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INTEGRAL CLOSURES OF IDEALS IN THE REES RING

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Introduction. The important ideas of reduction and integral closure of an ideal in a commutative Noetherian ring A (with identity) were introduced by Northcott and Rees [4]; a brief and direct approach to their theory is given in [6, (1.1)]. We begin by briefly summarizing some of the main aspects.

Let a be an ideal of A. We say that a is a *reduction* of the ideal b of A if $a \subseteq b$ and there exists $s \in \mathbb{N}$ such that $ab^s = b^{s+1}$. (We use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers.) An element x of A is said to be *integrally dependent* on a if there exist $n \in \mathbb{N}$ and elements $c_1, \ldots, c_n \in A$ with $c_i \in a^i$ for $i = 1, \ldots, n$ such that

$$x^n + c_1 x^{n-1} + \ldots + c_{n-1}x + c_n = 0\,.$$

In fact, this is the case if and only if a is a reduction of $a + Ax$; moreover,

$$\overline{a} = \{y \in A : y \text{ is integrally dependent on } a\}$$

is an ideal of A, called the *classical integral closure* of a, and it is the largest ideal of A which has a as a reduction in the sense that a is a reduction of $\overline{a}$ and any ideal of A which has a as a reduction must be contained in $\overline{a}$.

In [8], Sharp, Tiraş and Yassi introduced concepts of reduction and integral closure of an ideal I of a commutative ring R (with identity) relative to a Noetherian module M, and they showed that these concepts have properties which reflect those of the classical concepts outlined in the last paragraph. Again, we provide a brief review.

We say that I is a *reduction* of the ideal J of R *relative to* M if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $IJ^sM = J^{s+1}M$. An element x of R is said to be *integrally dependent* on I *relative to* M if there exists $n \in \mathbb{N}$ such that

$$x^n \cdot M \subseteq \Big(\sum_{i=1}^{n} x^{n-i} I^i\Big) \cdot M\,.$$

In fact, this is the case if and only if I is a reduction of $I + Rx$ relative to M

[8, (1.5)(iv)]; moreover,

$$I^{-} = \{y \in R : y \text{ is integrally dependent on } I \text{ relative to } M\}$$

is an ideal of R, called the *integral closure* of I *relative to* M, and is the largest ideal of R which has I as a reduction relative to M. In this paper, we indicate the dependence of I^{-} on the Noetherian R-module M by means of the extended notation $I^{-(M)}$.

Now we give the definition of the Rees ring. The classical reference is [5, p. 33]. Let R be a commutative ring with identity.

Let t be an indeterminate. Let $S = \{t^i : i \in \mathbb{N}_0\}$. Then S is a multiplicatively closed subset of $R[t]$. So we get the ring $S^{-1}(R[t])$. The homomorphism

$$\psi : R[t] \to S^{-1}(R[t]), \quad f \mapsto f/1,$$

is an injective ring homomorphism, and so we can consider $R[t]$ as a subring of $S^{-1}(R[t])$. Put $\frac{1}{t} = t^{-1}$. Then $R[t][t^{-1}] = S^{-1}(R[t])$ or $R[t, t^{-1}] = S^{-1}(R[t])$. Next, suppose that I is a proper ideal of R generated by $a_1, \ldots, a_n$ ($n \in \mathbb{N}$). Then $\mathcal{R} = R[a_1 t, \ldots, a_n t, t^{-1}] = R[It, t^{-1}]$ is a subring of $R[t, t^{-1}]$. $\mathcal{R}$ is called the *Rees ring* of R with respect to I (see [2, p. 120]). Note that each element of $\mathcal{R}$ is of the form $\sum_{i=m}^{n} b_i t^i$ where $m, n \in \mathbb{Z}$ (the set of integers), and, for $i > 0$, $b_i \in I^i$. Note also that for $i \le 0$ we interpret I^i as R.

Now we give another definition which will be helpful in this section.

