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SINGULAR INTEGRALS WITH HIGHLY OSCILLATING KERNELS ON THE PRODUCT DOMAINS

BY

LUNG-KEE CHEN (CORVALLIS, OREGON)

I. Introduction. The theory of singular integrals on product domains has been studied by several authors, e.g. [2], [3], [6], [7]. One of its applications is to the problem of almost everywhere convergence of double Fourier series (see [5]). For example, let f be in $L^p([-\pi,\pi]\times[-\pi,\pi])$, $p>1$, and let

$$S_{M,M^2}f(x,y)=\sum_{|n|\le M,\ |m|\le M^2} a_{n,m}e^{i(nx+my)}$$

be a partial sum of its Fourier series. Define a singular integral with highly oscillatory kernel,

$$L_1f(x,y)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{i(N(x,y)x'+N^2(x,y)y')}}{x'y'}f(x-x',y-y')\,dx'\,dy'\,,$$

where $N(x,y)$ is any real-valued measurable function on $\mathbb{R}^2$.

To show the convergence of the partial sums $S_{M,M^2}f$, it suffices to show the boundedness of the above singular integral L_1f, i.e. to show that there exists a constant C_p, depending only on p, such that

$$\|L_1f\|_p\le C_p\|f\|_p\,.$$

Here, we should remark that the convergence of S_{M,M^2} has been proved by C. Fefferman [1] if $p\ge 2$.

Let us look at two special cases of the operator L_1f. Suppose the function $N(x,y)$ is in $C^1(\mathbb{R}^2)$ and there exist three "large" positive constants A, B, C such that $A/2\le N(x,y)\le A$, $B/2\le\partial_xN\le B$ and $C/2\le\partial_yN\le C$. This case leads to the study of the singular integral with oscillating kernel (for more details, see [5])

$$L_2f(x,y)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{iN(y)x'}}{x'y'}f(x-x',y-y')\,dx'\,dy'\,.$$

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This operator is easily seen to be the double Hilbert transform. On the other hand [5], the case $N(x,y) = \lambda x y^\beta$ where λ is a large number, $\beta \geq 1$, leads us to consider a more general singular integral with variable integration domains, $\{D_y\}_{y\in\mathbb{R}}$,

$$L_3 f(x,y) = \iint\limits_{(x',y')\in D_y} \frac{1}{x'y'} f(x-x', y-y')\, dx'\, dy' ,$$

where D_y is a region symmetrical with respect to the x' and y' axes (the definition of D_y will be given later).

The motivations for our research stem basically from those two operators $L_2 f$ and $L_3 f$. In this paper, we would like to consider the boundedness of a singular integral with oscillating kernel and variable integration domains on a product domain.

Throughout this paper, we suppose $f(x,y) \in L^p(\mathbb{R}^2) \cap C_0^\infty(\mathbb{R}^2)$. For each $y \in \mathbb{R}$, let $\widehat{f}(\xi, y)$ denote the Fourier transform of f with respect to the first variable. Let $\|f(x,y)\|_{L^p(x)}$ and $\|f(x,y)\|_{L^p(y)}$ denote the L^p norms in the first and second variable, respectively, and let $\|f(x,y)\|_{L^p(x,y)}$ be the usual $L^p(\mathbb{R}^2)$ norm. C will denote some constants which may depend on p and may change at different occurrences.

Let

$$Tf(x,y) = \text{p.v.} \iint\limits_{D_y} \frac{e^{iN(y)x'}}{x'y'} f(x-x', y-y')\, dy'\, dx'$$

and consider the associated maximal singular integral

$$T^* f(x,y) = \sup_{\varepsilon>0} \left| \iint\limits_{D_y,\ |x'|>\varepsilon} \frac{e^{iN(y)x'}}{x'y'} f(x-x', y-y')\, dy'\, dx' \right| ,$$

where N is any real-valued measurable function defined on $\mathbb{R}$ and the definition of the domains $\{D_y\}_{y\in\mathbb{R}}$ is given below.

