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A CERTAIN PROPERTY OF ABELIAN GROUPS

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1. Statement of the result

1.1. DEFINITION. A subset A of an abelian group G is said to *cover not less than* $1/n$ of the group if there exist $g_1, \ldots, g_n \in G$ such that $\bigcup_{i=1}^n g_i A = G$.

1.2. Remark. $g_n = e$ may be assumed in the above definition since $G = g_n^{-1} G = \bigcup_{i=1}^n g_n^{-1} g_i A$.

1.3. THE MAIN THEOREM. *Let G be an abelian group and $A_1, \ldots, A_n$ its disjoint subsets, each covering not less than $1/n$ of the group. Then* $\bigcup_{i=1}^n A_i = G$.

1.4. Remark. For obvious arithmetical reasons the theorem holds for all finite (not necessarily abelian) groups. On the other hand, it is easy to construct a counterexample in a free group with two generators. The question arises about the class of groups for which the theorem holds, but we do not give any ideas for the answer in this paper.

2. Reduction of the main theorem to a special case. In the following only abelian groups will be considered and therefore we will use the additive notation.

2.1. LEMMA. *Let G be abelian and let sets A_i cover not less than $1/n$ of it. Assume that $A_i \cup \bigcup_{k=1}^{n-1} (g_{ik} + A_i) = G$ for $i = 1, \ldots, n$* (compare 1.2). *If Theorem* 1.3 *holds for the subgroup H of G generated by all elements g_{ik} and for the sets $A'_i = A_i \cap H$ then it holds for G.*

Proof. The action of H on its cosets in G is the same as on itself, so the theorem holds for each coset, and consequently for the whole G.

2.2. Remark. Let H be as in 2.1 and $A'_i = A_i \cap H$. Let $Z^{n(n-1)}$ be the free abelian group with generators $e_{ik}, i = 1, \ldots, n$, $k = 1, \ldots, n-1$, and let $\phi : Z^{n(n-1)} \to H$ be a homomorphism such that $\phi(e_{ik}) = g_{ik}$. Put $B_i = \phi^{-1}(A'_i)$. Then $B_i \cup \bigcup_{k=1}^{n-1} (e_{ik} + B_i) = Z^{n(n-1)}$ for $i = 1, \ldots, n$, thus each

B_i covers not less than $1/n$ of $Z^{n(n-1)}$. If we prove that $\bigcup_{i=1}^n B_i = Z^{n(n-1)}$, then by surjectivity of ϕ we get $\bigcup_{i=1}^n A'_i = H$.

According to 2.1 and 2.2, in order to prove the main theorem, it is enough to prove it in the following special case.

2.3. THEOREM (The special case of the main theorem). *Let $Z^{n(n-1)}$ be the free abelian group with generators e_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, n-1$, and let B_i be its disjoint subsets with $B_i \cup \bigcup_{k=1}^{n-1}(e_{ik} + B_i) = Z^{n(n-1)}$ for $1, \ldots, n$. Then $\bigcup_{i=1}^n B_i = Z^{n(n-1)}$.*

3. Proof of the special case (Theorem 2.3)

3.1. Let Z_i^{n-1} be the subgroup of $Z^{n(n-1)}$ generated by $e_{i1}, e_{i2}, \ldots, e_{i,n-1}$. Then $Z^{n(n-1)} = \bigoplus_{i=1}^n Z_i^{n-1}$ ($\bigoplus$ denotes the free abelian product) and therefore we may identify $Z^{n(n-1)}$ with the cartesian product $Z_1^{n-1} \times Z_2^{n-1} \times \ldots \times Z_n^{n-1}$.

3.2. DEFINITION. For $a \in Z^{n(n-1)}$ let $K_a^i = \{a, a - e_{i1}, a - e_{i2}, \ldots, a - e_{i,n-1}\}$.

3.3. Remark. Note that for each $a \in Z^{n(n-1)}$ we can find in K_a^i an element which belongs to B_i. If it is not a then, since $B_i \cup \bigcup_{k=1}^{n-1}(e_{ik} + B_i) = Z^{n(n-1)}$, $a \in e_{ik} + B_i$ for some k, and then $a - e_{ik} \in B_i$.

3.4. DEFINITION. Let $a \in Z^{n(n-1)}$ and let $a = a_1 + \ldots + a_n$ be its expansion with respect to the free abelian product of 3.1, i.e. $a_i \in Z_i^{n-i}$. Then define $K_a = K_{a_1}^1 \times \ldots \times K_{a_n}^n$.

3.5. LEMMA. *For each $i \in \{1, \ldots, n\}$ and each $a \in Z^{n(n-1)}$, K_a is a disjoint sum of n^{n-1} sets of the form K_x^i .*

Proof. Assume $i = n$. Then

$$K_a = (K_{a_1}^1 \times \ldots \times K_{a_{n-1}}^{n-1}) \times K_{a_n}^n = \bigcup_{y \in K_{a_1}^1 \times \ldots \times K_{a_{n-1}}^{n-1}} \{y\} \times K_{a_n}^n,$$

which is a disjoint sum of sets of the desired form. For other i the proof is the same.

3.6. Remark. According to 3.3 the sets K_a and B_i have at least n^{n-1} elements in common. Since B_i are disjoint an arithmetical argument proves that $K_a \subset \bigcup_{i=1}^n B_i$.

