $C(X)\}$ and the supremum is $\mathcal{A} \vee \mathcal{B} = \{A_\gamma \cup B_\gamma : \gamma$ is a l.o.a. in $C(X)\}$. Also it was shown that the functions $\wedge, \vee : N(X) \times N(X) \to N(X)$ are continuous [7, Thm. 1.9].

To prove Theorem 2 we use $\vee$ and $\wedge$ to endow $N(X)$ with a convex structure in the sense of Curtis [2, Definition 2.1]. We imitate Dugundji's proof in [3] to prove that $N(X)$ is a metric AR. First we need to introduce a new metric for $N(X)$.

DEFINITION 2.1. Let $H^* : N(X) \times N(X) \to \mathbb{R}$ be given by

$$H^*(\mathcal{A}, \mathcal{B}) = \sup\{H(A, B) : A \in \mathcal{A}, B \in \mathcal{B} \text{ and } A \subset B \text{ or } A \supset B\}.$$

LEMMA 2.2. (a) *H^* is a metric for $N(X)$ which is equivalent to H^2.*
(b) *If $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$ then $H^*(\mathcal{A}, \mathcal{B}), H^*(\mathcal{B}, \mathcal{C}) \leq H^*(\mathcal{A}, \mathcal{C})$.*
(c) *If $\mathcal{C} \leq \mathcal{B} \leq \mathcal{D}$ and $H^*(\mathcal{A}, \mathcal{C}), H^*(\mathcal{A}, \mathcal{D}) \leq \varepsilon$, then $H^*(\mathcal{A}, \mathcal{B}) \leq \varepsilon$.*
(d) *$H^*(\mathcal{C} \vee \mathcal{B}, \mathcal{D} \vee \mathcal{B}) \leq H^*(\mathcal{C}, \mathcal{D})$ for every $\mathcal{B}, \mathcal{C}, \mathcal{D} \in N(X)$.*

Proof. (a) Let $\mathcal{A},\mathcal{B},\mathcal{C} \in N(X)$ and let $A \in \mathcal{A}$ and $C \in \mathcal{C}$ such that $A \subset C$ or $A \supset C$. Then there exists a l.o.a. γ in $C(X)$ such that $A, C \in \gamma$. Let $B \in \gamma \cap \mathcal{B}$. Then $A \subset B$ or $A \supset B$ and $B \subset C$ or $B \supset C$. Hence $H(A,C) \le H(A,B) + H(B,C) \le H^*(\mathcal{A},\mathcal{B}) + H^*(\mathcal{B},\mathcal{C})$. Therefore $H^*(\mathcal{A},\mathcal{C}) \le H^*(\mathcal{A},\mathcal{B}) + H^*(\mathcal{B},\mathcal{C})$.

Clearly $H^2 \le H^*$. Let $\mathcal{A} \in N(X)$ and let $\varepsilon > 0$. By [7, 1.8] there exists $\delta > 0$ such that if $\mathcal{B} \in N(X)$, $H^2(\mathcal{A},\mathcal{B}) < \delta$, $A \in \mathcal{A}, B \in \mathcal{B}$ and $A \subset B$ or $A \supset B$ then $H(A,B) < \varepsilon$. Given $\mathcal{B} \in N(X)$ such that $H^2(\mathcal{A},\mathcal{B}) < \delta$, we have $H^*(\mathcal{A},\mathcal{B}) \le \varepsilon$. Hence H^* and H^2 are equivalent metrics for $N(X)$.

(b) This is evident.

(c) Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be such that $A \subset B$ or $A \supset B$. Let γ be a l.o.a. in $C(X)$ such that $A, B \in \gamma$. Let $C \in \gamma \cap \mathcal{C}$ and $D \in \gamma \cap \mathcal{D}$. Then $C \subset B \subset D$. If $A \subset B$ then $H(A,B) \le H(A,D) \le H^*(\mathcal{A},\mathcal{D}) \le \varepsilon$. If $A \supset B$, then $H(A,B) \le H(A,C) \le H^*(\mathcal{A},\mathcal{C}) \le \varepsilon$. Therefore $H^*(\mathcal{A},\mathcal{B}) \le \varepsilon$.

(d) Let $A \in \mathcal{C} \vee \mathcal{B}$ and $E \in \mathcal{D} \vee \mathcal{B}$ be such that $A \subset E$ or $A \supset E$. Let γ be a l.o.a. in $C(X)$ such that $A, E \in \gamma$. Let $C \in \mathcal{C} \cup \gamma$, $B \in \mathcal{B} \cup \gamma$ and $D \in \mathcal{D} \cup \gamma$. Suppose, for example, that $C \subset D$. If $B \subset C$ then $A = C$ and $E = D$, thus $H(A,E) \le H^*(\mathcal{C},\mathcal{D})$. If $C \subset B \subset D$, then $A = B$ and $E = D$, hence $H(A,E) \le H(C,D) \le H^*(\mathcal{C},\mathcal{D})$. If $D \subset B$ then $A = B = E$, so $H(A,E) \le H^*(\mathcal{C},\mathcal{D})$. Therefore $H^*(\mathcal{C} \vee \mathcal{B}, \mathcal{D} \vee \mathcal{B}) \le H^*(\mathcal{C},\mathcal{D})$.