For any two fixed numbers, $a > 1$, $b > 1$, take two non-negative smooth functions ψ and ϕ with compact supports in $\{1/a < r < a^2\}$ and $\{1/b < r < b^2\}$, respectively, such that

$$\sum_{h\in\mathbb{Z}} \psi(a^h r) = \sum_{k\in\mathbb{Z}} \phi(b^k r) = 1$$

for all $r > 0$. Let δ be a measurable function from $\mathbb{Z}\times\mathbb{R}$ to $\mathbb{R}^+$, i.e. $\delta(h,y) \geq 0$, $h \in \mathbb{Z}$, $y \in \mathbb{R}$. Define a family of measurable sets $\{D_y\}_{y\in\mathbb{R}}$ by

$$D_y = \Big\{ (x',y') \in \mathbb{R}^2 \Big| \sum_{(h,k)\in B_y} \psi(a^h|x'|)\phi(b^k|y'|) \neq 0 \Big\}$$

where $B_y = \{(h,k) \mid b^{-k} \leq \delta(h,y)\}$.

THEOREM. *For every measurable function $N(y)$ and the family of measurable sets $\{D_y\}_{y\in\mathbb{R}}$, $1<p<\infty$, there exists a constant C_p independent of f such that*

(i) $\|Tf\|_p \le C_p\|f\|_p$,
(ii) $\|T^*f\|_p \le C_p\|f\|_p$.

In the $p=2$ case, those operators have been studied by E. Prestini (see [7]).

II. Proof of Theorem. We need only show (i), since (ii) then follows from

LEMMA [7]. *Under the hypotheses of the Theorem, there exists a constant C such that*

$$T^*f(x,y) \le C\{M_xH_y^M f(x,y) + M_xTf(x,y)\}$$

where M_x denotes the classical Hardy–Littlewood maximal operator acting on x and H_y^M denotes the associated maximal Hilbert transform acting on y, i.e.

$$M_xf(x,y) = \sup_{\varepsilon>0}\frac{1}{\varepsilon}\int_{|x'|<\varepsilon}|f(x-x',y)|\,dx'$$

and

$$H_y^M f(x,y) = \sup_{\varepsilon>0}\Big|\int_{|y'|>\varepsilon} f(x,y-y')\,\frac{dy'}{y'}\Big|.$$

Without loss of generality, one assumes $a=b=2$. Then

$$\begin{aligned}Tf(x,y) &= \iint \sum_{(h,k)\in B_y} e^{iN(y)x'}\frac{\psi(2^h|x'|)}{x'}\frac{\phi(2^k|y'|)}{y'}f(x-x',y-y')\,dy'\,dx'\\ &\equiv \iint \sum_{(h,k)\in B_y} e^{iN(y)x'}\Psi_h(x')\Phi_k(y')f(x-x',y-y')\,dy'\,dx'\\ &= \int_{\mathbb{R}}\sum_{h\in\mathbb{Z}} e^{iN(y)x'}\Psi_h(x')\int_{\mathbb{R}}\sum_{2^{-k}\le\delta(h,y)}\Phi_k(y')f(x-x',y-y')\,dy'\,dx',\end{aligned}$$

where

$$\Psi_h(x') = \frac{\psi(2^h|x'|)}{x'},\qquad \Phi_k(y') = \frac{\phi(2^k|y'|)}{y'}.$$

Remark 1. Clearly, Ψ_h and Φ_k have the following properties:

(i) $\widehat{\Psi}_h(\xi) = \widehat{\Psi}_0(\xi/2^h)$,
(ii) $\widehat{\Psi}_h(0) = 0$,
(iii) $\widehat{\Psi}_0(\xi) \le C_l/|\xi|^l$ for any non-negative integer l,

(iv) $\widehat{\Psi}_0(\xi) \le C|\xi|$,
(v) Φ has the same properties (i)–(iv).

