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A SIERPIŃSKI–ZYGMUND FUNCTION WHICH HAS A PERFECT ROAD AT EACH POINT

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Perfect roads were defined by Maximoff in 1936 [M1]. They were studied in connection with derivatives and Darboux Baire class 1 functions [M2].

DEFINITION. Suppose $f : \mathbb{R} \to \mathbb{R}$. The function f *has a perfect road* at p means there is a Cantor set C such that p is a two-sided limit point of C and $f|C$ is continuous at p. When we say f is of *perfect road type*, we mean f has a perfect road at each point.

For Baire class 1 functions there are many conditions which are equivalent to a function being of perfect road type. We state some of them here.

THEOREM [Br]. *If f is of Baire class* 1, *then the following are equivalent.*

(a) *f is Darboux.*
(b) *f is of perfect road type.*
(c) *The graph of f is connected.*

Equivalence of (a) and (b) in the above theorem was first shown by Maximoff in [M1].

In the same paper, he raises the following natural question about functions of perfect road type: If f is of perfect road type, does there have to be a Cantor set C such that $f|C$ is continuous? Under the assumption of the continuum hypothesis, Maximoff gives a counterexample to this question [M1]. We answer this question in the negative in a strong way ([1]). We construct in ZFC a Sierpiński–Zygmund function that has a perfect road at each point. Let us recall the theorem of Sierpiński and Zygmund.

THEOREM [SZ]. *There is a function $f : \mathbb{R} \to \mathbb{R}$ such that if $M \subseteq \mathbb{R}$ and $|M| = 2^\omega$, then $f|M$ is not continuous.*

Now, we state the result of this paper.

([1]) This question was related to the author by Richard Gibson. At the time, neither the author nor Richard Gibson was aware of Maximoff's result.

THEOREM. *There exists in ZFC a function $h : \mathbb{R} \to \mathbb{R}$ such that h has a perfect road at each point but if M is any set such that $|M| = 2^\omega$, then $h|M$ is not continuous.*

Before we construct h, we will need three lemmas. We leave easy proofs of the first two lemmas to the reader.

LEMMA 1 [Ku]. *Suppose $U \subseteq \mathbb{R}$, and $f : U \to \mathbb{R}$ is continuous. Then there is a G_δ set M containing U and a continuous function g defined on M such that $g|U = f$. Moreover, if $G = \{h : \mathbb{R} \to \mathbb{R} \mid$ there is a G_δ set M such that $h|M$ is continuous and h is zero on $M^c\}$, then $|G| = 2^\omega$.*

LEMMA 2. *Let C be a Cantor set. Then there is a collection G of Cantor sets such that*

1) $\bigcup G = C$,
2) *if $g, h \in G$ and $g \neq h$, then $g \cap h = \emptyset$, and*
3) *if U is an open set in C, then there are 2^ω Cantor sets in G that intersect U.*

The third lemma is central to the proof of our theorem. It easily follows from the continuum hypothesis or from a consequence of Martin's axiom. However, we prove it in ZFC.

LEMMA 3. *Let $\{p_\alpha\}_{\alpha<2^\omega}$ be a well-ordering of $\mathbb{R}$. Then there exists a sequence $\{C_\alpha\}$ of Cantor sets such that*

1) *p_α is a two-sided limit point of C_α,*
2) *if $\beta < \alpha$ then $p_\beta \notin C_\alpha$,*
3) *if $\alpha \neq \beta$ then $[C_\alpha \backslash \{p_\alpha\}] \cap [C_\beta \backslash \{p_\beta\}] = \emptyset$.*

Proof. Let $\{F_n\}$ be a sequence of Cantor sets such that if $n \neq m$, then $F_n \cap F_m = \emptyset$ and $\bigcup F_n$ is dense in $\mathbb{R}$. Using Lemma 2 for each positive integer n, let G_n be a collection of Cantor sets such that $\bigcup G_n = F_n$, G_n is a pairwise disjoint collection, and if U is an open set in F_n, then there are 2^ω Cantor sets in G_n that intersect U. Let $G = \bigcup G_n$. Note that the collection G has the property that if O is an open subset of $\mathbb{R}$, then 2^ω elements of G intersect O; and also note that G is a pairwise disjoint collection.

Now, let B be a countable basis for $\mathbb{R}$. Let H_0 be a countable subcollection of G such that every element of B intersects some element of H_0. Suppose that $\alpha < 2^\omega$, and for each $\beta < \alpha$, H_β has been defined. Then we define H_α to be some countable subcollection G such that if $\beta < \alpha$ that $H_\beta \cap H_\alpha = \emptyset$ and every element of B intersects some element of H_α. H_α exists because $|\bigcup_{\beta<\alpha} H_\beta| < 2^\omega$ and 2^ω elements of G intersect each element of B.