Definition 2.3. Let

$$\Delta_n = \{(s_1,\ldots,s_n) \in I^n \mid s_1 + \ldots + s_n = 1\}.$$

Given $\mathcal{A}_1 \in N(X)$, let $M_1(\mathcal{A}_1, 1) = \mathcal{A}_1$. If $\mathcal{A}_1, \mathcal{A}_2 \in N(X)$ and $s \in I$, let

$$M_2(\mathcal{A}_1,\mathcal{A}_2,s,1-s) = \begin{cases} \mathcal{A}_2 \vee (\mu^{-1}(2s) \wedge \mathcal{A}_1) & \text{if } 0 \le s \le \frac{1}{2}, \\ \mathcal{A}_1 \vee (\mu^{-1}(2-2s) \wedge \mathcal{A}_2) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

Inductively, if $n \ge 3$, $\mathcal{A}_1,\ldots,\mathcal{A}_n \in N(X)$ and $(s_1,\ldots,s_n) \in \Delta_n$, let

$$M_n(\mathcal{A}_1,\ldots,\mathcal{A}_n,s_1,\ldots,s_n) = \begin{cases} M_2\left(M_{n-1}\left(\mathcal{A}_1,\ldots,\mathcal{A}_{n-1},\frac{s_1}{1-s_n},\ldots,\frac{s_{n-1}}{1-s_n}\right),\mathcal{A}_n,1-s_n,s_n\right) & \text{if } s_n < 1, \\ \mathcal{A}_n & \text{if } s_n = 1. \end{cases}$$

Lemma 2.4. (a) *$M_n : N(X)^n \times \Delta_n \to N(X)$ is continuous for every $n \in \mathbb{N}$.*

(b) *Suppose that $H^*(\mathcal{A},\mathcal{A}_1),\ldots,H^*(\mathcal{A},\mathcal{A}_n) \le \varepsilon$. Then for every $(s_1,\ldots,s_n) \in \Delta_n$, $H^*(M_n(\mathcal{A}_1,\ldots,\mathcal{A}_n,s_1,\ldots,s_n),\mathcal{A}) \le \varepsilon$.*

(c) *Suppose that $n \ge 2$ and $(s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n) \in \Delta_{n-1}$. Then*

$$M_n(\mathcal{A}_1,\ldots,\mathcal{A}_n,s_1,\ldots,s_{i-1},0,s_{i+1},\ldots,s_n) = M_{n-1}(\mathcal{A}_1,\ldots,\mathcal{A}_{i-1},\mathcal{A}_{i+1},\ldots,\mathcal{A}_n,s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n).$$

Proof. (a) Clearly M_1 and M_2 are continuous. Suppose that M_{n-1} is continuous ($n \geq 3$). Let $z = (\mathcal{A}_1, \ldots, \mathcal{A}_n, s_1, \ldots, s_n) \in N(X)^n \times \Delta_n$. If $s_n < 1$, the continuity of M_n at z is immediate. Suppose then that $s_n = 1$. Let $\varepsilon > 0$. Take $\delta > 0$ such that $\delta \leq 1/2$ and $H^*(F_1(X), \mu^{-1}(2t)) < \varepsilon/2$ for every $t \in [0, \delta)$. Let $w = (\mathcal{B}_1, \ldots, \mathcal{B}_n, t_1, \ldots, t_n) \in N(X)^n \times \Delta_n$ be such that $H^*(\mathcal{A}_1, \mathcal{B}_1), \ldots, H^*(\mathcal{A}_n, \mathcal{B}_n)$ and $1 - t_n$ are less than δ and $\varepsilon/2$. If $t_n = 1$, then $H^*(M_n(z), M_n(w)) = H^*(\mathcal{A}_n, \mathcal{B}_n) < \varepsilon$. If $t_n < 1$, then $M_n(w) = M_2(C, \mathcal{B}_n, 1 - t_n, t_n)$ where

$$C = M_{n-1}(\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}, t_1/(1 - t_n), \ldots, t_n/(1 - t_n))\,.$$

Thus $M_n(w) = \mathcal{B}_n \vee (\mu^{-1}(2(1-t_n)) \wedge C)$. Then $\mathcal{B}_n \vee F_1(X) \leq M_n(w) \leq \mathcal{B}_n \vee \mu^{-1}(2(1-t_n))$. Applying Lemma 2.2, we have $H^*(\mathcal{B}_n, M_n(w)) < \varepsilon/2$. Hence $H^*(M_n(z), M_n(w)) = H^*(\mathcal{A}_n, M_n(w)) < \varepsilon$. Therefore M_n is continuous.

(b) We only check this property for $n = 2$. Let $z = (\mathcal{A}_1, \mathcal{A}_2, s_1, s_2) \in N(X)^2 \times \Delta_2$ be such that $H^*(\mathcal{A}_1, \mathcal{A}), H^*(\mathcal{A}_2, \mathcal{A}) \leq \varepsilon$. Then $H^*(\mathcal{A}, \mathcal{A}_1 \vee \mathcal{A}_2) \leq \varepsilon$. Since $\mathcal{A}_2 \leq M_2(z) \leq \mathcal{A}_1 \vee \mathcal{A}_2$ or $\mathcal{A}_1 \leq M_2(z) \leq \mathcal{A}_1 \vee \mathcal{A}_2$, Lemma 2.2 implies that $H^*(\mathcal{A}, M_2(z)) \leq \varepsilon$.

Proof of Theorem 2. Let (Z, ϱ) be a metric space, let A be a closed subset of Z and let $g : A \to N(X)$ be a map.