Let us make a partition of unity, i.e. take a non-negative function $p \in C_0^\infty(\mathbb{R})$ with compact support contained in the set $\{1/4 < |\xi| < 2\}$ such that $\sum_{j\in\mathbb{Z}} p^2(2^{-j}|\xi|) = 1$ for all $\xi \in \mathbb{R}$, $\xi \neq 0$. For each y, define partial sum operators

$$\widehat{S_j f}(\xi, y) = p(2^{-j}|\xi - N(y)|)\widehat{f}(\xi, y)$$

where the Fourier transform acts on the first variable. Obviously, for every $h \in \mathbb{Z}$,

$$\sum_j S_{j+h}^2 f(x,y) \equiv \sum_j S_{j+h} S_{j+h} f(x,y) = f(x,y)\,,$$

in the sense of L^2 convergence. Let

$$\widehat{S_j^+ g}(\xi, y) = p(2^{-j}|\xi|)\widehat{g}(\xi, y)\,.$$

Since

$$\widehat{S_j f}(\xi + N(y), y) = p(2^{-j}|\xi|)\widehat{f}(\xi + N(y), y)$$

and

$$\widehat{S_j^2 f}(\xi + N(y), y) = p^2(2^{-j}|\xi|)\widehat{f}(\xi + N(y), y)\,,$$

one has

$$\begin{aligned} S_j f(x,y) &= e^{iN(y)x} S_j^+(e^{-iN(y)(\cdot)} f(\cdot, y))(x) \\ &= S_j^+(e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot, y))(x) \end{aligned}$$

and

$$S_j^2 f(x,y) = S_j^+ S_j^+(e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot, y))(x)\,.$$

Therefore, for each fixed $y \in \mathbb{R}$, by the Littlewood–Paley Theorem (see [8]),

$$\begin{aligned} &\Big\| \Big(\sum_j |S_j f|^2 \Big)^{1/2} \Big\|_{L^p(x)} \\ &\qquad = \Big\| \Big(\sum_j |S_j^+(e^{-iN(y)(\cdot)} f(\cdot, y))|^2 \Big)^{1/2} \Big\|_{L^p(x)} \approx \|f(x,y)\|_{L^p(x)}\,. \end{aligned}$$

Now, integrating both sides with respect to y, one has

$$\begin{aligned} (1)\quad &\Big\| \Big(\sum_j |S_j f|^2 \Big)^{1/2} \Big\|_{L^p(x,y)} \\ &\qquad = \Big\| \Big(\sum_j |S_j^+(e^{-iN(y)(\cdot)} f(\cdot, y))|^2 \Big)^{1/2} \Big\|_{L^p(x,y)} \approx \|f\|_{L^p(x,y)} \end{aligned}$$

for $1 < p < \infty$.

Let us write

$$
\begin{aligned}
Tf(x,y) &= \sum_{(h,k)\in B_y} [e^{iN(y)x'}\Psi_h(x')\Phi_k(y')] * \Big(\sum_j S_{j+h}^2 f(x,y)\Big) \\
&= \sum_{(h,k)\in B_y} [e^{iN(y)x'}\Psi_h(x')\Phi_k(y')] \\
&\quad * \Big[\sum_j (S_{j+h}^+ S_{j+h}^+ (e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot,y))(x))\Big] \\
&= \sum_j \Big\{ \sum_{(h,k)\in B_y} S_{j+h}^+ [(e^{iN(y)x'}\Psi_h(x')\Phi_k(y')) \\
&\quad * (S_{j+h}^+ (e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot,y))(x))]\Big\} \\
&\equiv \sum_j T_j f(x,y)\,.
\end{aligned}
$$

We rewrite $T_j f(x,y)$ as

$$
\sum_{h\in\mathbb{Z}} S_{j+h}^+ \Big\{ (e^{iN(y)x'}\Psi_h(x')) *_1 \Big[\sum_{2^{-k}\le\delta(h,y)} \Phi_k(y') *_2 (S_{j+h}^+ (e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot,y))(x))\Big]\Big\}
$$

where $*_1$ and $*_2$ denote the convolution operators acting on the first and second variables, respectively, and the variable index of the sum $\sum_{2^{-k}\le\delta(h,y)}$ is k.