(1.1) Definition. Let $(R_n)_{n \in \mathbb{Z}}$ be a family of subgroups of R. We say that R is a *graded ring* if the following conditions are satisfied.

(i) R is the direct sum of the subgroups R_n, i.e. $R = \sum_{n=-\infty}^{\infty} R_n$.
(ii) $R_q \cdot R_{q'} \subseteq R_{q+q'}$ for all $q, q' \in \mathbb{Z}$. (Observe that $R_q \cdot R_{q'}$ is the set of all elements x of R such that x is a sum of a finite number of elements of the form $a \cdot b$ with $a \in R_q$, $b \in R_{q'}$.)

The following proposition comes from [3].

(1.2) Proposition [3, Proposition 28, p. 115]. *Let A be a graded ring. If K is a submodule of the graded A-module $E = \sum_{i \in \mathbb{Z}} E^i$, then the following statements are equivalent*:

(a) $K = \sum_{i \in \mathbb{Z}} (E^i \cap K)$;
(b) *If $y \in K$, then all the homogeneous components of y belong to K*;
(c) *K can be generated by homogeneous elements.*

Next we give the notations and terminology which we will need throughout this paper.

(1.3) *Notations and terminology*. Let R be a commutative Noetherian ring and I be an ideal of R generated by $a_1, \ldots, a_s$, $I = (a_1, \ldots, a_s)$. Let

$\mathcal{R} = R[It, t^{-1}]$ be the Rees ring of R with respect to I. Let $\mathcal{R} = \bigoplus_{n\in\mathbb{Z}} R_n$ where R_n denotes the subgroup of $\mathcal{R}$ consisting of 0 and the homogeneous elements of $\mathcal{R}$ of degree n. For all $k \in \mathbb{N}$, by (1.2)(a),

$$\mathcal{R}t^{-k} = \bigoplus_{i\in\mathbb{Z}} (R_i \cap Rt^{-k}) .$$

Therefore

$$R_i \cap Rt^{-k} = \begin{cases} I^{i+k}t^i & \text{if } i > -k, \\ Rt^i & \text{if } i \le -k. \end{cases}$$

Let M be a finitely generated R-module. Then it is easy to see that $M[t] = \sum_{i=1}^{r} R[t]u_i$ where $u_1, \ldots, u_r$ is a generating set for M.

Let $S = \{t^i : i \in \mathbb{N}_0\}$ be a multiplicatively closed subset of $R[t]$. Then

$$M[t] \to S^{-1}(M[t]) , \qquad f \mapsto f/1 ,$$

is an injective module homomorphism. Now let

$$\mathcal{R}(R, I) = \bigoplus_n I^n t^n = \Big\{ \sum_{i=-q}^{p} a_i t^i \in R[t, t^{-1}] : a_i \in I^i \Big\} .$$

Then $\mathcal{R}(R, I)$ is a subring of $R[t, t^{-1}]$. Also let

$$\mathcal{R}(M, I) = \Big\{ \sum_{i=-r}^{s} m_i t^i \in M[t, t^{-1}] : m_i \in I^i M \Big\} .$$

We can regard $\mathcal{R}(M, I)$ as an $\mathcal{R}(R, I)$-module with the following scalar multiplication:

$$\mathcal{R}(R, I) \times \mathcal{R}(M, I) \to \mathcal{R}(M, I) ,$$
$$\Big(\sum_{i=-n}^{m} c_i t^i , \sum_{j=-q}^{p} m_j t^j \Big) \mapsto \sum_{i=-n}^{m} \sum_{j=-q}^{p} c_i m_j t^{i+j} ,$$

where $c_i m_j \in I^{i+j} M$.

Let $X_1, \ldots, X_s, X_{s+1}$ be indeterminates over R. Then $R[X_1, \ldots, X_{s+1}]$ is a Noetherian ring. It is readily seen that $M[X_1, \ldots, X_{s+1}]$ is a Noetherian $R[X_1, \ldots, X_{s+1}]$-module, and $M[a_1 t, \ldots, a_s t, t^{-1}]$ is a Noetherian $R[a_1 t, \ldots \ldots, a_s t, t^{-1}]$-module.