For each $x \in Z - A$, let $B_x = \{z \in Z \mid \varrho(x, z) < (1/2)\varrho(x, A)\}$. Let $U = \{U_\alpha \mid \alpha \in J\}$ be a neighborhood finite open refinement of $\{B_x \mid x \in Z - A\}$, indexed by a well ordered set J. Let $\{\phi_\alpha \mid \alpha \in J\}$ be a partition of unity on $Z - A$ subordinate to U. Given $\alpha \in J$, choose $x_\alpha \in Z - A$ such that $U_\alpha \subset B_{x_\alpha}$. Also choose $a_\alpha \in A$ such that $\varrho(x_\alpha, a_\alpha) < 2\varrho(x_\alpha, A)$. If $z \in U_\alpha$, then $(1/2)\varrho(x_\alpha, A) \leq \varrho(z, A)$, so $\varrho(z, a_\alpha) \leq 5\varrho(z, A)$.

Define $\widehat{g} : Z \to N(X)$ in the following way:

(a) For $x \in Z - A$, let $\alpha_1 < \ldots < \alpha_n$ be the ordering in J of those elements α for which $\phi_\alpha(x) > 0$, and define

$$\widehat{g}(x) = M_n(g(a_{\alpha_1}), \ldots, g(a_{\alpha_n}), \phi_{\alpha_1}(x), \ldots, \phi_{\alpha_n}(x))\,.$$

(b) For $x \in A$, define $\widehat{g}(x) = g(x)$.

If $x \in Z - A$, there exists an open subset U of Z and $\beta_1, \ldots, \beta_m \in J$ such that $x \in U \subset Z - A$, $\beta_1 < \ldots < \beta_m$ and $\phi_\alpha(z) = 0$ for every $z \in U$ and every $\alpha \notin \{\beta_1, \ldots, \beta_m\}$. Lemma 2.4(c) implies that

$$\widehat{g}(z) = M_m(g(a_{\beta_1}), \ldots, g(a_{\beta_m}), \phi_{\beta_1}(z), \ldots, \phi_{\beta_m}(z))$$

for every $z \in U$. Hence $\widehat{g}$ is continuous at x. If $x \in \mathrm{Fr}(A)$, let $\varepsilon > 0$. Let $\delta > 0$ be such that if $a \in A$ and $\varrho(a, x) \leq \delta$, then $H^*(g(a), g(x)) \leq \varepsilon$. Take $z \in Z$ such that $\varrho(z, x) \leq \delta/6$ and $z \notin A$. Let $\alpha_1 < \ldots < \alpha_n$ be those α's for which $\phi_\alpha(z) > 0$. Then $z \in U_{\alpha_1} \cap \ldots \cap U_{\alpha_n}$. Thus $\varrho(z, a_{\alpha_i}) \leq 5\varrho(z, A) \leq 5\varrho(z, x) < (5/6)\delta$ for each i. Hence $\varrho(x, a_{\alpha_i}) < \delta$ for each i. Lemma 2.4(b) implies that $H^*(\widehat{g}(z), \widehat{g}(x)) \le \varepsilon$. So $\widehat{g}$ is continuous at x, thus continuous and therefore X is an AR (metric).

We now make a start towards the proof of the Main Theorem with some preliminary technical results.

Some conventions. We consider the space 2^X of all nonempty closed subsets of X with the Hausdorff metric. Throughout this section ω will denote a fixed Whitney map for 2^X such that $\omega(X) = 1$ and if $A, B, C \in 2^X$ and $A \subset B$, then

$$\omega(B \cup C) - \omega(A \cup C) \le \omega(B) - \omega(A)$$

(such a map exists by [1]). Also β will denote a fixed l.o.a. in $C(X)$. Let $\beta^* = \beta - (\{X\} \cup F_1(X))$. Let $\sigma : I \to \beta$ denote the inverse of the map $\omega|\beta : \beta \to I$. Let $\phi : N(X) \to \beta$ be a continuous function defined by $\phi(\mathcal{A}) = A$ if and only if A is the unique element in $\mathcal{A} \cap \beta$. Finally, let $N(X)^* = N(X) - \{\{X\}, F_1(X)\}$.

Definition 3.1. Let $\psi : \beta^* \times (0,1] \times C(X) \to \mathbb{R}$ be given by

$$\psi(A, t, B) = \omega(A \cup B) - \omega(B) - t(\omega(B) - \omega(A)).$$

Lemma 3.2. (a) *ψ is continuous.*

(b) *If $A_1 \subsetneq A_2$, then $\psi(A_1, t, B) < \psi(A_2, t, B)$ for every $(t, B) \in (0,1] \times C(X)$.*

(c) *If $B_1 \subsetneq B_2$, then $\psi(A, t, B_1) > \psi(A, t, B_2)$ for every $(A, t) \in \beta^* \times (0,1]$.*

Definition 3.3. Given $(A, t) \in \beta^* \times (0,1]$, let

$$L(A, t) = \{B \in C(X) \mid \psi(A, t, B) = 0\}.$$

Lemma 3.4. (a) *$A \in L(A, t)$ and $L(A, t) \in N(X)$ for every $(A, t) \in \beta^* \times (0,1]$.*

(b) *If $0 < t_1 < t_2 \le 1$, then $L(A, t_1) \ge L(A, t_2)$.*

(c) *If $A_1 \subsetneq A_2$, then $L(A_1, t) \ll L(A_2, t)$.*

(d) *The function $L : \beta^* \times (0,1] \to N(X)$ is continuous.*

Proof. (a) Let $(A, t) \in \beta^* \times (0,1]$. Then $\psi(A, t, X) = -t(\omega(X) - \omega(A)) < 0$. Given $x \in X$, $\psi(A, t, \{x\}) = \omega(A \cup \{x\}) - \omega(A) + t\omega(A) > 0$. Then $L(A, t) \cap (\{X\} \cup F_1(X)) = \emptyset$ and $L(A, t)$ intersects every l.o.a. in $C(X)$. By Lemma 3.2(c), $L(A, t)$ is a compact antichain in $C(X)$. Therefore ([6, Thm. 1.2]), $L(A, t) \in N(X)$.