By the Littlewood–Paley Theorem, it follows that the $L^p(x)$ norm of $T_j f$, $\|T_j f(\cdot,y)\|_{L^p(x)}$, is dominated by

$$
\Big\| \Big[\sum_{h\in\mathbb{Z}} \Big| (e^{iN(y)x'}\Psi_h(x')) *_1 \Big(\sum_{2^{-k}\le\delta(h,y)} \Phi_k(y') *_2 (S_{j+h}^+ (e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot,y))(x))\Big)\Big|^2\Big]^{1/2}\Big\|_{L^p(x)}\,. \tag{2}
$$

Define

$$
\begin{aligned}
g_{\{y,h,j\}}(x) &= \sum_{2^{-k}\le\delta(h,y)} \Phi_k(y') *_2 (S_{j+h}^+ (e^{iN(y)x} e^{-iN(y)(\cdot)} f(\cdot,y))(x)) \\
&= \sum_{2^{-k}\le\delta(h,y)} \Phi_k(y') *_2 S_{j+h} f(x,y)\,.
\end{aligned}
$$

Hence, by (2),

$$
\|T_j f(x,y)\|_{L^p(x)} \le C \Big\| \Big[\sum_{h\in\mathbb{Z}} |(e^{iN(y)x'}\Psi_h(x')) *_1 g_{\{y,h,j\}}(x)|^2\Big]^{1/2}\Big\|_{L^p(x)}. \tag{3}
$$

R e m a r k 2. It is clear that

$$|g_{\{y,h,j\}}(x)| \le \sup_l \Big| \sum_{2^{-k}\le l} \Phi_k(y') *_2 (S^+_{j+h}(e^{iN(y)x}e^{-iN(y)(\cdot)}f(\cdot,y))(x)) \Big|$$
$$\le 2H^M_y(S^+_{j+h}(e^{iN(y)x}e^{-iN(y)(\cdot)}f(\cdot,y))(x))$$

where H^M_y is the associated maximal Hilbert transform acting on the second variable.

The proof of the Theorem is now divided into three parts, according as $p=2$, $2<p<\infty$, and $1<p<2$.

For the first part, $p=2$, applying Plancherel's Theorem to the right hand side of (3), one has

$$\|T_jf(x,y)\|_{L^2(x)} \le C\Big\|\Big[\sum_{h\in\mathbb{Z}}|\widehat{\Psi}_h(\xi-N(y))\widehat{g}_{\{y,h,j\}}(\xi)|^2\Big]^{1/2}\Big\|_{L^2(\xi)}$$
$$= C\Big\|\Big[\sum_{h\in\mathbb{Z}}|\widehat{\Psi}_h(\xi)\widehat{g}_{\{y,h,j\}}(\xi+N(y))|^2\Big]^{1/2}\Big\|_{L^2(\xi)}.$$

Before computing the Fourier transform of $g_{\{y,h,j\}}(x)$, let us note that the convolution operator $*_2$ in the next three equalities is only acting on the second component "y". It has nothing to do with the y in the function $N(y)$, for example

$$\Phi_k(y') *_2 \widehat{f}(\xi+N(y),y) = \int_{\mathbb{R}} \Phi_k(y')\widehat{f}(\xi+N(y),y-y')\,dy'.$$

Since

$$\widehat{g}_{\{y,h,j\}}(\xi+N(y)) = \sum_{2^{-k}\le\delta(h,y)} \Phi_k(y') *_2 ((S_{j+h}f(\cdot,y))^\wedge(\xi+N(y)))$$
$$= \sum_{2^{-k}\le\delta(h,y)} \Phi_k(y') *_2 (p(2^{-j-h}|\xi|)\widehat{f}(\xi+N(y),y))$$
$$= \sum_{2^{-k}\le\delta(h,y)} p(2^{-j-h}|\xi|)(\Phi_k(y') *_2 \widehat{f}(\xi+N(y),y)),$$

