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CONTACT CR-SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF A SASAKIAN SPACE FORM

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Introduction. The purpose of this paper is to study contact CR-submanifolds with nonvanishing parallel mean curvature vector immersed in a Sasakian space form.

In §1 we state general formulas on contact CR-submanifolds of a Sasakian manifold, especially those of a Sasakian space form. §2 is devoted to the study of contact CR-submanifolds with nonvanishing parallel mean curvature vector and parallel f-structure in the normal bundle immersed in a Sasakian space form. Moreover, we suppose that the second fundamental form of a contact CR-submanifold commutes with the f-structure in the tangent bundle, and compute the restricted Laplacian for the second fundamental form in the direction of the mean curvature vector. As applications of this, in §3, we prove our main theorems.

1. Preliminaries. Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure tensors (φ,ξ,η,g). The structure tensors of $\widetilde{M}$ satisfy

$$\varphi^2X=-X+\eta(X)\xi\,,\quad \varphi\xi=0\,,\quad \eta(\xi)=1\,,\quad \eta(\varphi X)=0\,,$$
$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y)\,,\qquad \eta(X)=g(X,\xi)$$

for any vector fields X and Y on $\widetilde{M}$. We denote by $\widetilde{\nabla}$ the operator of covariant differentiation with respect to the metric g on $\widetilde{M}$. We then have

$$\widetilde{\nabla}_X\xi=\varphi X\,,\quad (\widetilde{\nabla}_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X=\widetilde{R}(X,\xi)Y\,,$$

$\widetilde{R}$ denoting the Riemannian curvature tensor of $\widetilde{M}$.

Let M be an $(n+1)$-dimensional submanifold of $\widetilde{M}$. *Throughout this paper, we assume that the submanifold M of $\widetilde{M}$ is tangent to the structure vector field ξ.*

We denote by the same g the Riemannian metric tensor field induced on M from that of $\widetilde{M}$. The operator of covariant differentiation with respect to the induced connection on M will be denoted by ∇. Then the Gauss and

Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^\perp$ of M. A and B appearing here are both called the *second fundamental forms* of M and are related by

$$g(B(X,Y),V) = g(A_V X, Y)\,.$$

The second fundamental form A_V in the direction of the normal vector V can be considered as a symmetric $(n+1,n+1)$-matrix.

The covariant derivative $\nabla_X A$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V \nabla_X Y\,.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of* V. If the second fundamental form is parallel in any direction, it is said to be *parallel.*

The *mean curvature vector* ν of M is defined to be $\nu = (\operatorname{Tr} B)/(n+1)$, where $\operatorname{Tr} B$ denotes the trace of B. If $\nu = 0$, then M is said to be *minimal*. If the second fundamental form A vanishes identically, then M is said to be *totally geodesic*. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M. A parallel normal vector field V ($\neq 0$) is called an *isoperimetric section* if $\operatorname{Tr} A_V$ is constant, and is called a *minimal section* if $\operatorname{Tr} A_V$ is zero.

For any vector field X tangent to M, we put

$$\varphi X = PX + FX\,,$$

where PX is the tangential part and FX the normal part of φX. Then P is an endomorphism of the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. Similarly, for any vector field V normal to M, we put

$$\varphi V = tV + fV\,,$$

where tV is the tangential part and fV the normal part of φV. We then have

$$g(PX,Y) + g(X,PY) = 0\,, \qquad g(fV,U) + g(V,fU) - 0\,,$$
$$g(FX,V) + g(X,tV) = 0\,.$$

Moreover,

$$P^2 = -I - tF + \eta \otimes \xi\,, \qquad FP + fF = 0\,,$$
$$Pt + tf = 0\,, \qquad f^2 = -I - Ft\,.$$

We define the covariant derivatives of P, F, t and f by

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y\,, \qquad (\nabla_X F)Y = D_X(FY) - F\nabla_X Y\,,$$
$$(\nabla_X t)V = \nabla_X(tV) - tD_X V\,, \qquad (\nabla_X f)V = D_X(fV) - fD_X V\,,$$

respectively.