(b) Let $B \in L(A, t_2)$ and let γ be a l.o.a. in $C(X)$ such that $B \in \gamma$. Let $A_1 \in \gamma \cap \omega^{-1}(\omega(A))$. Since $\psi(A, t_2, A_1) = \omega(A \cup A_1) - \omega(A_1) \ge 0 = \psi(A, t_2, B)$, by Lemma 3.2(c), we have $A_1 \subset B$. Then $\psi(A, t_1, B) \ge \psi(A, t_2, B) = 0$. Let $C \in \gamma \cap L(A, t_1)$. Then $\psi(A, t_1, C) = 0 \le \psi(A, t_1, B)$. So Lemma 3.2(c) implies that $B \subset C$. Hence $L(A, t_2) \le L(A, t_1)$.

(c) This follows from Lemma 3.2.

(d) Let $((A_n, t_n))_n \subset \beta^* \times (0,1]$ and let $(A,t) \in \beta^* \times (0,1]$ be such that $A_n \to A$ and $t_n \to t$. Take $B \in L(A,t)$. Let γ be a l.o.a. in $C(X)$ such that $B \in \gamma$. For each $n \in \mathbb{N}$, take $B_n \in \gamma \cap L(A_n, t_n)$. If $(B_n)_n$ does not converge to B, since γ is compact, there exists a subsequence $(B_{n_k})_k$ of $(B_n)_n$ and $C \in \gamma$ such that $B_{n_k} \to C \neq B$. Then $0 = \psi(A_{n_k}, t_{n_k}, B_{n_k}) \to \psi(A,t,C)$. So $\psi(A,t,C) = \psi(A,t,B)$. Lemma 3.2(c) implies that $C = B$. This contradiction proves that $B_n \to B$. Hence $B \in \liminf L(A_n, t_n)$. Therefore $L(A,t) \subset \liminf L(A_n, t_n)$. Now take $B \in \limsup L(A_n, t_n)$. Then there exists a sequence $n_1 < n_2 < \ldots$ and elements $B_k \in L(A_{n_k}, t_{n_k})$ such that $B_k \to B$. Then $0 = \psi(A_{n_k}, t_{n_k}, B_k) \to \psi(A,t,B)$. Thus $B \in L(A,t)$. Hence $\limsup L(A_n, t_n) \subset L(A,t)$. Therefore $L(A_n, t_n) \to L(A,t)$. Consequently, L is continuous.

LEMMA 3.5. *Let $\mathcal{A}, \mathcal{B} \in N(X)^*$. Let $r, s > 0$ be such that $r < \omega(\phi(\mathcal{A}))$ and $s < \omega(\phi(\mathcal{B}))$. Suppose $t_1, t_2 \in (0,1]$ are such that $L(\sigma(r), t_1) \wedge \mathcal{A} = L(\sigma(s), t_2) \wedge \mathcal{B}$. Then $t_1 = t_2$.*

Proof. Since $r < \omega(\phi(\mathcal{A}))$, we have $\sigma(r) \subset \phi(\mathcal{A}) \neq \sigma(r)$. Then $\sigma(r) \in (L(\sigma(r), t_1) \wedge \mathcal{A}) \cap \beta$. Similarly, $\sigma(s) \in (L(\sigma(s), t_2) \wedge \mathcal{B}) \cap \beta$. Thus $\sigma(r) = \sigma(s)$. Since $\sigma(r)$ is a proper subset of $\phi(\mathcal{A})$ and $\phi(\mathcal{B})$, we have $\sigma(r) \notin \mathcal{A} \cup \mathcal{B}$. Therefore there exists $B \in L(\sigma(r), t_1) \wedge \mathcal{A}$ such that $B \neq \sigma(r)$ and $B \notin \mathcal{A} \cup \mathcal{B}$. Thus $B \in L(\sigma(r), t_1) \cap L(\sigma(s), t_2)$ and $\sigma(r)$ is not contained in B. Consequently, $\psi(\sigma(r), t_1, B) = \psi(\sigma(s), t_2, B) = 0$. So

$$\begin{aligned} &\omega(\sigma(r) \cup B) - \omega(B) - t_1(\omega(B) - \omega(\sigma(r))) \\ &\qquad = \omega(\sigma(r) \cup B) - \omega(B) - t_2(\omega(B) - \omega(\sigma(r))) = 0\,. \end{aligned}$$

Thus $(t_1 - t_2)(\omega(B) - \omega(\sigma(r))) = 0$. If $\omega(B) - \omega(\sigma(r)) = 0$, then $\omega(\sigma(r) \cup B) = \omega(B)$. Hence $\sigma(r) \subset B$. This contradiction proves that $t_1 = t_2$.