one has

$$\|T_jf(x,y)\|_{L^2(x)}$$
$$\le C\Big\|\Big[\sum_{h\in\mathbb{Z}}\Big|\widehat{\Psi}_h(\xi)p(2^{-j-h}|\xi|)\sum_{2^{-k}\le\delta(h,y)}\Phi_k *_2 \widehat{f}(\xi+N(y),y)\Big|^2\Big]^{1/2}\Big\|_{L^2(\xi)}$$
$$= C\Big\|\Big[\sum_{h\in\mathbb{Z}}\Big|\widehat{\Psi}_0(2^{-h}\xi)p(2^{-j-h}|\xi|)\sum_{2^{-k}\le\delta(h,y)}\Phi_k *_2 \widehat{f}(\xi+N(y),y)\Big|^2\Big]^{1/2}\Big\|_{L^2(\xi)}.$$

Employing Remark 1, one has

$$\|T_j f(x,y)\|_{L^2(x)} \le C\Big\|\Big[\sum_{h\in\mathbb{Z}}\Big|\min\{|2^{-h}\xi|,\ |2^{-h}\xi|^{-1}\}p(2^{-j-h}|\xi|) \times \sum_{2^{-k}\le\delta(h,y)} \Phi_k *_2 \widehat{f}(\xi+N(y),y)\Big|^2\Big]^{1/2}\Big\|_{L^2(\xi)}.$$

By the hypothesis on the support of p, the function $p(2^{-j-h}|\xi|)$ is supported in $2^{j+h-2} < |\xi| < 2^{j+h+1}$, i.e. $2^{j-2} < |2^{-h}\xi| < 2^{j+1}$. This implies $\min\{|2^{-h}\xi|,\ |2^{-h}\xi|^{-1}\} \le 4\min\{2^{-j},\ 2^j\}$. Therefore,

$$\begin{aligned}\|T_j f(x,y)\|_{L^2(x)} &\le C\min\{2^{-j},\ 2^j\}\Big\|\Big[\sum_{h\in\mathbb{Z}}\Big|\sum_{2^{-k}\le\delta(h,y)}\Phi_k * \widehat{S_{j+h}f}(\xi+N(y),y)\Big|^2\Big]^{1/2}\Big\|_{L^2(\xi)}\\ &\le C\min\{2^{-j},\ 2^j\}\Big\|\Big[\sum_{h\in\mathbb{Z}}|H_y^M(\widehat{S_{j+h}f}(\xi+N(y),y))|^2\Big]^{1/2}\Big\|_{L^2(\xi)},\end{aligned}$$

where the last inequality is obtained by using the ideas in Remark 2.

Finally, to finish the proof of this case, take the L^2 norm with respect to y in the last inequality and apply Fubini's Theorem to get

$$\text{(4)}\quad \|T_j f(x,y)\|_{L^2(x,y)} \le C\min\{2^{-j},\ 2^j\}\Big\|\Big\|\Big(\sum_{h\in\mathbb{Z}}|H_y^M(\widehat{S_{j+h}f}(\xi+N(y),y))|^2\Big)^{1/2}\Big\|_{L^2(y)}\Big\|_{L^2(\xi)}.$$

By the fact that the vector-valued Hilbert transform is bounded on $L^p(y)$, $1 < p < \infty$ (see [4]), and the Plancherel Theorem, one concludes that

$$\begin{aligned}\|T_j f(x,y)\|_{L^2(x,y)} &\le C\min\{2^{-j},\ 2^j\}\Big\|\Big\|\Big(\sum_{h\in\mathbb{Z}}|\widehat{S_{j+h}f}(\xi+N(y),y)|^2\Big)^{1/2}\Big\|_{L^2(y)}\Big\|_{L^2(\xi)}\\ &= C\min\{2^{-j},\ 2^j\}\Big\|\Big\|\Big(\sum_{h\in\mathbb{Z}}|\widehat{S_{j+h}f}(\xi+N(y),y)|^2\Big)^{1/2}\Big\|_{L^2(\xi)}\Big\|_{L^2(y)}\\ &= C\min\{2^{-j},\ 2^j\}\Big\|\Big(\sum_{h\in\mathbb{Z}}|e^{-iN(y)x}S_{j+h}f(x,y)|^2\Big)^{1/2}\Big\|_{L^2(x,y)}\\ &\le C\min\{2^{-j},\ 2^j\}\|f\|_{L^2(x,y)},\end{aligned}$$

where the last inequality is obtained by using (1).