For any vector field X tangent to M, we have

$$\widetilde{\nabla}_X \xi = \varphi X = \nabla_X \xi + B(X,\xi)\,,$$

and hence

$$\nabla_X \xi = PX\,, \tag{1.1}$$

$$A_V \xi = -tV\,, \qquad B(X,\xi) = FX\,. \tag{1.2}$$

Furthermore,

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y) - g(X,Y)\xi + \eta(Y)X\,, \tag{1.3}$$

$$(\nabla_X F)Y = -B(X,PY) + fB(X,Y)\,, \tag{1.4}$$

$$(\nabla_X t)V = A_{fV}X - PA_V X\,, \tag{1.5}$$

$$(\nabla_X f)V = -FA_V X - B(X,tV)\,. \tag{1.6}$$

A submanifold M of a Sasakian manifold $\widetilde{M}$ tangent to the structure vector field ξ is called a *contact CR-submanifold* of $\widetilde{M}$ if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M satisfying the following conditions (see [6]–[8]):

(1) H is invariant with respect to φ, i.e. $\varphi H_x \subset H_x$ for each x in M, and

(2) the complementary orthogonal distribution $H^\perp : x \to H_x^\perp \subset T_x(M)$ is anti-invariant with respect to φ, i.e. $\varphi H_x^\perp \subset T_x(M)^\perp$ for each x in M.

For a contact CR-submanifold M, the structure vector field ξ satisfies $\xi \in H$ or $\xi \in H^\perp$.

We put $\dim H = h$, $\dim H^\perp = p$ and $\operatorname{codim} M = 2m - n = q$. If $p = 0$, then a contact CR-submanifold M is called an *invariant submanifold* of $\widetilde{M}$, and if $h = 0$, then M is called an *anti-invariant submanifold* of $\widetilde{M}$ tangent to ξ. If $p = q$ and $\xi \in H$, then a contact CR-submanifold M is called a *generic submanifold* of $\widetilde{M}$ (see [2], [3], [5]).

In the following, we suppose that M is a contact CR-submanifold of a Sasakian manifold $\widetilde{M}$. Then

$$FP = 0\,, \quad fF = 0\,, \quad tf = 0\,, \quad Pt = 0\,, \tag{1.7}$$

$$P^3 + P = 0\,, \qquad f^3 + f = 0\,. \tag{1.8}$$

The equations in (1.8) show that P is an f-structure in M and f is an f-structure in the normal bundle of M (see [4]). From (1.3) we obtain

$$A_{FX}Y - A_{FY}X = \eta(Y)X - \eta(X)Y \qquad \text{for } X, Y \in H^\perp\,. \tag{1.9}$$

We denote by $\widetilde{M}^{2m+1}(c)$ a $(2m+1)$-dimensional Sasakian space form of constant φ-sectional curvature c. Then the Gauss and Codazzi equations of M are respectively

$$\begin{aligned}(1.10)\quad R(X,Y)Z &= \tfrac{1}{4}(c+3)[g(Y,Z)X - g(X,Z)Y] \\ &+ \tfrac{1}{4}(c-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + g(PY,Z)PX - g(PX,Z)PY + 2g(X,PY)PZ] \\ &+ A_{B(Y,Z)}X - A_{B(X,Z)}Y\,,\end{aligned}$$

where R is the Riemannian curvature tensor of M, and

$$\begin{aligned}(1.11)\quad & g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) \\ &= g((\nabla_X B)(Y,Z), V) - g((\nabla_Y B)(X,Z), V) \\ &= \tfrac{1}{4}(c-1)[g(PY,Z)g(FX,V) - g(PX,Z)g(FY,V) \\ &\quad + 2g(X,PY)g(FZ,V)]\,.\end{aligned}$$

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V\,.$$

Then we have the Ricci equation

$$\begin{aligned}(1.12)\quad & g(R^{\perp}(X,Y)V,U) + g([A_U,A_V]X,Y) \\ &= \tfrac{1}{4}(c-1)[g(FY,V)g(FX,U) - g(FX,V)g(FY,U) \\ &\quad + 2g(X,PY)g(fV,U)]\,.\end{aligned}$$