2. Some related results. Throughout this section, unless otherwise stated, R will denote a commutative ring with identity. Essentially our aim is to investigate some interrelations between the Rees ring of R with respect to an ideal of R and the ground ring R.

We begin with a well-known lemma which gives us the connection between the integral closure of an ideal in the Rees ring and the integral closure of the ideal in the ground ring R.

(2.1) Lemma. *Let R be a commutative Noetherian ring, I be an ideal of R, and $\mathcal{R}$ be the Rees ring of R with respect to I. Let t be an indeterminate and $u = t^{-1}$. Then*

$$(\overline{u^i\mathcal{R}}) \cap R = (\overline{I^i})$$

where the bar refers to classical integral closure. ■

From now on, let $\mathbb{M} = \mathcal{R}(M, I) = M[a_1t, \ldots, a_st, t^{-1}]$. Also, $(\mathcal{R}t^{-k})^{-(\mathbb{M})}$, for $k \in \mathbb{N}$, is the integral closure of $\mathcal{R}t^{-k}$ relative to $\mathbb{M}$.

Now for all $i > -k$, $k \in \mathbb{N}$, define

$$C_{i,k} = \{x \in R : xt^i \in R_i \cap (\mathcal{R}t^{-k})^{-(\mathbb{M})}\}.$$

It is clear that for all $i > -k$, $C_{i,k}$ is an ideal of R. In particular,

$$C_{0,k} = R \cap (\mathcal{R}t^{-k})^{-(\mathbb{M})}.$$

Now we give the relation between $(I^k)^{-(M)}$ and $(\mathcal{R}t^{-k})^{-(\mathbb{M})}$. The following theorem can be used to reduce problems about the integral closure of the powers of I relative to M to the corresponding problems for powers of the principal ideal $\mathcal{R}t^{-k}$ in $\mathcal{R}$.

(2.2) Theorem. *Let R be a commutative Noetherian ring and I be an ideal of R. Let $\mathcal{R}$ be the Rees ring of R with respect to I. Let M be a Noetherian R-module and $\mathbb{M} = \mathcal{R}(M, I)$. Then for $k \in \mathbb{N}$,*

$$(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cap R = (I^k)^{-(M)}.$$

Proof. Let $x \in (I^k)^{-(M)}$. Then there is an $n \in \mathbb{N}$ such that

$$x^n \cdot M \subseteq \Big(\sum_{i=1}^{n} x^{n-i} I^i\Big) \cdot M.$$

It is enough to show that an element of the form $x^n m_j t^j$, where $m_j \in I^j M$, is in $(\sum_{i=1}^n x^{n-i}(\mathcal{R}t^{-k})^i)\mathbb{M}$. Since

$$x^n m_j \in x^n I^j M \subseteq \sum_{i=1}^{n} x^{n-i} I^j (I^k)^i M,$$

we have $x^n m_j = \sum_{i=1}^n x^{n-i}\beta_i$ where $\beta_i \in I^j(I^k)^i M$, and the result follows.

For the converse, let us first give some useful ideas about the ideal $C_{i,k}$ we have just defined. It is easy to see that $I \cdot C_{i,k} \subseteq C_{i+1,k}$ for all $i \geq 1$ and $C_{i+1,k} \subseteq C_{i,k}$ for all $i \geq 0$.

Also $I^i \subseteq C_{i-k,k} \subseteq I^{i-k}$ for $i - k > -k$ ($k \in \mathbb{N}$). Indeed, if $x \in I^i$, then $xt^{-k+i} \in R_{-k+i}$. Therefore $xt^{-k+i} \in R_{i-k} \cap (Rt^{-k})^{-(\mathbb{M})}$. Since the second inclusion is clear, we omit its proof.

