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TAME L^p-MULTIPLIERS

BY

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0. Introduction. Let G be a compact abelian group and let Γ be its discrete dual group. Let $1 \le p, q \le \infty$. A function $m : \Gamma \to \mathbb{C}$ is called an (L^p, L^q) *multiplier* (or L^p *multiplier* if $p = q$) if for every $f \in L^p(G)$ there is a function $T_m f \in L^q(G)$ such that

$$\widehat{T_m f}(\gamma) = m(\gamma)\widehat{f}(\gamma)$$

for all $\gamma \in \Gamma$. The space of (L^p, L^q) multipliers will be denoted by $M(p, q)$ (or $M(p)$ if $p = q$).

It is well known that $M(p, q) = M(q', p')$ (where $p' = p/(p-1)$, $q' = q/(q-1)$) and that $M(G) = M(1) \subsetneq M(p) \subsetneq M(2) = l^\infty$ if $p \ne 1, 2, \infty$. It is also known that $M(1, q) = L^q(G)$. For choices of p and q other than these few special cases, fundamental questions such as characterizing $M(p, q)$ remain unsolved. For background information on (L^p, L^q) multipliers we refer the reader to [4, Ch. 16] and [16].

A concept which has proved useful in the study of measures is tameness.

DEFINITION [1]. A measure μ is called *tame* if for each $\chi \in \Delta M(G)$, the maximal ideal space of $M(G)$, there is a $\gamma_0 \in \Gamma$ and $a \in \mathbb{C}$ with $|a| \le 1$ such that $\chi_\mu = a\gamma_0$ μ-a.e., where χ_μ is the μ-measurable function on G such that $\chi(\nu) = \int \chi_\mu \, d\nu$ for all $\nu \ll \mu$.

Notice that this implies that $\chi(\gamma\mu) = a\widehat{\mu}(\gamma_0\gamma)$ for all $\gamma \in \Gamma$. Motivated by this observation we propose:

DEFINITION. Given $m \in M(p)$ and $\gamma \in \Gamma$ let γm denote the L^p multiplier defined by $\gamma m(\alpha) = m(\gamma\alpha)$ for $\alpha \in \Gamma$. We will call a multiplier $m \in M(p)$ *tame* if for every $\chi \in \Delta M(p)$, the maximal ideal space of $M(p)$, there exist $\gamma_0 \in \Gamma$ and $|a| \le 1$ such that for all $\gamma \in \Gamma$, $\chi(\gamma m) = am(\gamma_0\gamma)$.

An example of a tame multiplier which is not a measure is a one-sided Riesz product (see Section 1).

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In this paper we study tame multipliers and show interesting similarities to measures. For example, our main theorem (2.2) is that any tame idempotent multiplier on L^p is the Fourier transform of a measure. We obtain estimates on the size of tame multipliers which belong to $M(2,p)$ for some $p>2$ (Section 3). These are similar to estimates obtained in [8] and [5] for measures, and are false for non-tame multipliers.

In Section 4 we prove that $E \subseteq \Gamma$ has the property that every tame multiplier supported on E vanishes at infinity if and only if E does not contain the translate of the support of a one-sided Riesz product, a result which is analogous to the Host and Parreau [14] characterization of Rajchman sets. We also prove a result analogous to their characterization of sets of continuity [13].

One could also define tame multipliers on the Hardy space $H^1(T)$ and we show in Section 5 that any such multiplier is either a measure or an element of c_0.

1. Examples of tame multipliers. Since $M(G) \subseteq M(p)$ for all $1 \le p \le \infty$, any $\chi \in \Delta M(p)$ induces an element in $\Delta M(G)$, and thus any tame measure is a tame multiplier on L^p. A consequence of [6, 10.2.14] is that if $m \in M(p) \cap c_0$ for some $1 \le p < 2$, then m is a tame multiplier on L^q for all $p < q \le 2$.

An example of a multiplier on $L^p(T)$, $1<p<\infty$, which is not tame is $m = 1_{\mathbb{N}}$. This is immediate from Theorem 2.1 but can also be easily proved directly. Just note that if for some increasing sequence of integers $\{n_k\}$ we have $\lim n_k m(n) = am(n_0+n)$ for all n, then setting $n=0$ we see that $a \neq 0$ and $n_0 \in \mathbb{N}$; but evaluating at $-n_0-1$ contradicts this.

Notice that $\Gamma \subseteq \Delta M(p)$ in the sense that $\gamma \in \Gamma$ can be identified with the complex homomorphism (also called γ) which maps the multiplier m to $m(\gamma)$. We will write $\overline{\Gamma}_p$ for the weak* closure of Γ in $\Delta M(p)$.