2. Parallel mean curvature vector. In this section we prepare some lemmas for later use.

Let M be an $(n+1)$-dimensional contact CR-submanifold of a $(2m+1)$-dimensional Sasakian manifold $\widetilde{M}$. We have the following decomposition of the tangent space $T_x(M)$ at each point x in M:

$$T_x(M) = H_x(M) + \{\xi\} + N_x(M)\,,$$

where $H_x(M) = \varphi H_x(M)$ and $N_x(M)$ is the orthogonal complement of $H_x(M) + \{\xi\}$ in $T_x(M)$. Then $\varphi N_x(M) = FN_x(M) \subset T_x(M)^{\perp}$. Similarly,

$$T_x(M)^{\perp} = FN_x(M) + N_x(M)^{\perp}\,,$$

where $N_x(M)^{\perp}$ is the orthogonal complement of $FN_x(M)$ in $T_x(M)^{\perp}$. Then $\varphi N_x(M)^{\perp} = fN_x(M)^{\perp} = N_x(M)^{\perp}$.

We take an orthonormal basis $e_1,\ldots,e_{2m+1}$ of $\widetilde{M}$ such that, when restricted to M, $e_1,\ldots,e_{n+1}$ are tangent to M. Then $e_1,\ldots,e_{n+1}$ form an orthonormal basis of M. We can choose them so that $e_1,\ldots,e_p$ form an orthonormal basis of $N_x(M)$ and $e_{p+1},\ldots,e_n$ form an orthonormal basis of $H_x(M)$ and $e_{n+1} = \xi$. Moreover, we can take $e_{n+2},\ldots,e_{2m+1}$ of an orthonormal basis of $T_x(M)^\perp$ such that $e_{n+2},\dots,e_{n+1+p}$ form an orthonormal basis of $FN_x(M)$ and $e_{n+2+p},\dots,e_{2m+1}$ form an orthonormal basis of $N_x(M)^\perp$. In case of need, we can take $e_{n+2},\dots,e_{n+1+p}$ such that $e_{n+2}=Fe_1,\dots,e_{n+1+p}=Fe_p$. Unless otherwise stated, we use the convention that the ranges of indices are respectively:

$$i,j,k=1,\dots,n+1;\quad x,y,z=1,\dots,p;\quad a,b,c=n+2,\dots,2m+1\,.$$

LEMMA 2.1. *Let M be a contact CR-submanifold of a Sasakian manifold $\widetilde{M}$. If the f-structure f in the normal bundle of M is parallel*, *i.e.* $\nabla f=0$, *then*

$$A_U tV = A_V tU \tag{2.1}$$

for any vector fields U and V normal to M, *and the mean curvature vector ν satisfies*

$$f\nu = 0\,. \tag{2.2}$$

Proof. From (1.6) we have

$$g(A_V tU,X)=g(B(X,tV),U)=g(A_U tV,X)\,.$$

This gives (2.1). Since $fF=0$, (1.4) implies

$$0=-fB(X,PY)+f^2B(X,Y)\,.$$

Hence we obtain $f^2\sum B(e_i,e_i)=0$. From this and the equation $f^3+f=0$, we get (2.2).

From (2.2) we see that the mean curvature vector ν of M is in $FN_x(M)$.

In the following, we suppose that M is an $(n+1)$-dimensional contact CR-submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector ν and parallel f-structure f in the normal bundle of M. Furthermore, we assume that the second fundamental form A and the f-structure P on M commute, $PA=AP$, which means that $PA_V=A_VP$ for any vector field V normal to M. In this case, the contact CR-structure P induced on M is normal (see [3]).

We put $\mu=\nu/|\nu|$. Then μ is a nonvanishing parallel unit normal vector with $f\mu=0$, i.e. μ is an isoperimetric section in the normal bundle of M. We notice that $(\nabla_X A)_\mu Y=(\nabla_X A_\mu)Y$ for any vector fields X and Y tangent to M.