Now to complete the proof we show that I^i is a reduction of $C_{i-k,k}$ relative to M. It is enough to show that each element of $C_{i-k,k}$ is integrally dependent on I^i relative to M by the preceding paragraph and [8, (1.5)(v)].

Let $x \in C_{i-k,k}$. Then $xt^{i-k} \in (\mathcal{R}t^{-k})^{-(\mathbb{M})}$. Thus there exists an $n \in \mathbb{N}$ such that

$$(*) \qquad \mathcal{R}(xt^{i-k})^n \cdot \mathbb{M} \subseteq \Big(\sum_{r=1}^{n} (\mathcal{R}t^{-k})^r (\mathcal{R}xt^{i-k})^{n-r}\Big) \cdot \mathbb{M}\,.$$

We claim that

$$x^n \cdot M \subseteq \Big(\sum_{r=1}^{n} x^{n-r} (I^k)^r\Big) \cdot \mathbb{M}\,.$$

Let $y \in x^n \cdot M$. Then $y = x^n \cdot m$ for some $m \in M$. Hence $(xt^{i-k})^n \cdot m \in \mathcal{R}(xt^{i-k})^n \cdot \mathbb{M}$. By $(*)$,

$$(xt^{i-k})^n \cdot m \in \Big(\sum_{r=1}^{n} (\mathcal{R}t^{-k})^r \mathcal{R}(xt^{i-k})^{n-r}\Big) \cdot M\,.$$

Therefore

$$x^n t^{n(i-k)} m = \sum_{r=1}^{n} x^{n-r} t^{(i-k)(n-r)-kr} \gamma_r \quad \text{ with } \gamma_r \in \mathbb{M}\,.$$

By comparing components of degree $n(i-k)$, we get

$$x^n \cdot m \in \Big(\sum_{r=1}^{n} x^{n-r} (I^i)^r\Big) \cdot M\,.$$

This means x is integrally dependent on I^i relative to M. Then by [8, (1.5)(v)], I^i is a reduction of $C_{i-k,k}$ relative to M for all $i \ge 1$. Now the result follows from [8, (1.5)(vii)]. ∎

One could naturally ask whether there exist any relations, as in (2.2), between $(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M}$ and $(I^k)^{-(M)} \cdot M$. It will be shown in (2.5) that the answer is yes, and to prove this we need to show first that the integral closure of a homogeneous ideal in a graded ring is homogeneous.

(2.3) Proposition. *Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded Noetherian ring and let I be a homogeneous ideal in R. Then $\bar{I}$, the integral closure of I in R, is a homogeneous ideal of R.*

Proof. Let $T = R[t, t^{-1}]$. Consider

$$T_n = \Big\{ \sum_{i=-p}^{q} r_{ni} t^i \in T : r_{ni} \in R_n \Big\}, \qquad n \in \mathbb{Z}\,.$$

For $n \in \mathbb{Z}$, T_n is an additive subgroup of T. Also $T_n \cdot T_m \subseteq T_{m+n}$. Let $\mathcal{R} = R[It, t^{-1}]$. Then $\mathcal{R} = \bigoplus_{n\in\mathbb{Z}} (R[It, t^{-1}])_n$ is a graded subring of $R[t, t^{-1}]$.

Now let $x = \sum_{i=-p}^{q} x_i \in \bar{I}$. Then by [6, (1.1)(ii)], xt is integral over $R[It, t^{-1}]$. By [1, Proposition 20, p. 321] all homogeneous components of xt are integral over $R[It, t^{-1}]$. This completes the proof. ∎

(2.4) Corollary. *Let* $R = \bigoplus_{n\in\mathbb{Z}} R_n$ *be a graded ring and let* I *be a homogeneous ideal of* R. *Suppose that* M *is a Noetherian graded* R*-module. Then* $I^{-(M)}$ *is a homogeneous ideal of* R.