Recall $\{\gamma_j\}_{j=1}^{\infty} \subseteq \Gamma$ is called *dissociate* if $\prod_{j=1}^{N} \gamma_j^{\varepsilon_j} = 1$ for $\varepsilon_j = 0, \pm 1, \pm 2$ implies $\gamma_j^{\varepsilon_j} = 1$ for all j. Given a dissociate set of characters $\{\gamma_j\}_{j=1}^{\infty}$ and a sequence of complex numbers $\{a_j\}_{j=1}^{\infty}$, define $m : \Gamma \to \mathbb{C}$ by

$$m(\gamma) = \begin{cases} \prod_{k=1}^{N} a_{j_k} & \text{if } \gamma = \prod_{k=1}^{N} \gamma_{j_k}, \\ 0 & \text{else.} \end{cases}$$

We will write $m = \prod (1 + a_j \gamma_j)$ for short. When $|a_j| \le 1$ for all j then $m \in l^{\infty}(\Gamma)$ and hence is a multiplier on L^2, and we will refer to m as a *one-sided Riesz product*. When $\gamma_j^2 = 1$ for all j then m is actually a Riesz product, but if, for example, $\Gamma = \mathbb{Z}$ and $a_j = 1/2$ then m is not a measure. Our first result characterizes the tame one-sided Riesz products which belong to $M(p)$ for some $p \neq 2$.

Proposition 1.1. *Assume $\gamma_j^2 \neq 1$ for any j. Then the one-sided Riesz product m (notation as above) is a tame L^p multiplier for some $1 < p < \infty$, $p \neq 2$, if and only if* $\limsup |a_j| < 1$.

Proof. First we will prove that to be an L^p multiplier for some $p < 2$ it is necessary to have $\limsup |a_j| < 1$. This requires a minor improvement on a result in [12].

Lemma 1.2. *For $|b|$ real and sufficiently small and $|r| \leq 1$,*

$$\Big\|\prod_{j=1}^{N}(1+b\gamma_j+r\bar{b}\gamma_j^{-1})\Big\|_p=\bigg(1+|b^2|\bigg(\frac{1+|r|^2}{2}+\Big(\frac{p}{2}-1\Big)\frac{|1+r|^2}{2}\bigg)+O(|b|^3)\bigg)^N.$$

Proof. It is routine to see that

$$\Big\|\prod_{j=1}^{N}(1+b\gamma_j+r\bar{b}\gamma_j^{-1})\Big\|_p^p=(1+|b^2|(1+|r|^2))^{Np/2}\int\prod_{j=1}^{N}(1+X_j)^{p/2}$$

where

$$X_j=\frac{2\operatorname{Re}\gamma_j(\bar{r}b+b)+2\operatorname{Re}\gamma_j^2|b^2|\bar{r}}{1+|b^2|(1+|r|^2)}\,.$$

When $|b|$ is sufficiently small a Taylor series expansion gives

$$\begin{aligned}\int\prod_{j=1}^{N}(1+X_j)^{p/2}&=\int\prod_{j=1}^{N}\bigg(1+\frac{p}{2}X_j+\frac{p}{2}\Big(\frac{p/2-1}{2}\Big)X_j^2+O(\|X_j\|_\infty^3)\bigg)\\&=\int\prod_{j=1}^{N}\bigg(1+\frac{p}{2}\Big(\frac{p}{2}-1\Big)|\bar{r}b+b|^2+P_j+O(|b|^3)\bigg)\end{aligned}$$

where $P_j=c_j\operatorname{Re}\gamma_j+d_j\operatorname{Re}\gamma_j^2$ for certain coefficients c_j and d_j. Because of the dissociateness condition

$$\int\prod_k P_{j_k}=0$$

(for all but the empty product), thus

$$\begin{aligned}&\Big\|\prod_{j=1}^{N}(1+b\gamma_j+r\bar{b}\gamma_j^{-1})\Big\|_p^p\\&\qquad=(1+|b^2|(1+|r|^2))^{Np/2}\bigg(1+\frac{p}{2}\Big(\frac{p}{2}-1\Big)|\bar{r}b+b|^2+O(|b|^3)\bigg)^N\end{aligned}$$

and one further application of Taylor series completes the proof. ∎

Proof of Proposition 1.1 (ctd.). Suppose $m \in M(p)$ for some $p \neq 2$. Let $t < \limsup |a_j|$ and choose $|a_{j_k}| \geq t$. Let

$$f = \prod_{k=1}^{N} (1 + b\gamma_{j_k} + r\bar{b}\gamma_{j_k}^{-1})$$

with $r = 2/p - 1$. By setting $r = 0$ in the lemma we obtain

$$\|T_m f\|_p = \Big\| \prod_{k=1}^{N} (1 + a_{j_k} b\gamma_{j_k}) \Big\|_p = \prod_{k=1}^{N} (1 + |a_{j_k} b|^2 p/4 + O(|b|^3)) .$$

Combining this estimate with the estimate on $\|f\|_p$ from the lemma, we see that for $|b|$ small

$$\frac{\|T_m f\|_p}{\|f\|_p} \geq \left(\frac{1 + t^2|b|^2 p/4 + O(|b|^3)}{1 + |b|^2/p' + O(|b|^3)} \right)^N$$

and this tends to infinity as $N \to \infty$ unless $t^2 p/4 \leq 1/p'$. But since $m \in M(p)$, the operator norm of m dominates $\|T_m f\|_p / \|f\|_p$, and hence $\lim\sup |a_j| \leq 4/(pp')$.