LEMMA 3.6. *Let $(\mathcal{A}_n)_n$ be a sequence in $N(X)$, let $\mathcal{A} \in N(X)$, let $(A_n)_n$ be a sequence in $\beta - \{X\}$, let $A \in \beta - \{X\}$ and let $(t_n)_n$ be a sequence in $(0,1]$. If $t_n \to 0$, $A_n \to A$ and $\mathcal{A}_n \wedge L(A_n, t_n) \to \mathcal{A}$, then $\mathcal{A}_n \to \mathcal{A}$.*

Proof. Let $B \in \limsup L(A_n, t_n)$. Then there exists a sequence $n_1 < n_2 < \ldots$ and elements $B_k \in L(A_{n_k}, t_{n_k})$ such that $B_k \to B$. Then

$$\begin{aligned} 0 = \psi(A_{n_k}, t_{n_k}, B_k) &= \omega(B_k \cup A_{n_k}) - \omega(B_k) - t_{n_k}(\omega(B_k) - \omega(A_{n_k})) \\ &\to \omega(B \cup A) - \omega(B)\,. \end{aligned}$$

Hence $A \subset B$.

For each $n \in \mathbb{N}$, $A_n \in L(A_n, t_n)$, so there exists $B_n \in \mathcal{A}_n \wedge L(A_n, t_n)$ such that $B_n \subset A_n$. It follows that there exists $A_0 \in \mathcal{A}$ such that $A_0 \subset A$.

Now we prove that $\mathcal{A} \subset \liminf \mathcal{A}_n$. Let $B \in \mathcal{A} - \{A_0\}$. Then there exists a sequence $(B_n)_n$ such that $B_n \in \mathcal{A}_n \wedge L(A_n, t_n)$ for each n and $B_n \to B$. Since A_0 is not contained in B, we have $B \notin \limsup L(A_n, t_n)$. Then there exists $N \in \mathbb{N}$ such that $B_n \in \mathcal{A}_n$ for every $n \geq N$. Therefore $B \in \liminf \mathcal{A}_n$. Since $A_0 \neq X$, $\mathcal{A}$ is a nondegenerate continuum. Hence $\mathcal{A} \subset \liminf \mathcal{A}_n$.

Now we show that $\limsup \mathcal{A}_n \subset \mathcal{A}$. Let $B \in \limsup \mathcal{A}_n$. Then there exists a sequence $n_1 < n_2 < \ldots$ and elements $B_k \in \mathcal{A}_{n_k}$ such that $B_k \to B$. For each k, choose $C_k \in L(A_{n_k}, t_{n_k})$ such that $B_k \subset C_k$ or $C_k \subset B_k$. If $B_k \subset C_k$ for infinitely many k, then $B_k \in \mathcal{A}_{n_k} \wedge L(A_{n_k}, t_{n_k})$ for infinitely many k. Thus $B \in \mathcal{A}$. Suppose then that $C_k \subset B_k$ for every k. Let $C \in C(X)$ be the limit of some subsequence of $(C_k)_k$. Then $C \in \limsup L(A_n, t_n)$. Thus $A_0 \subset A \subset C \subset B$. If $B = A_0$, then $B \in \mathcal{A}$. Suppose then that $A_0 \neq B$.

Choose a point $x_0 \in B - A_0$. Since $\mathcal{A} \in N(X)$, there exists a Whitney map $\nu : 2^X \to I$ and there exists $s \in I$ such that $(\nu \mid C(X))^{-1}(s) = \mathcal{A}$ (see [11]). Choose $r \in I$ such that $s < r < \nu(A_0 \cup \{x_0\})$. Take a sequence $(x_k)_k$ such that $x_k \in B_k$ for all k and $x_k \to x_0$. Since $\nu(B) \geq \nu(A_0 \cup \{x_0\}) > r$, there exists $K \in \mathbb{N}$ such that $\nu(B_k) > r$ for every $k \geq K$.

Given $k \geq K$, choose a l.o.a. γ_k in $C(X)$ such that $\{x_k\}$, $B_k \in \gamma_k$. Take $D_k \in \gamma_k \cap \nu^{-1}(r)$ and $E_k \in \gamma_k \cap L(A_{n_k}, t_{n_k})$. Let $(D_{k_l})_l$ and $(E_{k_l})_l$ be subsequences of $(D_k)_k$ and $(E_k)_k$ respectively which converge to elements D and E respectively. Then $x_0 \in D \cap E$ and $\nu(D) = r$. Since $E \subset \limsup L(A_n, t_n)$, it follows that $A_0 \subset E$. If $E \subset D$, we have $\nu(D) \geq \nu(A_0 \cup \{x_0\}) > r$. This contradiction proves that E is not contained in D. Since $D_{k_l} \subset E_{k_l}$ or $E_{k_l} \subset D_{k_l}$ for every l, we have $D \subsetneq E$. So $\nu(E) > r$. Thus there exists $L \in \mathbb{N}$ such that $\nu(E_{k_l}), \nu(B_{k_l}) > r$ for all $l \geq L$. Then $\nu(E_{k_l} \cap B_{k_l}) \geq r$ for all $l \geq L$. Hence $\nu(E \cap B) \geq r$. But

$$E \cap B \in \limsup \mathcal{A}_n \wedge L(A_n, t_n) = \mathcal{A} = (\nu \mid C(X))^{-1}(s)$$

and $s < r$. This contradiction proves that $B \in \mathcal{A}$.

Therefore $\limsup \mathcal{A}_n \subset \mathcal{A}$. Hence $\mathcal{A}_n \to \mathcal{A}$.