For the second part, $2 < p < \infty$, by (3) since

$$\|T_j f(x,y)\|_{L^p(x)} \le C\Big\|\Big[\sum_{h\in\mathbb{Z}}|(e^{iN(y)x'}\Psi_h(x')) *_1 g_{\{y,h,j\}}(x)|^2\Big]^{1/2}\Big\|_{L^p(x)},$$

there exists a $G \in L^{(p/2)'}(\mathbb{R})$ with norm one such that

$$\begin{aligned}\|T_j f(x,y)\|_{L^p(x)} &\le C\Big(\int \sum_{h\in\mathbb{Z}} |(e^{iN(y)x'}\Psi_h(x')) *_1 g_{\{y,h,j\}}(x)|^2 G(x)\,dx\Big)^{1/2} \\ &\le C\Big(\sum_{h\in\mathbb{Z}} \int |\Psi_h| *_1 |g_{\{y,h,j\}}(x)|^2 |G(x)|\,dx\Big)^{1/2} \\ &= C\Big(\sum_{h\in\mathbb{Z}} \int |g_{\{y,h,j\}}(x)|^2 |\Psi_h(-\cdot)| *_1 |G(x)|\,dx\Big)^{1/2} \\ &\le C\Big(\int \sum_{h\in\mathbb{Z}} |g_{\{y,h,j\}}(x)|^2 MG(x)\,dx\Big)^{1/2},\end{aligned}$$

where $MG(x)$ denotes the classical Hardy–Littlewood maximal function in one dimension. Applying Hölder's inequality, one has

$$\begin{aligned}\|T_j f(x,y)\|_{L^p(x)} &\le C\Big\|\sum_{h\in\mathbb{Z}} |g_{\{y,h,j\}}(x)|^2\Big\|^{1/2}_{L^{p/2}(x)} \|MG\|^{1/2}_{L^{(p/2)'}(x)} \\ &\le C\Big\|\Big(\sum_{h\in\mathbb{Z}} |g_{\{y,h,j\}}(x)|^2\Big)^{1/2}\Big\|_{L^p(x)},\end{aligned}$$

where the last inequality is obtained by using the L^p boundedness of the Hardy–Littlewood maximal function (see [8]) and the definition of G. Applying Remark 2, one has

$$\begin{aligned}&\|T_j f(x,y)\|_{L^p(x)} \\ &\quad\le C\Big\|\Big(\sum_{h\in\mathbb{Z}} |H_y^M(S_{j+h}^+(e^{iN(y)x}e^{-iN(y)(\cdot)}f(\cdot,y))(x))|^2\Big)^{1/2}\Big\|_{L^p(x)}.\end{aligned}$$

Now, one takes the L^p norm with respect to y in the last inequality. By Fubini's Theorem and, again, the L^p boundedness of the vector-valued Hilbert transform, one has

$$\begin{aligned}(5)\quad &\|T_j f(x,y)\|_{L^p(x,y)} \\ &\le C\Big\|\Big(\sum_{h\in\mathbb{Z}} |H_y^M(S_{j+h}^+(e^{iN(y)x}e^{-iN(y)(\cdot)}f(\cdot,y))(x))|^2\Big)^{1/2}\Big\|_{L^p(x,y)} \\ &\le C\Big\|\Big(\sum_{h\in\mathbb{Z}} |e^{iN(y)x}S_{j+h}^+(e^{-iN(y)(\cdot)}f(\cdot,y))(x)|^2\Big)^{1/2}\Big\|_{L^p(x,y)} \\ &\le C\|f\|_{L^p(x,y)} \qquad \text{(by (1))}.\end{aligned}$$