Now assume $\limsup |a_j| < 1$. It is known [22] that the Riesz product $\mu = \prod (1 + \frac{1}{2}(\gamma_j + \gamma_j^{-1})) \in M(p, 2)$ for some $p < 2$. Choose a positive integer k and constant C so that $|m^k(\gamma)| \leq C|\widehat{\mu}(\gamma)|$ for all $\gamma \in \Gamma$. It follows that $m^k \in M(p, 2)$ and an application of Stein's analytic interpolation theorem (see [10, 1.3] for the details of how the interpolation theorem is applied in this context) proves that $m \in M(q, 2)$ for some $q < 2$. In particular, $m \in M(q)$.

Brown in [1, 5.1] proved that a Riesz product μ satisfying $\limsup |\widehat{\mu}(\gamma)| < 1$ was a tame measure. We can prove that a one-sided Riesz product m with $\limsup |a_j| < 1$ is a tame L^q multiplier for q chosen as above, by appropriately modifying the proof for tameness of Riesz products given in [6, 7.3], replacing $\widehat{\mu}$ there by m. We will briefly outline the necessary changes.

Given a subset Φ of the infinite dissociate set $\Theta = \{\gamma_j\}$ we define

$$m_\Phi \equiv \prod_{\gamma_j \in \Theta \setminus \Phi} (1 + a_j \gamma_j) .$$

For $\chi \in \Delta M(q)$ and $\gamma \in \Gamma$ we define $m_\Phi(\chi\gamma)$ to be $\chi(\gamma m_\Phi)$.

Replace $\Omega(\Phi)$ by

$$\Omega'(\Phi) \equiv \Big\{ \prod_{j=1}^{N} \gamma_j^{\varepsilon_j} : \varepsilon_j = 0, 1,\ \gamma_j \in \Phi,\ N \in \mathbb{N} \Big\} .$$

Analogous to Riesz products, for a finite subset Φ of Θ we have

$$m = \sum \{ m(\gamma)\bar{\gamma} m_\Phi : \gamma \in \Omega'(\Phi) \} .$$

For $M \in M(p,q)$ denote by $\|M\|_{p,q}$ the operator norm. The main point of the proof of the theorem requires knowing that $|m_\Phi(\chi\gamma)|$ is uniformly bounded over all finite subsets Φ. But this is true since $|m_\Phi(\chi\gamma)| \le \|m_\Phi\|_{q,q} \le \|m_\Phi\|_{q,2} \le \|m\|_{q,2} < \infty$. The reader should have no trouble seeing how the remainder of the proof is modified. ∎

Remarks. 1. Given any $1 < q < 2$ there exists $\varepsilon > 0$ such that $\prod(1+\varepsilon(\gamma_j+\gamma_j^{-1}))$ belongs to $M(q,2)$ (cf. [8]), and hence $\prod(1+\varepsilon\gamma_j)$ belongs to $M(q)$ when ε is sufficiently small.

2. When $\limsup|a_j| = 1$ then m is a non-tame multiplier on L^2. To see this consider a weak* cluster point χ of $\{\prod_{k=1}^N \gamma_{j_k}\}_{N=1}^\infty$ where the sequence $J = \{j_k\}$ is chosen so that $\lim_{N\to\infty}\prod_{k=1}^N |a_{j_k}| \ne 0$. If $\chi(\gamma m) = am(\gamma\gamma_0)$ for all $\gamma \in \Gamma$, then since $|\chi(m)| \ne 0$ we must have $a \ne 0$ and γ_0 in the support of m, say $\gamma_0 = \prod_{l=1}^n \gamma_{i_l}$. Choose $j \in J\backslash\{i_1,\ldots,i_n\}$. Then $m(\prod_{k=1}^N \gamma_{j_k}\gamma_j) = 0$ whenever $j_N \ge j$, so $\chi(\gamma_j m) = 0$, while $am(\gamma_0\gamma_j) \ne 0$.

One-sided Riesz products are of fundamental importance in understanding the structure of tame multipliers as our next result shows. We will be using this proposition in both Sections 3 and 4.

We will denote by $M^{\mathrm{t}}(p)$ the set of tame multipliers on L^p.