LEMMA 3.7. *If $\mathcal{A} \in N(X)^*$ and $\alpha > 0$, then there exists $\varepsilon \in (0, 1]$ such that $H^*(\mathcal{A} \wedge L(\phi(\mathcal{A}), \varepsilon), \mathcal{A}) < \alpha$.*

Proof. Let $A = \phi(\mathcal{A})$. It is enough to prove that $\mathcal{A} \wedge L(A, 1/n) \to \mathcal{A}$. Let $B \in \mathcal{A} - \{A\}$. Choose a l.o.a. γ in $C(X)$ such that $\mathcal{B} \in \gamma$. For each n, let $B_n \in \gamma \cap L(A, 1/n)$. Since A is not contained in B, it follows that $0 < \omega(B \cup A) - \omega(B) = \limsup \psi(A, 1/n, B)$. Thus there exists $N \in \mathbb{N}$ such that $0 < \psi(A, 1/n, B)$ for every $n \geq N$. Since $\psi(A, 1/n, B_n) = 0$, we obtain $B \subset B_n$ for every $n \geq N$. So $B \in \mathcal{A} \wedge L(A, 1/n)$ for all $n \geq N$. Hence $B \in \liminf \mathcal{A} \wedge L(A, 1/n)$. Therefore $\mathcal{A} \subset \liminf \mathcal{A} \wedge L(A, 1/n)$.

Now take $B \in \limsup \mathcal{A} \wedge L(A, 1/n)$. Then there exists a sequence $n_1 < n_2 < \ldots$ and elements $B_k \in \mathcal{A} \wedge L(A, 1/n_k)$ such that $B_k \to B$. Then each $B_k = A_k \cap C_k$ where $A_k \in \mathcal{A}$, $C_k \in L(A, 1/n_k)$ and $A_k \subset C_k$ or $C_k \subset A_k$. If $B_k = A_k$ for infinitely many k, then $B \in \mathcal{A}$. Suppose then that $B_k = C_k \subset A_k$ for every k. Then $0 = \psi(A, 1/n_k, B_k) \to \omega(A \cup B) - \omega(B)$.

Thus $A \subset B$. Let $(A_{k_m})_m$ be a subsequence of $(A_k)_k$ which converges to an $A_0 \in \mathcal{A}$. Then $A \subset B \subset A_0$. Hence $A = B = A_0$, so $B \in \mathcal{A}$. Thus $\limsup \mathcal{A} \wedge L(A, 1/n) \subset \mathcal{A}$.

Therefore $\mathcal{A} \wedge L(A, 1/n) \to \mathcal{A}$.

LEMMA 3.8. *Let* $\alpha : N(X) \to (0, \infty)$ *be a map. Then*:

(a) *There exists a map* $\varepsilon : N(X)^* \to (0, 1]$ *such that*

$$H^*(\mathcal{A} \wedge L(\phi(A), \varepsilon(\mathcal{A})), \mathcal{A}) < \alpha(\mathcal{A})$$

for every $\mathcal{A} \in N(X)^*$.

(b) *There exist maps* $\varepsilon, h : N(X)^* \to (0, \infty)$ *such that*, *for each* $\mathcal{A} \in N(X)^*$, $\varepsilon(\mathcal{A}) \le 1$, $h(\mathcal{A}) \le \omega(\phi(\mathcal{A}))/2$ *and*

$$H^*(\mathcal{A}, \mathcal{A} \wedge L(\sigma[\phi(\mathcal{A}) - h(\mathcal{A})],\ \varepsilon(\mathcal{A}))) < \alpha(\mathcal{A})\,.$$

(c) *There exists a map* $k : N(X) \to (0, 1/2]$ *such that*, *for every* $\mathcal{A} \in N(X)$,

$$H^*(\mathcal{A}, \mathcal{A} \vee \omega^{-1}(k(\mathcal{A})) < \alpha(\mathcal{A})$$

and

$$H^*(\mathcal{A}, \mathcal{A} \wedge \omega^{-1}(1 - k(\mathcal{A}))) < \alpha(\mathcal{A})\,.$$

(d) *If* $\alpha_0 : N(X) \to (0, \infty)$ *is a map*, *then there exists a map* $\delta : N(X) \to (0, \infty)$ *such that* $H^*(\mathcal{A}, \mathcal{B}) < \delta(\mathcal{A})$ *implies that* $|\alpha(\mathcal{A}) - \alpha(\mathcal{B})| < \alpha_0(\mathcal{A})$.

Proof. (a) Let $\varepsilon_0 : N(X)^* \to (0, \infty)$ be given by

$$\varepsilon_0(\mathcal{A}) = \sup\{t \in (0, 1] : H^*(\mathcal{A}, \mathcal{A} \wedge L(\phi(\mathcal{A}), t)) < \alpha(\mathcal{A})\}\,.$$

By Lemma 3.7, ε_0 is well defined. Let $t \in (0, 1]$ be such that $H^*(\mathcal{A}, \mathcal{A} \wedge L(\phi(\mathcal{A}), t)) < \alpha(\mathcal{A})$ and let $(\mathcal{A}_n)_n$ be a sequence such that $\mathcal{A}_n \to \mathcal{A}$. Then $H^*(\mathcal{A}_n, \mathcal{A}_n \wedge L(\phi(\mathcal{A}_n), t)) \to H^*(\mathcal{A}, \mathcal{A} \wedge L(\phi(\mathcal{A}), t))$ and $\alpha(\mathcal{A}_n) \to \alpha(\mathcal{A})$. It follows that ε_0 is a lower semi-continuous positive function. Then (see [4, Ch. VIII, 4.3]) there exists a map $\varepsilon : N(X)^* \to (0, \infty)$ such that $0 < \varepsilon(\mathcal{A}) < \varepsilon_0(\mathcal{A})$ for every $\mathcal{A} \in N(X)^*$.