LEMMA 2.2. *The second fundamental forms of M satisfy*

$$\begin{aligned} g(A_\mu X,A_V Y) &= \tfrac14(c+3)g(X,Y)g(\mu,V)\\ &\quad -\tfrac14(c-1)[g(FX,\mu)g(FY,V)+\eta(X)\eta(Y)g(\mu,V)]\\ &\quad +\sum g(A_\mu tV,te_a)g(A_aX,Y)\,, \end{aligned} \tag{2.3}$$

where A_a denotes the second fundamental form in the direction of e_a.

Proof. By the assumption $PA = AP$, we have $g(A_\mu PX, tV) = 0$ for any vector field X tangent to M and any vector field V normal to M. We then have

$$g((\nabla_Y A)_\mu PX, tV) + g(A_\mu(\nabla_Y P)X, tV) + g(A_\mu PX, (\nabla_Y t)V) = 0\,,$$

and hence

$$\begin{aligned} g((\nabla_{PY} A)_\mu PX, tV) + g(X, PY)g(\mu, V) + \sum g(A_\mu tV, te_a)g(A_a X, PY) \\ + g(A_\mu PX, A_V Y) + g(A_\mu PX, A_{fV} PY) = 0\,. \end{aligned}$$

Using the Codazzi equation (1.11) and the Ricci equation (1.12) gives

$$\begin{aligned} \tfrac{1}{4}(c+3)g(PX, Y)g(\mu, V) + \sum g(A_\mu tV, te_a)g(A_a X, PY) \\ + g(A_\mu PX, A_V Y) = 0\,. \end{aligned}$$

Hence

$$\begin{aligned} g(A_\mu PX, A_V PY) = \tfrac{1}{4}(c+3)g(PX, PY)g(\mu, V) \\ + \sum g(A_\mu tV, te_a)g(A_a X, P^2 Y)\,. \end{aligned}$$

On the other hand,

$$g(A_\mu PX, A_V PY) = g(A_\mu X, A_V Y) + g(A_\mu X, A_{FY} tV) + \eta(Y)g(A_\mu X, tV)\,,$$

$$\begin{aligned} -\sum g(A_\mu tV, te_a)g(A_a X, P^2 Y) \\ = \sum g(A_\mu tV, te_a)g(A_a X, Y) + g(A_\mu tV, A_{FY} X) \\ - g(A_\mu tV, \xi)g(\xi, A_{FY} X) + \eta(Y)g(A_\mu tV, X) \\ - \eta(Y)g(A_\mu tV, \xi)g(\xi, X) \\ = \sum g(A_\mu tV, te_a)g(A_a X, Y) + g(A_\mu tV, A_{FY} X) \\ + g(FX, FY)g(\mu, V) + \eta(X)g(\mu, V) + \eta(Y)g(A_\mu X, tV)\,. \end{aligned}$$

From the above equations, we find

$$\begin{aligned} g(A_\mu X, A_V Y) = \sum g(A_\mu tV, te_a)g(A_a X, Y) \\ + \tfrac{1}{4}(c-1)g(PX, PY)g(\mu, V) + g(X, Y)g(\mu, V) \\ + g([A_{FY}, A_\mu]tV, X)\,. \end{aligned}$$

Since, by the Ricci equation (1.12),

$$g([A_{FY}, A_\mu]tV, X) = \tfrac{1}{4}(c-1)[g(FX, FY)g(\mu, V) - g(FX, \mu)g(FY, V)]\,,$$

the equation above becomes our result (2.3).

Since the mean curvature vector of M is parallel, we see that, by the Codazzi equation (1.11),

$$\sum (\nabla_i A)_\mu e_i = 0\,, \tag{2.4}$$

where ∇_i denotes the covariant differentiation in the direction of e_i.