PROPOSITION 1.3. *Suppose Γ has no elements of order 2 and $m \in M^{\mathrm{t}}(p)\backslash c_0(\Gamma)$. There exists a one-sided Riesz product ϱ and $\gamma_0 \in \Gamma$ with $m(\gamma_0) \ne 0$ such that $|m(\gamma_0)|\,|\varrho(\gamma)| \le |m(\gamma_0\gamma)|$ for all $\gamma \in \Gamma$. In particular, a translate of the support of m contains the support of a one-sided Riesz product.*

Remark. We have no reason to believe this result is not true if Γ has elements of order 2, however, in the proof we use the fact that if Γ has no elements of order 2 then any infinite subset of Γ contains an infinite dissociate set.

Proof. Choose $\chi \in \overline{\Gamma}_p$ with $\chi(m) \ne 0$. Assume $\chi(\gamma m) = am(\gamma_0\gamma)$ for all $\gamma \in \Gamma$ and suppose the net $\{\gamma_\alpha\} \subseteq \Gamma$ converges weak* to χ in $\Delta M(p)$. Denote the multiplier $\gamma_0 m$ by m_1. As $\chi(\gamma_0^{-1}m_1) = am_1(1) \ne 0$, if $0 < \varepsilon < |a|$ is fixed, we may choose $\gamma_{j_1} \in \{\gamma_\alpha\}$, $\gamma_{j_1} \ne \gamma_0$, such that $|m_1(\gamma_0^{-1}\gamma_{j_1})| \ge (|a|-\varepsilon)|m_1(1)|$. Now assume we have inductively constructed a dissociate set

$$\{\gamma_0^{-1}\gamma_{j_1},\ldots,\gamma_0^{-1}\gamma_{j_n}\} \subseteq \{\gamma_0^{-1}\gamma_\alpha\}\,.$$

such that

$$\Big|m_1\Big(\prod_{i=1}^n(\gamma_0^{-1}\gamma_{j_i})^{\varepsilon_i}\Big)\Big| \ge (|a|-\varepsilon)^k|m_1(1)|$$

whenever $\varepsilon_i = 0$ or 1 and $\sum_{i=1}^n \varepsilon_i = k$.

For each $(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1\}^n$ we have the inequality

$$\Big|\chi\Big(\gamma_0^{-1}\prod_{i=1}^{n}(\gamma_0^{-1}\gamma_{j_i})^{\varepsilon_i}m_1\Big)\Big| = \Big|am_1\Big(\prod_{i=1}^{n}(\gamma_0^{-1}\gamma_{j_i})^{\varepsilon_i}\Big)\Big| \geq |a|(|a|-\varepsilon)^{\Sigma_{i=1}^{n}\varepsilon_i}|m_1(1)|\,.$$

Thus we can choose $\gamma_{j_{n+1}}\in\{\gamma_\alpha\}$ so that

$$\Big|m_1\Big(\gamma_0^{-1}\gamma_{j_{n+1}}\prod_{i=1}^{n}(\gamma_0^{-1}\gamma_{j_i})^{\varepsilon_i}\Big)\Big| \geq (|a|-\varepsilon)(|a|-\varepsilon)^{\Sigma_{i=1}^{n}\varepsilon_i}|m_1(1)|$$

for all $(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1\}^n$, and so that the sequence $\{\gamma_0^{-1}\gamma_{j_1},\ldots,\gamma_0^{-1}\gamma_{j_{n+1}}\}$ is dissociate. This completes the induction step. Then

$$\varrho=\prod_i(1+(|a|-\varepsilon)\gamma_0^{-1}\gamma_{j_i})$$

is the one-sided Riesz product which works. ■

2. Tame ε-idempotent multipliers

DEFINITION. An L^p multiplier m is called *ε-idempotent* ($\varepsilon<1/2$) if for every $\gamma\in\Gamma$, either $|m(\gamma)-1|\leq\varepsilon$ or $|m(\gamma)|\leq\varepsilon$.

We will denote by $E(m)$ the set $\{\gamma:|m(\gamma)|>\varepsilon\}$.

The celebrated Cohen Idempotent Theorem [2] states that the characteristic function of a set $E\subseteq\Gamma$ is the Fourier transform of an idempotent measure if and only if E belongs to the coset ring of Γ, the Boolean ring generated by all cosets of subgroups of Γ. This was later generalized to ε-idempotent measures μ by Méla [17] who proved that if the norm of μ was small enough then $E(\widehat{\mu})$ belonged to the coset ring. The purpose of this section is to prove a similar result for tame ε-idempotent multipliers. Our proof was inspired by the paper of Ramsey and Wells [20] on strongly continuous ε-idempotent measures.

THEOREM 2.1. *If m is a tame ε-idempotent multiplier on L^p with $\varepsilon<1/3$, then $E(m)$ is a finite union of cosets of a subgroup of Γ.*

Combined with Cohen's theorem we immediately have

COROLLARY 2.2. *If m is a tame idempotent multiplier on L^p then m is* (*the Fourier transform of*) *a measure.*

Remark. Without tameness such a result is false of course. Consider for example $m=1_{\mathbb{N}}$.