Proof. Let the bar refer to the natural ring homomorphism $R \to R/0 :_R M$. By [8, (1.6)], $\overline{I^{-(M)}} = (\bar{I})^{-(\bar{R})}$, the integral closure $\bar{I}$ in $\overline{R}$. By (2.3), $\overline{I^{-(M)}}$ is a homogeneous ideal. Now the result follows from the definition of the graded ring structure on the residue class ring. ∎

Now we are able to give an answer to the question asked just after (2.2).

(2.5) Theorem. *Let* $\mathcal{R}$ *and* $\mathbb{M}$ *be as in* (2.2). *Then for all* $k \in \mathbb{N}$,

$$(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M} \cap M = (I^k)^{-(M)} \cdot M .$$

Proof. By the result about $C_{i,k}$ given in (2.2), the zero component of $(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M}$ is $C_{0,k} \cdot M$. This gives us $(I^k)^{-(M)} \cdot M = C_{0,k} \cdot M \subseteq (\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M} \cap M$.

Let $m \in (\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M} \cap M$. Since m is a homogeneous element of $(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M}$ of degree 0, it belongs to $C_{0,k} \cdot M$. This completes the proof. ∎

We conclude this paper by giving the interrelation between the associated primes in $\mathcal{R}$ and in R. To do this we need the following proposition.

(2.6) Proposition [3, Proposition 20, p. 99]. *Let* N *be a* p*-primary submodule of an* R*-module* E *and let* K *be an arbitrary submodule of* E. *If* $K \not\subseteq N$, *then* $(N : K)$ *is a* p*-primary ideal. If* $K \subseteq N$, *then* $(N : K) = R$. ∎

(2.7) Proposition. *Let* $\mathcal{R}$ *and* $\mathbb{M}$ *be as in* (2.5). *Let*

$$p \in \mathrm{Ass}_R\left(\frac{M}{(I^k)^{-(M)} \cdot M}\right)$$

for $k \in \mathbb{N}$. *Then there exists*

$$\mathcal{P} \in \mathrm{Ass}_{\mathcal{R}}\left(\frac{\mathbb{M}}{(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M}}\right)$$

such that $\mathcal{P} \cap R = p$.

Proof. Let

$$\mathcal{G} = \frac{\mathbb{M}}{(\mathcal{R}t^{-k})^{-(\mathbb{M})} \cdot \mathbb{M}} = \bigoplus_{n\in\mathbb{Z}} G_n .$$

We have shown that

$$G_0 = \frac{M}{(I^k)^{-(M)} \cdot M} .$$

Let $p \in \mathrm{Ass}_R G_0$. Then there exists $g_0 \in G_0$ such that $(0 :_R g_0) = p$. Now consider $\mathcal{R}g_0$, a homogeneous submodule of $\mathcal{G}$. Take a minimal primary decomposition for 0 in $\mathcal{R}g_0$ (because $\mathcal{R}$ is Noetherian). Then $0 = \bigcap_{i=1}^{n} \alpha_i$, with α_i being $\mathcal{P}_i$-primary homogeneous submodules of $\mathcal{R}g_0$ $(1 \leq i \leq n)$. Then

$$p = (0 :_R g_0) = R \cap (0 :_{\mathcal{R}} \mathcal{R}g_0) = \bigcap_{\substack{i=1 \\ g_0 \notin \alpha_i}}^{n} (R \cap (\alpha_i :_{\mathcal{R}} \mathcal{R}g_0)) .$$

Thus by (2.6), $(\alpha_i :_{\mathcal{R}} \mathcal{R}g_0)$ is a $\mathcal{P}_i$-primary ideal and by [7, (9.33)(ii)], $\mathcal{P}_i \in \mathrm{Ass}_{\mathcal{R}}\, g_0 \subseteq \mathrm{Ass}_{\mathcal{R}}\, \mathcal{G}$. Now the result follows by [7, 3.50]. ∎

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