LEMMA 2.3. *The restricted laplacian for A_μ is given by*

$$(2.5)\qquad (\nabla^2 A)_\mu X = \sum (R(e_i, X)A)_\mu e_i + \tfrac{1}{4}(c-1) \times \Big[-A_{FX}t\mu - tB(t\mu, X) + 3PA_\mu PX + g(t\mu, X)t \operatorname{Tr} B - 2(\operatorname{Tr} A_{FX})t\mu - g(X, t\mu)\sum A_a te_a - 2\sum g(A_a te_a, X)t\mu - (n-1)g(t\mu, X)\xi - (2n+1)\eta(X)t\mu \Big].$$

P r o o f. From (1.11) and (2.4) we have

$$\begin{aligned}(\nabla^2 A)_\mu X = &\sum (\nabla_i \nabla_i A)_\mu X \\ = &\sum (R(e_i, X)A)_\mu e_i \\ &+ \tfrac{1}{4}(c-1)\sum [g((\nabla_i F)e_i, \mu)PX + g(Fe_i, \mu)(\nabla_i P)X \\ &- g((\nabla_i F)X, \mu)Pe_i - g(FX, \mu)(\nabla_i P)e_i \\ &+ 2g((\nabla_i P)e_i, X)t\mu + 2g(Pe_i, X)(\nabla_i t)\mu].\end{aligned}$$

Using (1.2)–(1.5) and Lemma 2.1, we find (2.5).

From (2.5) we have

$$(2.6)\qquad \begin{aligned} g((\nabla^2 A)_\mu, A_\mu) = &\sum g((\nabla_i \nabla_i A)_\mu e_j, A_\mu e_j) \\ = &\sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) \\ &+ \tfrac{3}{4}(c-1)\Big[\operatorname{Tr}(A_\mu P)^2 - \sum g(A_\mu t\mu, A_a te_a) \\ &+ \sum g(A_\mu t\mu, te_a)\operatorname{Tr} A_a + n\Big].\end{aligned}$$

On the other hand, by the Gauss equation (1.10),

$$(2.7)\qquad \begin{aligned}\sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) = &\tfrac{1}{4}(c+3)(n+1)\operatorname{Tr} A_\mu^2 \\ &- \tfrac{1}{4}(c-1)\operatorname{Tr} A_\mu^2 - \tfrac{1}{4}(c+3)(\operatorname{Tr} A_\mu)^2 - \tfrac{1}{4}(c-1)(n+1) \\ &+ \sum \operatorname{Tr}(A_\mu A_a)^2 - \sum \operatorname{Tr} A_\mu^2 A_a^2 + \sum \operatorname{Tr} A_a \operatorname{Tr} A_\mu^2 A_a \\ &- \sum (\operatorname{Tr} A_\mu A_a)^2.\end{aligned}$$

LEMMA 2.4. *The curvature tensor R of M satisfies*

$$(2.8)\qquad \sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) = \tfrac{1}{16}(c-1)^2(n-p).$$

P r o o f. From the Ricci equation (1.12) we have

$$(2.9)\qquad \sum \operatorname{Tr}(A_\mu A_a)^2 - \sum \operatorname{Tr} A_\mu^2 A_a^2 = -\tfrac{1}{16}(c-1)^2(p-1).$$

On the other hand, (2.3) implies

$$\sum \operatorname{Tr} A_a \operatorname{Tr} A_\mu^2 A_a = \tfrac{1}{4}(c+3)(\operatorname{Tr} A_\mu)^2 + \sum \operatorname{Tr} A_a \operatorname{Tr} A_\mu A_b g(A_\mu te_a, te_b) - \tfrac{1}{4}(c-1)\sum g(A_\mu t\mu, te_a)\operatorname{Tr} A_a\,,$$

$$\sum (\operatorname{Tr} A_\mu A_a)^2 = \tfrac{1}{4}(c+3)(n+1)A_\mu^2 - \tfrac{1}{2}(c-1)A_\mu^2 + \sum \operatorname{Tr} A_a \operatorname{Tr} A_\mu A_b g(A_\mu te_a, te_b)\,.$$