We need some preliminary ideas first.

DEFINITION. Recall that $m\in l^\infty(\Gamma)$ is called *weakly almost periodic* (*wap*) if Γm is relatively weakly compact in $l^\infty(\Gamma)$.

LEMMA 2.3. *A tame multiplier is wap.*

Proof. We verify the Grothendieck criterion [7]. Assume that both $\lim_i \lim_j m(\gamma_i\alpha_j)$ and $\lim_j \lim_i m(\gamma_i\alpha_j)$ exist. Let χ and ψ be weak* cluster points in $\Delta M(p)$ of $\{\gamma_i\}$ and $\{\alpha_j\}$ respectively. Because of the tameness of m there exist $a, b \in \mathbb{C}$ and $\alpha_0, \gamma_0 \in \Gamma$ such that $\lim_i \lim_j m(\gamma_i\alpha_j) = \lim_i \psi(\gamma_i m)$ $= a \lim_i m(\gamma_i\alpha_0) = a\chi(\alpha_0 m) = abm(\alpha_0\gamma_0)$. A similar argument gives the same answer for $\lim_j \lim_i m(\gamma_i\alpha_j)$. ∎

Next we introduce an idea from the geometry of Banach spaces.

DEFINITION. A subset D of a Banach space is called *dentable* if for every $\varepsilon > 0$ there exists an $x \in D$ which does not belong to $\overline{\mathrm{co}}(D\backslash B_\varepsilon(x))$, the closed convex hull of $D\backslash B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y : \|y - x\| < \varepsilon\}$.

LEMMA 2.4. *If $m \in l^\infty(\Gamma)$ is wap, $E \subseteq \Gamma$ and Em is weakly compact in $l^\infty(\Gamma)$, then Em is norm compact.*

Proof. Let $\{\gamma_\alpha m\}$ be a net in Em and take a weakly convergent subnet, say $\{\gamma_\beta m\}$ with limit $\gamma m \in Em$.

As m is a wap multiplier Γm is relatively weakly compact in $l^\infty(\Gamma)$. Certainly Γm is a bounded subset of $l^\infty(\Gamma)$ and hence it is dentable in $l^\infty(\Gamma)$ [3, p. 138]. Thus for each $\varepsilon > 0$ there is a point $\gamma_0 m \in \Gamma m$ which is not in $\overline{\mathrm{co}}(\Gamma m\backslash B_\varepsilon(\gamma_0 m))$. A translation argument proves that

$$\gamma m \notin \overline{\mathrm{co}}(\Gamma m\backslash B_\varepsilon(\gamma m)) \equiv C\,.$$

Applying a separation theorem we can find $f \in l^\infty(\Gamma)^*$ such that

$$\operatorname{Re} f(\gamma m) < t \le \inf_{s\in C} \operatorname{Re} f(s)\,.$$

Our converging subnet is eventually in the weakly open neighbourhood $\{w \in l^\infty : \operatorname{Re} f(w) < t\}$ of γm. Thus eventually $\operatorname{Re} f(\gamma_\beta m) < t$ and so $\gamma_\beta m \notin C$. This implies that $\gamma_\beta m$ belongs to $B_\varepsilon(\gamma m)$ eventually and as this holds for all $\varepsilon > 0$ the subnet $\{\gamma_\beta m\}$ is converging in norm to γm. ∎

Proof of Theorem 2.1. Denote by E the set $E(m)$ and let 1_E denote the characteristic function of E. If $\chi \in \Delta M(2)$ then χ restricted to $M(p)$ is an element of $\Delta M(p)$, thus there is some $a \in \mathbb{C}$ and $\gamma_0 \in \Gamma$ with $\chi(\gamma m) = am(\gamma_0\gamma)$ for all $\gamma \in \Gamma$. As $\Delta M(2) = \overline{\Gamma}_2$ ([21]) we can find $\{\gamma_\alpha\} \subseteq \Gamma$ converging weak* in $\Delta M(2)$ to χ.

The ε-idempotency of m ensures that if $\gamma \notin \gamma_0^{-1}E$ then $\gamma_\alpha\gamma \notin E$ eventually, and if there is some $\gamma \in \gamma_0^{-1}E$ with $\gamma_\alpha\gamma \notin E$ eventually, then $\gamma_\alpha\tau \notin E$ eventually for all $\tau \in \Gamma$. In this case $\chi(\tau 1_E) = 0 = 0 \cdot 1_E(\gamma_0\tau)$ for all τ, otherwise $\chi(\tau 1_E) = 1_E(\gamma_0\tau)$. Thus 1_E is a tame, idempotent multiplier on L^2.