For the third part, $1 < p < 2$, for each fixed $y \in \mathbb{R}$, let us compute $\|T_j f(x,y)\|_{L^p(x)}$. Again, using (3), by duality, there exists a sequence of functions $\{q_h(x)\}_{h\in\mathbb{Z}} \in L^{p'}(l^2)$ with mixed norm one such that

$$\begin{aligned}
&\|T_j f(x,y)\|_{L^p(x)} \\
&\quad \le C \sum_{h\in\mathbb{Z}} \int_{\mathbb{R}} (e^{iN(y)(\cdot)}\Psi_h(\cdot)) *_1 g_{\{y,h,j\}}(x) q_h(x)\,dx \\
&\quad = C \sum_{h\in\mathbb{Z}} \int_{\mathbb{R}} g_{\{y,h,j\}}(x) (e^{-iN(y)(\cdot)}\Psi_h(-\cdot)) *_1 q_h(x)\,dx \\
&\quad \le C \int_{\mathbb{R}} \sum_{h\in\mathbb{Z}} H_y^M(S_{j+h}^+(e^{iN(y)x}e^{-iN(y)(\cdot)} f(\cdot,y))(x)) \\
&\qquad\qquad\qquad\qquad \times |(e^{-iN(y)(\cdot)}\Psi_h(-\cdot)) *_1 q_h(x)|\,dx \\
&\quad \le C \Big\| \Big(\sum_{h\in\mathbb{Z}} |H_y^M(S_{j+h}^+(e^{iN(y)x}e^{-iN(y)(\cdot)} f(\cdot,y))(x))|^2 \Big)^{1/2} \Big\|_{L^p(x)} \\
&\qquad \times \Big\| \Big(\sum_{h\in\mathbb{Z}} |(e^{-iN(y)(\cdot)}\Psi_h(-\cdot)) *_1 q_h(x)|^2 \Big)^{1/2} \Big\|_{L^{p'}(x)} .
\end{aligned}$$

It is clear that the term $|(e^{-iN(y)(\cdot)}\Psi_h(-\cdot)) *_1 q_h(x)|$ is bounded by the classical Hardy–Littlewood maximal function $Mq_h(x)$, which does not depend on y. Again, by the boundedness of the vector-valued Hardy–Littlewood maximal function and the definition of $\{q_h(x)\}_{h\in\mathbb{Z}} \in L^{p'}(l^2)$, one concludes that

$$\begin{aligned}
&\|T_j f(x,y)\|_{L^p(x)} \\
&\qquad \le C \Big\| \Big(\sum_{h\in\mathbb{Z}} |H_y^M(S_{j+h}^+(e^{iN(y)x}e^{-iN(y)(\cdot)} f(\cdot,y))(x))|^2 \Big)^{1/2} \Big\|_{L^p(x)} .
\end{aligned}$$

From now on, one uses the same ideas as in the proof of the case $2 < p < \infty$ to get

$$\|T_j f\|_{L^p(x,y)} \le C\|f\|_{L^p(x,y)} \quad (1 < p < 2)\,. \tag{6}$$

Employing the real interpolation theorem between (4) and (5), and (4) and (6), together with Minkowski's inequality, one obtains

$$\begin{aligned}
\|Tf\|_{L^p(x,y)} &\le \sum_j \|T_j f\|_{L^p(x,y)} \\
&\le C \sum_j \min\{2^{-j\alpha},\ 2^{j\alpha}\} \|f\|_{L^p(x,y)} \le C\|f\|_{L^p(x,y)}
\end{aligned}$$

for some $\alpha = \alpha(p) > 0$, $1 < p < \infty$. Thus (i) is proved. Hence, the Theorem is proved.

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DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY
CORVALLIS, OREGON 97331
U.S.A.
E-mail: CHEN@MATH.ORST.EDU

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