Hence

$$\sum \operatorname{Tr} A_a \operatorname{Tr} A_\mu^2 A_a - \sum (\operatorname{Tr} A_\mu A_a)^2 = -\tfrac{1}{4}(c+3)(n+1)A_\mu^2 + \tfrac{1}{2}(c-1)A_\mu^2 + \tfrac{1}{4}(c+3)(\operatorname{Tr} A_\mu)^2 - \tfrac{1}{4}(c-1)\sum g(A_\mu t\mu, te_a)\operatorname{Tr} A_a\,. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.7), we find

$$\sum g((R(e_i,e_j)A)_\mu e_i, A_\mu e_j) = \tfrac{1}{4}(c-1)\Big[\operatorname{Tr} A_\mu^2 - \sum g(A_\mu t\mu, te_a)\operatorname{Tr} A_a - (n+1) - \tfrac{1}{4}(c-1)(p-1)\Big]\,. \tag{2.11}$$

Since, by (2.3),

$$\operatorname{Tr} A_\mu^2 = \tfrac{1}{4}(c-1)(n-1) + (n+1) + \sum g(A_\mu t\mu, te_a)\operatorname{Tr} A_a\,, \tag{2.12}$$

equation (2.11) becomes (2.8).

LEMMA 2.5. *For the second fundamental form A_μ we have*

$$g((\nabla^2 A)_\mu, A_\mu) = -\tfrac{1}{8}(c-1)^2(n-p)\,. \tag{2.13}$$

Proof. First of all,

$$\operatorname{Tr}(A_\mu P)^2 = -\operatorname{Tr} A_\mu^2 + 1 + \sum g(A_\mu te_a, A_\mu te_a)\,. \tag{2.14}$$

Furthermore, (2.3) implies

$$\sum g(A_\mu te_a, A_\mu te_a) = \tfrac{1}{4}(c-1)(p-1) + \sum g(A_\mu t\mu, A_a te_a)\,. \tag{2.15}$$

From (2.12), (2.13) and (2.15) we obtain

$$\operatorname{Tr}(A_\mu P)^2 - \sum g(A_\mu t\mu, A_a te_a) + \sum g(A_\mu t\mu, te_a)\operatorname{Tr} A_a + n = -\tfrac{1}{4}(c-1)(n-p)\,. \tag{2.16}$$

Substituting (2.8) and (2.16) into (2.6) yields (2.13).

3. Theorems. Let M be an $(n+1)$-dimensional contact CR-submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. We suppose that $\nabla f = 0$ and $PA = AP$.

First of all, we prove that $\Delta \operatorname{Tr} A_\mu^2 = 0$. We take an orthonormal basis $\{A_a\}$ such that $e_{n+2} = \mu$ and $\operatorname{Tr} A_a = 0$, $a = n+3, \ldots, 2m+1$. Then (2.12) becomes

$$\operatorname{Tr} A_\mu^2 = \tfrac{1}{4}(c-1)(n-1) + (n+1) + g(A_\mu t\mu, t\mu) \operatorname{Tr} A_\mu \,.$$

Hence

$$\Delta \operatorname{Tr} A_\mu^2 = \sum \nabla_i \nabla_i \operatorname{Tr} A_\mu^2 = \sum g((\nabla^2 A)_\mu t\mu, t\mu) \operatorname{Tr} A_\mu + 2 \sum g((\nabla_i A)_\mu t\mu, (\nabla_i t)\mu) \operatorname{Tr} A_\mu \,. \tag{3.1}$$

On the other hand, (2.5) implies

$$g((\nabla^2 A)_\mu t\mu, t\mu) = \sum g((R(e_i, t\mu)A)_\mu e_i, t\mu) + \tfrac{3}{4}(c-1)\Big[\operatorname{Tr} A_\mu - \sum g(A_\mu t e_a, t e_a)\Big] \,. \tag{3.2}$$