3.7. LEMMA. *$Z^{n(n-1)}$ is a disjoint sum of sets of the form K_a.*

Proof. It will be proved in the next section that Z_i^{n-1} is a disjoint sum of sets of the form K_x^i, i.e. $Z^{n-1} = \bigcup_{x \in I_i} K_x^i$ for each i and some $I_i \subset Z_i^{n-1}$.

Thus

$$Z^{n(n-1)} = \bigcup_{(x_1,\ldots,x_n)\in I_1\times\ldots\times I_n} K^1_{x_1}\times\ldots\times K^n_{x_n}$$

and the lemma follows.

Theorem 2.3 now follows from 3.6 and 3.7.

4. Combinatorial fact used in the proof of Lemma 3.7

4.1. PROPOSITION (used in 3.7). *Let Z^{n-1} be a free abelian group with generators $e_1,\ldots,e_{n-1}$. For $x\in Z^{n-1}$ put $K_x=\{x,x-e_1\ldots,x-e_{n-1}\}$. Then Z^{n-1} is a disjoint sum of sets of the form K_x.*

Before proving the proposition we give some notations and lemmas.

4.2. *Notation.* Let Z_n be the group of integers (mod n), i.e. a cyclic group of n elements. Denote by a_i, $i\in\{0,1,\ldots,n-1\}$, the generators in $(Z_n)^{n-1}$ of the form $(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)$. Put $x_i=a_i+ia_{n-1}$ for $i=1,\ldots,n-2$, and denote by V the subgroup of $(Z_n)^{n-1}$ generated by $x_1,\ldots,x_{n-2}$.

4.3. LEMMA. *The elements $x_1,\ldots,x_{n-2}$ are free abelian generators of V over Z_n.*

Proof. Suppose $\sum_{i=1}^{n-2}k_ix_i=0$, $0\le k_i<n$. Then $\sum_{i=1}^{n-2}k_i(a_i+ia_{n-1})=\sum_{i=1}^{n-2}k_ia_i+\big(\sum_{i=1}^{n-2}ik_i\big)a_{n-1}=0$, which implies $k_i=0$ for $i=1,\ldots,n-2$ and the lemma follows.

4.4. Remark. Lemma 4.3 implies that V consists of n^{n-2} elements.

4.5. LEMMA. *For $1\le i\le n-1$, $a_i\notin V$.*

Proof. Suppose $a_i=\sum_{l=1}^{n-2}k_lx_l=\sum_{l=1}^{n-2}k_la_l+\big(\sum_{l=1}^{n-2}lk_l\big)a_{n-1}$. If $i\ne n-1$ then $k_i=1$ and $k_l=0$ for $l\ne i$, but then $\sum_{l=1}^{n-2}lk_l=i\ne 0$ (mod n), a contradiction.

If $i=n-1$ then $k_l=0$ for each l, which contradicts $\sum_{l=1}^{n-2}lk_l=1$ (mod n).

4.6. LEMMA. *For $1\le i,\ j\le n$, $i\ne j$, $a_i-a_j\notin V$.*

Proof. Case 1: $i\ne j$, $i\ne n-1$ and $j\ne n-1$. Suppose $a_i-a_j=\sum_{l=1}^{n-2}k_lx_l=\sum_{l=1}^{n-2}k_la_l+\big(\sum_{l=1}^{n-2}lk_l\big)a_{n-1}$. Then $k_i=1$, $k_j=-1$ (mod n), $k_l=0$ for $l\ne i,j$ and consequently $\sum_{l=1}^{n-2}lk_l=i-j\ne 0$ (mod n), a contradiction.

Case 2: $i\ne j$, $i=n-1$. By the argument of the previous case we get $k_j=-1$ (mod n), $k_l=0$ for $l\ne j$, and consequently $\sum_{l=1}^{n-2}lk_l=-j$ (mod n), which contradicts $\sum_{l=1}^{n-2}lk_l=1$ (mod n) since $1\le j<n-1$. This completes the proof.

4.7. LEMMA. *For* $x \in (Z_n)^{n-1}$ *define* $K_x = \{x, x - a_1, \ldots, x - a_{n-1}\}$. *Then* $\{K_x : x \in V\}$ *is a disjoint family and covers* $(Z_n)^{n-1}$.

Proof. Suppose that $K_{x_1} \cap K_{x_2} \neq \emptyset$ for some $x_1, x_2 \in V$, $x_1 \neq x_2$. Then either $x_1 = x_2 - a_i$, $x_2 = x_1 - a_i$ or $x_1 - a_i = x_2 - a_j$ for some $i, j \in \{1, \ldots, n-1\}$. The first two cases imply $a_i \in V$ and the third implies $i \neq j$ (since $x_1 \neq x_2$) and $a_i - a_j \in V$, but none of these is possible according to 4.5 and 4.6. It follows from 4.4 that $\bigcup\{K_x : x \in V\}$ consists of n^{n-1} elements and thus equals $(Z_n)^{n-1}$.

4.8. Proof of 4.1. Let $\phi : Z^{n-1} \to (Z_n)^{n-1}$ be the canonical homomorphism satisfying $\phi(e_i) = a_i$, and $V' = \phi^{-1}(V)$. Then from 4.7 it follows that the family $\{K_x : x \in V'\}$ is a disjoint cover of Z^{n-1}, which completes the proof.

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