Let $\{\tau_\alpha 1_E\}$ be a net in $E1_E$. As 1_E is wap, $E1_E$ is relatively weakly compact in $l^\infty(\Gamma)$, thus it is possible to find a net $\{\tau_\beta\}$ with $\tau_\beta 1_E \to w$

weakly in $l^\infty(\Gamma)$ and $\tau_\beta \to \psi$ weak* in $\Delta M(2)$. Since $\{\tau_\beta\} \subseteq E$ and 1_E is tame and idempotent there is some $\gamma_0 \in E$ with $\psi(\gamma 1_E) = \lim 1_E(\tau_\beta \gamma) = 1_E(\gamma_0\gamma)$ for all $\gamma \in \Gamma$. But evaluation at $\gamma \in \Gamma$ is a continuous linear functional on $l^\infty(\Gamma)$ and thus $w = \gamma_0 1_E \in E1_E$. Hence $E1_E$ is weakly compact and so is norm compact by the lemma.

Finally, a norm compactness argument proves that the equivalence relation, $\gamma_1 \sim \gamma_2$ if $\gamma_1 1_E = \gamma_2 1_E$, partitions E into finitely many equivalence classes. Each of these is clearly a translate of the subgroup $\{\gamma \in \Gamma : \gamma 1_E = 1_E\}$. ∎

Remarks. 1. The same argument works if m is a tame L^p multiplier with the property that for all $\gamma \in \Gamma$ either $|m(\gamma)| \le \delta_0$ or $\delta_1 \le |m(\delta)| \le 1$ where $\delta_1^2 > \delta_0$. This is the best possible result since when $m = \prod_{n=1}^\infty (1 + \varepsilon e^{i3^n x})$, then the set $\{n : |m(n)| \ge \varepsilon\}$ is not a union of finitely many arithmetic progressions.

2. We thank the referee for pointing out a simplification in our original proof.

Definition. A multiplier m is called *quasi-idempotent* if there is some $\delta > 0$ so that $\operatorname{supp} m = \{\gamma : |m(\gamma)| \ge \delta\}$.

Corollary 2.5. *If m is a tame quasi-idempotent multiplier then $\{\gamma : |m(\gamma)| > 0\}$ is a union of finitely many cosets of a subgroup of Γ.*

Proof. Apply the previous remark with $\delta_0 = 0$. ∎

3. L^p-improving tame multipliers

Definition. If $m \in M(2, q)$ for some $q > 2$ then m is called *L^p-improving.*

Examples of L^p-improving measures include the Cantor–Lebesgue measure [18], and most Riesz products [22]. For background information and basic properties of L^p-improving measures see [5]. L^p-improving multipliers have been characterized in terms of the "size" of the sets $\{\gamma : |m(\gamma)| > \varepsilon\}$ as $\varepsilon \to 0$ ([8], [10]).

Many other properties of L^p-improving multipliers are known. For example, if a measure μ maps L^2 to L^p then $\limsup_{\gamma\in\Gamma} |\widehat{\mu}(\gamma)|^2 \le (2/p)\|\mu\|^2_{M(G)}$ ([8]). Such an estimate is not true for multipliers. Indeed, it need not even be the case that $\limsup_{\gamma\in\Gamma} |m(\gamma)| < 1$ for $m \in M(2, p)$ for all $p > 2$, as the example $m = 1_E$ for E an infinite Sidon set illustrates. However, for tame multipliers there is a similar estimate.

Proposition 3.1. *Suppose Γ has no elements of order 2, and $m \in M^{\rm t}(p) \cap M(2, p)$. Then $|\chi(m)|^2 \le (2/p)\|m\|^2_{l^\infty}$ for all $\chi \in \overline{\Gamma}_p \backslash \Gamma$.*

Proof. Assume there is some $\chi \in \overline{\Gamma}_p \setminus \Gamma$ with $\chi(m) = am(\gamma_0) \neq 0$ and construct for $\varepsilon > 0$ the one-sided Riesz product ϱ as in the proof of Proposition 1.3. Since $\gamma_0 m \in M(2,p) = M(p',2)$ where $1/p' + 1/p = 1$, it is clear that $\varrho \in M(2,p)$.

By [12, 1.5]

$$2/p \geq \limsup_{\gamma \in \Gamma} |\varrho(\gamma)|^2 = (|a| - \varepsilon)^2 .$$

Let $\varepsilon \to 0$ to finish the proof. ∎

From this we easily get the following interesting results when Γ has no elements of order 2.

Corollary 3.2. *If* $m \in M^{\mathrm{t}}(p) \cap M(2,p)$ *then* $\limsup_{\gamma \in \Gamma} |m(\gamma)|^2 \leq (2/p)\|m\|_{l^\infty}^2$.

Remark. The better estimate $\limsup_{\gamma \in \Gamma} |\widehat{\mu}(\gamma)|^2 \leq \|\mu\|_{M(G)}^2/(p-1)$ is known for tame measures in $M(2,p)$ [12, 1.3], but for tame multipliers our result is best possible since the one-sided Riesz product $\prod (1 + (\sqrt{2}/\sqrt{p}) e^{ix_j})$ defined on T^∞ belongs to $M(2,p)$ [12, 2.3].