From (1.5) and (1.11) we also have

$$\sum g((\nabla_i A)_\mu t\mu, (\nabla_i t)\mu) \operatorname{Tr} A_\mu = -\tfrac{1}{4}(c-1)\Big[\operatorname{Tr} A_\mu - \sum g(A_\mu t e_a, t e_a)\Big] \,. \tag{3.3}$$

Using the Gauss equation, we see that

$$\begin{aligned} &\sum g((Re_i, t\mu)A)_\mu e_i, t\mu) \\ &\quad = \sum g(R(e_i, t\mu)A_\mu e_i, t\mu) - \sum g(R(e_i, t\mu)e_i, A_\mu t\mu) \\ &\quad = \tfrac{1}{4}(c+3)(n+1) g(A_\mu t\mu, t\mu) - \tfrac{1}{4}(c-1) g(A_\mu t\mu, t\mu) \\ &\qquad - \tfrac{1}{4}(c+3)\operatorname{Tr} A_\mu + \sum g(A_a t\mu, [A_\mu, A_a] t\mu) \\ &\qquad - \sum g(A_a t\mu, t\mu) \operatorname{Tr} A_\mu A_a + g(A_\mu t\mu, A_\mu t\mu) \operatorname{Tr} A_\mu \,. \end{aligned}$$

By the Ricci equation (1.12) and the equation

$$\begin{aligned} \operatorname{Tr} A_\mu A_a = {} & \tfrac{1}{4}(c+3)(n+1) g(\mu, e_a) - \tfrac{1}{2}(c-1) g(\mu, e_a) \\ & + g(A_\mu t e_a, t\mu) \operatorname{Tr} A_\mu \,, \end{aligned}$$

we find

$$\sum g((R(e_i, t\mu)A)_\mu e_i, t\mu) = \tfrac{1}{4}(c-1)[g(A_\mu t e_a, t e_a) - \operatorname{Tr} A_\mu] \,. \tag{3.4}$$

From (3.1)–(3.4) we have the following

Lemma 3.1. $\Delta \operatorname{Tr} A_\mu^2 = 0$.

We next prove

Theorem 3.1. *Let M be an $(n+1)$-dimensional contact CR-submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If the f-structure f in the normal bundle is parallel, and if*

$PA = AP$, *then*

$$|(\nabla A)_\mu|^2 = \tfrac{1}{8}(c-1)^2(n-p)\,.$$

Proof. Generally,

$$\tfrac{1}{2}\Delta\,\mathrm{Tr}\,A_\mu^2 = g((\nabla^2 A)_\mu, A_\mu) + |(\nabla A)_\mu|^2\,.$$

Thus our assertion follows by Lemmas 2.5 and 3.1.

Let us put

$$T(X,Y) = (\nabla_X A)_\mu Y + \tfrac{1}{4}(c-1)[g(FY,\mu)PX - g(PX,Y)t\mu]\,.$$

Then, by the Codazzi equation (1.11),

$$|T|^2 = |(\nabla A)_\mu|^2 - \tfrac{1}{8}(c-1)^2(n-p) \geq 0\,.$$

Therefore, T vanishes identically if and only if

$$|(\nabla A)_\mu|^2 = \tfrac{1}{8}(c-1)^2(n-p)\,.$$

COROLLARY 3.1. *Under the same assumptions as in Theorem* 3.1,

$$(\nabla_X A)_\mu Y = -\tfrac{1}{4}(c-1)[g(FY,\mu)PX - g(PX,Y)t\mu] \tag{3.5}$$

for any vector fields X and Y tangent to M.

THEOREM 3.2. *Let M be an $(n+1)$-dimensional generic submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If $PA = AP$, then*

$$|(\nabla A)_\mu|^2 = \tfrac{1}{8}(c-1)^2(n-p)\,,$$

or equivalently

$$(\nabla_X A)_\mu Y = -\tfrac{1}{4}(c-1)[g(FY,\mu)PX - g(PX,Y)t\mu]$$

for any vector fields X and Y tangent to M.

Theorems 3.1 and 3.2 are generalizations of some theorems in [1] and [2].