Now, we have a sequence $\{H_\alpha\}_{\alpha<2^\omega}$ such that (a) H_α is a countable collection of Cantor sets, (b) $\bigcup H_\alpha$ is dense in $\mathbb{R}$, and (c) if $\alpha \neq \beta$ then $(\bigcup H_\alpha) \cap (\bigcup H_\beta) = \emptyset$.

Fix α. Let $\{s_n\}$ and $\{t_n\}$ be two sequences, one increasing and the other decreasing, and both converging to p_α. Now, let A_n be a Cantor set such that $A_n \subseteq (s_n, s_{n+1}) \cap \bigcup H_\alpha$ and $A'_n \subseteq (t_{n+1}, t_n) \cap \bigcup H_\alpha$. We may assume that $(A_n \cup A'_n) \cap \{p_\beta \mid \beta < \alpha\} = \emptyset$, because otherwise we could write $A_n \cup A'_n$ as a disjoint union of 2^ω many Cantor sets, and one of them would have to miss $\{p_\beta \mid \beta < \alpha\}$. Now, we let $C_\alpha = \bigcup_{n=1}^{\infty}(A_n \cup A'_n) \cup \{p_\alpha\}$. So, for each $\alpha < 2^\omega$, we have defined a Cantor set C_α.

$\{C_\alpha\}$ obviously satisfies the first two conditions of the lemma. The third condition is also satisfied because $C_\alpha \backslash \{p_\alpha\} \subseteq \bigcup H_\alpha$ and if $\alpha \neq \beta$ then $(\bigcup H_\alpha) \cap (\bigcup H_\beta) = \emptyset$. Thus, the proof of the lemma is complete.

Now, we proceed to construct a function h of the theorem. Let $\{p_\alpha\}_{\alpha<2^\omega}$ be a well-ordering of $\mathbb{R}$, and let $\{C_\alpha\}_{\alpha<2^\omega}$ be a sequence of Cantor sets as described in Lemma 3. Let $\{f_\alpha\}_{\alpha<2^\omega}$ be a well-ordering of all the functions described in Lemma 1. Finally, let $D_\alpha = C_\alpha \backslash \{p_\alpha\}$.

We construct h inductively. At the αth stage, we will define h on $D_\alpha \cup \{p_\alpha\}$. Let $h(p_0) \in \mathbb{R}\backslash\{f_0(p_0)\}$. For each $p_\alpha \in D_0$, let $h(p_\alpha)$ be such that $|h(p_0) - h(p_\alpha)| < |p_0 - p_\alpha|$ and $h(p_\alpha) \notin \{f_\beta(p_\alpha) \mid \beta \leq \alpha\}$. Now, suppose γ is an ordinal and h is defined on $\{p_\delta \mid \delta < \gamma\} \cup (\bigcup_{\delta<\gamma} D_\delta)$. If $h(p_\gamma)$ is not defined yet, then we let $h(p_\gamma) \in \mathbb{R}\backslash\{f_\delta(p_\gamma) \mid \delta \leq \gamma\}$. Otherwise, we leave $h(p_\gamma)$ unchanged. We note that h is not defined at any point of D_γ yet because $\{C_\alpha\}_{\alpha<2^\omega}$ satisfies the second and the third condition of Lemma 3. If $p_\beta \in D_\gamma$, then let $h(p_\beta)$ be such that $|h(p_\gamma) - h(p_\beta)| < |p_\gamma - p_\beta|$ and $p_\beta \notin \{f_\delta(p_\beta) \mid \delta \leq \beta\}$.

Now h is a well-defined function; and for each $\alpha, h|C_\alpha$ is continuous at p_α. Thus, h has a perfect road at each one of its points. We want to show that if $M \subseteq \mathbb{R}$ and $|M| = 2^\omega$, then $h|M$ is not continuous. To get a contradiction, assume $h|M$ is continuous for some $M \subseteq \mathbb{R}$ and $|M| = 2^\omega$. By Lemma 1, there is an α such that $f_\alpha|M = h|M$. Since $|M| = 2^\omega$, let $\beta > \alpha$ be such that $p_\beta \in M$. But, by the definition of the function h, $h(p_\beta) \notin \{f_\delta(p_\beta) \mid \delta \leq \beta\}$. Since $f_\alpha(p_\beta) = h(p_\beta)$, we have a contradiction. Thus, h is not continuous on any set of size 2^ω; and this completes the proof of the theorem.

Now we state some questions. The class of Perfect Road (PR), Extendable (Ext), Almost Continuous (AC), Connectivity (Conn), Darboux (D), Peripherally Continuous (PC) functions are related in the following fashion on the real line. Refer to [BHL] for definitions.

Ext → AC → Conn → D → PC

↘ ↗

PR

QUESTION. Is there an AC Sierpiński–Zygmund function? Or even a D Sierpiński–Zygmund function? It is shown in [RGR] that if f is an extendable function then f has the "strong Cantor intermediate value property" which implies that every open set contains a Cantor set C such that $f|C$ is continuous. So, there is no extendable Sierpiński–Zygmund function.

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