Corollary 3.3. *If m is a tame multiplier on L^p for all $p > 2$ and* $m \in \bigcap_{p>2} M(2,p)$ *then* $m \in c_0$.

Corollary 3.4. *If m is a tame multiplier on L^q for all $1 < q < 2$ and* $m \in M(p,q)$ *for all* $1 < p < q < 2$ *then* $m \in c_0$.

Proof. An interpolation argument proves $m \in M^{\mathrm{t}}(s)$ for all $s > 2$. ∎

4. Tame Rajchman sets

Definition. Recall that a subset E of Γ is called a *Rajchman set* if for all $\mu \in M(G)$, $\limsup_{\gamma \in E^{\mathrm{c}}} |\widehat{\mu}(\gamma)| = 0$ implies $\limsup_{\gamma \in \Gamma} |\widehat{\mu}(\gamma)| = 0$.

The classical result of Rajchman [19] to the effect that $\mathbb{Z}^+$ and $\mathbb{Z}^-$ are Rajchman sets inspired this definition. A beautiful result of Host and Parreau characterizes Rajchman sets.

Theorem [14]. *A subset E of Γ is a Rajchman set if and only if E does not contain any translate of the support of a Riesz product.*

There is a similar result for tame multipliers, with one-sided Riesz products replacing Riesz products, in the case when Γ has no elements of order 2.

Theorem 4.1. *Assume Γ has no elements of order* 2. *The following are equivalent*:

(1) *For all* $1 < p < \infty$ *and for all* $m \in M^{\mathrm{t}}(p)$, *if* $\limsup_{\gamma \in E^{\mathrm{c}}} |m(\gamma)| = 0$, *then* $\limsup_{\gamma \in \Gamma} |m(\gamma)| = 0$;

(2) *For all* $1 < p < \infty$ *and for all* $m \in M^{\mathrm{t}}(p)$, *if* $m = 0$ *on* E^{c} *then* $\limsup_{\gamma \in \Gamma} |m(\gamma)| = 0$;

(3) *For some* $1 < p < \infty$ *and for all* $m \in M^{\mathrm{t}}(p)$, *if* $m = 0$ *on* E^{c} *then* $\limsup_{\gamma\in\Gamma} |m(\gamma)| = 0$;

(4) E *does not contain any translate of the support of a one-sided Riesz product.*

Proof. (1)⇒(2) and (2)⇒(3) are trivial.

(3)⇒(4). If (4) fails then any translated one-sided Riesz product $m \in M(p)$ supported on E with $\limsup_{\gamma\in\Gamma} |m(\gamma)| > 0$ gives a contradiction of (3).

(4)⇒(1). Fix p. Suppose there is an $m \in M^{\mathrm{t}}(p)$ with $\limsup_{\gamma\in E^{\mathrm{c}}} |m(\gamma)| = 0$ but $\limsup_{\gamma\in\Gamma} |m(\gamma)| \neq 0$. From Proposition 1.3 we can find $\gamma_0 \in \Gamma$ with $m(\gamma_0) \neq 0$, a dissociate set $\{\gamma_j\} \subseteq \Gamma$ and a constant $\delta > 0$ such that whenever $\varepsilon_j \in \{0, 1\}$ then

$$\Big|m\Big(\gamma_0 \prod \gamma_j^{\varepsilon_j}\Big)\Big| \geq \frac{\delta^{\Sigma\varepsilon_j}}{|m(\gamma_0)|}\,.$$

Note that $\gamma_0\gamma_j \in E$ for some j, say j_0, for otherwise

$$\limsup_{\gamma\in E^{\mathrm{c}}} |m(\gamma)| \geq \limsup_{j} |m(\gamma_0\gamma_j)| \geq \frac{\delta}{|m(\gamma_0)|} > 0\,.$$

A similar argument shows we may inductively pick $\{\gamma_{j_i}\}_{i=0}^{\infty} \subseteq \{\gamma_j\}$ with $\{j_i\}$ increasing and

$$\gamma_0\gamma_{j_0} \prod_{i=1}^{n} \gamma_{j_i}^{\varepsilon_i} \in E$$

for all $\varepsilon_i = 0, 1$ and $n \in \mathbb{N}$, contradicting (4). ∎

Remark. As usual these results fail without tameness. Consider $E = \{3^n\} \subseteq \mathbb{Z}$ and $m = 1_E$.

Call a set E satisfying these equivalent properties a *tame Rajchman set.* We do not know if the union of two tame Rajchman sets is another such set. It is the case that the union of a tame Rajchman set and a $\Lambda(p)$ set is another tame Rajchman set. Just argue as in [11, Proof of Theorem A] replacing Proposition 1.1 there by [9, 2.2].