THEOREM 3.3. *Let M be an $(n+1)$-dimensional contact CR-submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If the f-structure f in the normal bundle is parallel, and if $PA = AP$, then each eigenvalue of A_μ is constant.*

Proof. We suppose that $A_\mu X = \lambda X$. Then $A_\mu PX = PA_\mu X = \lambda PX$. Using (3.5), we also have

$$(Y\lambda)g(X,X) = \tfrac{1}{2}(c-1)g(PY,X)g(t\mu,X)\,.$$

Replacing X by PX, we obtain $(Y\lambda)g(PX,PX) = 0$. If $PX = 0$, then $(Y\lambda)g(X,X) = 0$ and hence $Y\lambda = 0$. If $PX \neq 0$, we also have $Y\lambda = 0$. Consequently, λ is constant.

THEOREM 3.4. *Let M be an $(n+1)$-dimensional generic submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If $PA = AP$, then each eigenvalue of A_μ is constant.*

THEOREM 3.5. *Let M be an $(n+1)$-dimensional complete and simply connected contact CR-submanifold with nonvanishing parallel mean curvature vector and with parallel f-structure f in the normal bundle in a unit sphere S^{2m+1}. If $PA = AP$, then M is a product of Riemannian manifolds, $M_1 \times \ldots \times M_s$, where s is the number of the distinct eigenvalues of A_μ, and the mean curvature vector of M is an umbilical section of M_t $(t = 1, \ldots, s)$.*

Proof. From Theorems 3.1 and 3.3 we see that the smooth distribution T_t $(t = 1, \ldots, s)$ which consists of all eigenspaces associated with the eigenvalues of A_μ can be defined and is parallel. M is assumed to be simply connected and complete, and therefore our assertion follows from the de Rham decomposition theorem.

THEOREM 3.6. *Let M be an $(n+1)$-dimensional complete and simply connected generic submanifold with nonvanishing parallel mean curvature vector in a unit sphere S^{2m+1}. If $PA = AP$, then M is a product of Riemannian manifolds, $M_1 \times \ldots \times M_s$, where s is the number of the distinct eigenvalues of A_μ, and the mean curvature vector of M is an umbilical section of M_t $(t = 1, \ldots, s)$.*

THEOREM 3.7. *Let M be an $(n+1)$-dimensional contact CR-submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector and parallel f-structure f in the normal bundle. If $PA = AP$, and if the sectional curvature of M is nonpositive, then the second fundamental form in the direction of the mean curvature vector is parallel. Moreover, either $c = 1$, or $P = 0$ and M is anti-invariant in $\widetilde{M}^{2m+1}(c)$ with respect to φ.*

Proof. We take an orthonormal basis $e_1, \ldots, e_{n+1}$ such that $A_\mu e_i = \lambda_i e_i$ $(i = 1, \ldots, n+1)$. We denote by K_{ij} the sectional curvature of M spanned by e_i and e_j. Then

$$\sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) = \tfrac{1}{4} \sum (\lambda_i - \lambda_j)^2 K_{ij} \,.$$

Substituting this into (2.8), we obtain

$$\sum (\lambda_i - \lambda_j)^2 K_{ij} = \tfrac{1}{8}(c-1)^2(n-p) \geq 0 \,.$$

Thus, if $K_{ij} \leq 0$, then $(c-1)^2(n-p) = 0$, and hence $(\nabla A)_\mu = 0$ by Theorem 3.1. Moreover, we have either $c = 1$ or $n = p$. If $n = p$, then $P = 0$ and M is an anti-invariant submanifold of $\widetilde{M}^{2m+1}(c)$ tangent to the structure vector field ξ.

THEOREM 3.8. *Let M be an $(n+1)$-dimensional generic submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If $PA = AP$, and if the sectional curvature of M is nonpositive, then the second fundamental form in the direction of the mean curvature vector is parallel. Moreover, either $c = 1$, or $P = 0$ and M is anti-invariant in $\widetilde{M}^{2m+1}(c)$ with respect to φ.*

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