(b) By (a) there exists a map $\varepsilon : N(X)^* \to (0, 1]$ such that

$$H^*(\mathcal{A}, \mathcal{A} \wedge L(\phi(\mathcal{A}), \varepsilon(\mathcal{A}))) < \alpha(\mathcal{A})/2$$

for every $\mathcal{A} \in N(X)^*$. Let $h_0 : N(X)^* \to (0, 1]$ be given by

$$h_0(\mathcal{A}) = \sup\{t \in (0, \omega(\phi(\mathcal{A}))/2] : H^*(\mathcal{A} \wedge L(\sigma[\omega(\phi(\mathcal{A})) - t], \varepsilon(\mathcal{A})), \mathcal{A}) < \alpha(\mathcal{A})\}\,.$$

Then h_0 is a positive lower semi-continuous function, so there exists a map $h : N(X)^* \to (0, 1]$ such that $0 < h(\mathcal{A}) < h_0(\mathcal{A})$ for every $\mathcal{A} \in N(X)^*$.

The proof of (c) is similar. Claim (d) was proved in [8, Lemma 1.13].

P r o o f o f t h e M a i n T h e o r e m. We will use Toruńczyk's characterization of the Hilbert space l_2 ([10, p. 248]): Let Y be a complete separable AR space. Then Y is homeomorphic to l_2 if and only if given a map $f : \mathbb{N} \times Q \to Y$ (Q denotes the Hilbert cube) and a map $\alpha : Y \to (0, \infty)$, there is a map $g : \mathbb{N} \times Q \to Y$ with $\{g(\{n\} \times Q)\}_{n\in\mathbb{N}}$ discrete in Y and $d_Y(f(z), g(z)) < \alpha(f(z))$ for every $z \in \mathbb{N} \times Q$.

Take maps $f : \mathbb{N} \times Q \to N(X)$ and $\alpha : N(X) \to (0, \infty)$. Lemma 3.8 implies that:

(a) There exists a map $\delta : N(X) \to (0, \infty)$ such that $H^*(\mathcal{A}, \mathcal{B}) < \delta(\mathcal{A})$ implies that $|\alpha(\mathcal{A}) - \alpha(\mathcal{B})| < \alpha(\mathcal{A})/2$.

(b) There exists a map $k : N(X) \to (0, 1/2]$ such that $H^*(\mathcal{A}, \mathcal{A} \vee \omega^{-1}(k(\mathcal{A})))$ and

$$H^*(\mathcal{A},\, \mathcal{A} \wedge \omega^{-1}(1 - k(\mathcal{A}))) < \alpha(\mathcal{A})/4,\ \delta(\mathcal{A})$$

for every $\mathcal{A} \in N(X)$.

(c) There exist maps $\varepsilon, h : N(X)^* \to (0, \infty)$ such that, for each $\mathcal{A} \in N(X)^*$, $h(\mathcal{A}) \le \omega(\phi(\mathcal{A}))/2$, $\varepsilon(\mathcal{A}) \le 1$ and

$$H^*(\mathcal{A} \wedge L(\sigma[\omega(\phi(\mathcal{A})) - h(\mathcal{A})], \varepsilon(\mathcal{A})), \mathcal{A}) < \alpha(\mathcal{A})/8\,.$$

Define $G_1, G_2 : N(X) \to N(X)$ by $G_1(\mathcal{A}) = \mathcal{A} \vee \omega^{-1}(k(\mathcal{A}))$ and $G_2(\mathcal{A}) = \mathcal{A} \wedge \omega^{-1}(1 - k(\mathcal{A}))$. Then G_1, G_2 are continuous and $G_2(G_1(\mathcal{A})) \in N(X)^*$ for each $\mathcal{A} \in N(X)$. Given $\mathcal{A} \in N(X)$ with $|\alpha(\mathcal{A}) - \alpha(G_i(\mathcal{A}))| < \alpha(\mathcal{A})/2$, then $\alpha(G_i(\mathcal{A})) < (3/2)\alpha(\mathcal{A})$ for $i = 1, 2$. Then $\alpha(G_2(G_1(\mathcal{A}))) < (9/4)\alpha(\mathcal{A})$. Furthermore,

$$\begin{aligned} H^*(\mathcal{A}, G_2(G_1(\mathcal{A}))) &\le H^*(\mathcal{A}, G_1(\mathcal{A})) + H^*(G_1(\mathcal{A}), G_2(G_1(\mathcal{A}))) \\ &< \alpha(\mathcal{A})/4 + \alpha(G_1(\mathcal{A}))/4 < (5/8)\alpha(\mathcal{A})\,. \end{aligned}$$

Define $f_0 = G_2 \circ G_1 \circ f$. Let $t_1 = \min(\varepsilon(f_0(\{1\} \times Q)) \cup \{1/2\})$ and, for $n \ge 2$, let $t_n = \min(\varepsilon(f_0(\{n\} \times Q)) \cup \{t_{n-1}/2\})$. Then $t_n \to 0$ and $0 < t_{n+1} < t_n/2 < t_n < 1$ for every n.

For each $n \in \mathbb{N}$, define $g_n : N(X)^* \to N(X)$ by $g_n(\mathcal{A}) = \mathcal{A} \wedge L(\sigma[\omega(\phi(\mathcal{A})) - h(\mathcal{A})], t_n)$, and define $g : \mathbb{N} \times Q \to N(X)$ by $g(n, x) = g_n(f_0(n, x))$. Then g is continuous.