We can also use Proposition 1.3 to prove a result analogous to Host and Parreau's characterization of sets of continuity [13].

THEOREM 4.2. *Assume* Γ *has no elements of order* 2. *The following are equivalent*:

(1) *For each* $1 < p < \infty$ *and for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $m \in M^{\mathrm{t}}(p)$, $\|m\|_{l^\infty} \leq 1$ *and* $\limsup_{\gamma\in E^{\mathrm{c}}} |m(\gamma)| < \delta$, *then* $\limsup_{\gamma\in E} |m(\gamma)| < \varepsilon$;

(2) *For some* $1 < p < \infty$ *and for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $m \in M^{\mathrm{t}}(p)$, $\|m\|_{l^\infty} \le 1$ *and* $\limsup_{\gamma\in E^{\mathrm{c}}} |m(\gamma)| < \delta$, *then* $\limsup_{\gamma\in E} |m(\gamma)| < \varepsilon$;

(3) *For some positive integer* n, E *does not contain*

$$\gamma\theta_n(\{\gamma_j\}) \equiv \Big\{ \prod \gamma_j^{\varepsilon_j} : \varepsilon_j = 0,1 \ \textit{ for all } j, \textit{ and } \sum \varepsilon_j \le n \Big\}$$

for any $\gamma \in \Gamma$ *and infinite dissociate set* $\{\gamma_j\}$.

Proof. (1)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(3). Suppose $E \supseteq \gamma_0\theta_n(\{\gamma_j\})$ and choose $\varepsilon > 0$ so that $m = \gamma_0 \prod(1+\varepsilon\gamma_j) \in M^{\mathrm{t}}(p)$. Then $\limsup_{\gamma\in E^{\mathrm{c}}} |m(\gamma)| \le \varepsilon^{n+1}$ but $\limsup_{\gamma\in E} |m(\gamma)| = \varepsilon$.

(3)$\Rightarrow$(1). Suppose (1) fails. Then for some $\varepsilon > 0$ and each $n \in \mathbb{N}$ there is a tame L^p multiplier m with $\|m\|_{l^\infty} \le 1, \limsup_{\gamma\in E^{\mathrm{c}}} |m(\gamma)| < \varepsilon^{n+1}$ but $\limsup_{\gamma\in E} |m(\gamma)| > \varepsilon$. From the latter property we deduce the existence of $\chi \in \overline{\Gamma}_p \backslash \Gamma$ such that $|\chi(m)| > \varepsilon$. Assume $\chi(\gamma m) = am(\gamma_0\gamma)$ for all $\gamma \in \Gamma$. Since $\|m\| \le 1$ we have $|a| > \varepsilon$, and as $|a| \le 1$, $|m(\gamma_0)| > \varepsilon$. From the proof of Proposition 1.3 we see that the one-sided Riesz product $\varrho = \prod(1 + \varepsilon\gamma_j)$ (built on some appropriate dissociate set $\{\gamma_j\}$) satisfies $|m(\gamma_0)\varrho(\gamma)| \le |m(\gamma_0\gamma)|$ for all $\gamma \in \Gamma$. It follows that if $\gamma \in \theta_n(\{\gamma_j\})$ then $|m(\gamma_0\gamma)| \ge \varepsilon^{n+1}$, and so only finitely many words in $\gamma_0\theta_n(\{\gamma_j\})$ can belong to E^{c}. After removing the finitely many γ_j on which these words are built we conclude that $\gamma_0\theta_n(\{\gamma_j\}_{j=k}^\infty) \subseteq E$ for some k, contradicting (3). ∎

5. Tame H^1 multipliers. One could similarly define tame multipliers on $H^1(T)$, however, these turn out to be trivial.

PROPOSITION 5.1. *Any tame multiplier on* H^1 *is either a measure or it belongs to* c_0.

Proof. Assume the tame multiplier $m \notin c_0$. Choose an increasing sequence of positive integers $\{n_k\}$ with $|m(n_k)| \ge \delta > 0$. As in [15] consider $g_k(x) = e^{-in_k x} m(e^{in_k x} F_{n_k}(x))$ where F_n is the nth Fejér kernel. Since $\|g_k\|_{L^1} \le \|m\|_{H^1,H^1}$ we can find a weak* converging subsequence (not renamed) converging to $\mu \in M(T)$. Clearly $m(n_k + j) \to \widehat{\mu}(j)$ for all $j \in \mathbb{Z}$.

Take a further subnet of $\{n_k\}$ converging weak* in $\Delta M(H^1)$. As m is tame it follows that $\widehat{\mu}(j) = am(n_0 + j)$ for some $a \in \mathbb{C}$, $n_0 \in \mathbb{Z}$. Since $\widehat{\mu}(0) = \lim m(n_k) \ne 0$, we have $a \ne 0$, and thus m is the Fourier transform of the measure $(1/a)e^{in_0 x}\mu$. ∎

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