Let $y = (n, x) \in \mathbb{N} \times Q$. Since $t_n \le \varepsilon(f_0(y))$, we have

$$\begin{aligned} &f_0(y) \wedge L(\sigma[\phi(f_0(y)) - h(f_0(y))], \varepsilon(f_0(y))) \\ &\qquad \le f_0(y) \wedge L(\sigma[\phi(f_0(y)) - h(f_0(y))], t_n) \le f_0(y)\,. \end{aligned}$$

Then $H^*(f_0(y), g_n(f_0(y))) < \alpha(f_0(y))/8 < (9/32)\alpha(f(y))$. Thus

$$\begin{aligned} H^*(f(y), g(y)) &\le H^*(f(y), f_0(y)) + H^*(f_0(y), g(y)) \\ &< (5/8)\alpha(f(y)) + (9/32)\alpha(f(y)) < \alpha(f(y))\,. \end{aligned}$$

Therefore $H^*(f(y), g(y)) < \alpha(f(y))$.

Notice that Lemma 3.5 implies that the sets $g(\{1\} \times Q)$, $g(\{2\} \times Q), \dots$ are pairwise disjoint.

Now we prove that $F_1(X), \{X\} \notin \mathrm{Cl}_{N(X)}\, G_2(G_1(N(X)))$. Suppose that there exists a sequence $(C_n)_n$ in $N(X)$ such that $G_2(G_1(C_n)) \to F_1(X)$. Then

$$(C_n \vee \omega^{-1}(k(C_n)) \wedge \omega^{-1}(1 - k(C_n \vee \omega^{-1}(k(C_n)))) \to F_1(X)\,.$$

Since $\omega^{-1}(1 - k(C_n \vee \omega^{-1}(k(C_n)))) \geq \omega^{-1}(1/2)$ for each n, we then have $C_n \vee \omega^{-1}(k(C_n)) \to F_1(X)$. Thus C_n and $\omega^{-1}(k(C_n)) \to F_1(X)$. Hence $F_1(X) = \omega^{-1}(k(F_1(X)))$. Thus $k(F_1(X)) = 0$. This contradiction proves that $F_1(X) \notin \mathrm{Cl}_{N(X)}\, G_2(G_1(N(X)))$. Now suppose that there exists a sequence $(C_n)_n$ in $N(X)$ such that $G_2(G_1(C_n)) \to \{X\}$. Then $C_n \vee \omega^{-1}(k(C_n)) \to \{X\}$ and $\omega^{-1}(1 - k(C_n \vee \omega^{-1}(k(C_n)))) \to \{X\}$, so

$$\{X\} = \omega^{-1}(1 - k(\{X\} \vee \omega^{-1}(k(\{X\})))) = \omega^{-1}(1 - k(\{X\}))\,.$$

It follows that $k(\{X\}) = 0$. This contradiction proves that $\{X\} \notin \mathrm{Cl}_{N(X)}\, G_2(G_1(N(X)))$.

Finally, we prove that the family $\{g(\{n\} \times Q)\}_{n \in \mathbb{N}}$ is discrete in $N(X)$. Suppose that this is not true. Then there exists $\mathcal{A} \in N(X)$, a sequence $n_1 < n_2 < \dots$ and elements $\mathcal{B}_k \in g(\{n_k\} \times Q)$ such that $\mathcal{B}_k \to \mathcal{A}$. For each k, put $\mathcal{B}_k = g(n_k, x_k)$, let $\mathcal{A}_k = f_0(n_k, x_k)$ and $A_k = \sigma[\omega(\phi(\mathcal{A}_k)) - h(\mathcal{A}_k)] \in \beta^*$. Then $\mathcal{B}_k = \mathcal{A}_k \wedge L(A_k, t_{n_k})$. Suppose, by taking a subsequence if necessary, that $A_k \to A$ for some $A \in \beta$.

We will show that $A \neq X$. Suppose $A = X$. Since $A_k \subset \sigma(\omega(\phi(\mathcal{A}_k))) = \phi(\mathcal{A}_k)$, we have $A_k \in \mathcal{B}_k$. Now, $\mathcal{B}_k \to \mathcal{A}$ implies $\mathcal{A} = \{X\}$. Thus $\mathcal{A}_k \to \{X\}$ and $L(A_k, t_{n_k}) \to \{X\}$. This is a contradiction since $\{X\} \notin \mathrm{Cl}_{N(X)}\, G_2(G_1(N(X)))$. Therefore $A \in \beta - \{X\}$.

Applying Lemma 3.6 we see that $\mathcal{A}_k \to \mathcal{A}$. Since $\mathcal{A}_k \in G_2(G_1(N(X)))$, we have $\mathcal{A} \in N(X)^*$. Given k, $A_k = \sigma(\omega(\phi(A_k)) - h(A_k)) \subset \sigma(\omega(\phi(\mathcal{A}_k))) = \phi(\mathcal{A}_k) \in \mathcal{A}_k$. Then A_k is an element of $L(A_k, t_{n_k})$ contained in an element of $\mathcal{A}_k$. Thus $A_k \in \mathcal{A}_k \wedge L(A_k, t_{n_k}) = \mathcal{B}_k$. This implies that $A \in \mathcal{A}$. Thus $A \notin F_1(X) \cup \{X\}$. Since $A_k \to \sigma(\omega(\phi(\mathcal{A})) - h(\mathcal{A}))$, we get $A = \sigma(\omega(\phi(\mathcal{A})) - h(\mathcal{A}))$. But $A \in \mathcal{A} \cap \beta$ implies that $A = \phi(\mathcal{A})$. Thus $h(\mathcal{A}) = 0$. This contradiction proves that the family $\{g(\{n\} \times Q)\}_{n \in \mathbb{N}}$ is discrete and ends the proof of the theorem.

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