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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF MULTIPOINT BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

BY

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0. Introduction. This paper presents conditions for the existence and uniqueness of solutions for multipoint boundary value problems of the form

(0.1) $$x' = f(t,x)\,, \quad x = (x_1,\ldots,x_n),$$

(0.2) $$x_s(t_1) = \alpha_m\,, \quad x_j(\tau) = \alpha_j\,, \quad x_r(t_2) = \alpha_p \quad (j = 1,\ldots,n,\ j \ne m,p,\ m \ne p)$$

where $f : [t_1,t_2] \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau \in [t_1,t_2]$, $s,m,r,p \in \{1,\ldots,n\}$, $n \ge 3$, and $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$. Specifically, we present conditions where some restriction on the signs of the entries in the Jacobian matrix of f plays a role.

In [8] the author has given such a criterion for a certain class of two-point boundary value problems for Eq. (0.1). The first results of this nature were established by Garner [5, 6] and Garner and Burton [7]. Their theorems only concern the situation when (0.1) is linear and $(s,m,r,p) = (1,1,n,n)$. Results in the same spirit, with an nth-order ($n \ge 3$) differential equation in place of (0.1), have been obtained in [4] for linear cases and in [1, 2, 9] for nonlinear cases. The principal result of the present paper is Theorem 3.1, which generalizes the theorems in [5–7]. One can also derive as applications of this theorem various results which, in some cases, improve the theorems in [1, 2, 4, 9]. These applications are presented in the last section.

1. Notation and definitions. We shall assume that all matrices introduced in this paper are real $n \times n$ matrices. For a matrix $A = (a_{ij})$ the matrix $\operatorname{diag}(a_{11},\ldots,a_{nn})$ will be denoted by $\operatorname{diag} A$. The symbol A° will denote the matrix $A - \operatorname{diag} A$. The set of all matrices $\Lambda = (\lambda_{ij})$ with entries of the form

$$\lambda_{ij} = \lambda_i\lambda_j\,, \quad i,j = 1,\ldots,n\,,$$

where $\lambda_i = \pm 1$, $i = 1,\ldots,n$, will be denoted by Δ. It is clear that Δ

consists of exactly 2^{n-1} different symmetric matrices having all entries equal to 1 or -1.

Definition 1.1. Let Λ be a matrix in Δ and let $J \subset \mathbb{R}$. A matrix $A = (a_{ij}(t))$, $t \in J$, belongs to the class $\mathcal{D}_\Lambda^+(J)$ if

$$\lambda_{ij} a_{ij}(t) \geq 0 \quad \text{on } J, \quad i,j = 1, \ldots, n\,,$$

and A is in $\mathcal{D}_\Lambda^-(J)$ if $-A \in \mathcal{D}_\Lambda^+(J)$. The class $\mathcal{D}_\Lambda^+(J)$ was used in [8].

Definition 1.2. Let $\Lambda \in \Delta$ and $J \subset \mathbb{R}$. A matrix $U \in \mathcal{D}_\Lambda^+(J)$ ($U \in \mathcal{D}_\Lambda^-(J)$) is said to be a *minor-matrix* of a family $\mathcal{F} \subset \mathcal{D}_\Lambda^+(J)$ (resp. $\mathcal{F} \subset \mathcal{D}_\Lambda^-(J)$) if for each $F \in \mathcal{F}$

$$\lambda_{ij} f_{ij}(t) \geq \lambda_{ij} u_{ij}(t) \quad \text{on } J, \quad i,j = 1, \ldots, n\,,$$

where $F = (f_{ij})$ and $U = (u_{ij})$. If

$$\lambda_{ij} f_{ij}(t) \leq \lambda_{ij} u_{ij}(t) \quad \text{on } J, \quad i,j = 1, \ldots, n\,,$$

for each $F \in \mathcal{F}$, then U is said to be a *major-matrix* of $\mathcal{F}$.

It is clear that for every $\mathcal{F} \subset \mathcal{D}_\Lambda^+(J)$ ($\mathcal{F} \subset \mathcal{D}_\Lambda^-(J)$) the matrix with all entries zero is the trivial minor-matrix (resp. major-matrix) of $\mathcal{F}$.

Definition 1.3. A matrix $U = (u_{ij}(t))$, $t \in J$, is said to be *upper* (*lower*) *irreducible* on J with respect to a pair of indices (s,m), $s \neq m$ (sth row and mth column) if there exist a finite sequence $k_0, k_1, \ldots, k_{l-1}$ of indices in $\{1, \ldots, n\}$ and a nondecreasing (resp. nonincreasing) sequence $\xi_1, \ldots, \xi_{l-1}$ of points in J such that

$$u_{k_q k_{q+1}}(\xi_{q+1}) \neq 0 \quad \text{ for } q = 0, 1, \ldots, l-2\,,$$
$$k_q \neq k_{q+1} \quad \text{for } q = 0, 1, \ldots, l-2\,, \quad k_0 = s, \quad k_{l-1} = m\,.$$

Clearly it is sufficient to consider $l \leq n$.

U is *irreducible* on J with respect to (s,m), $s \neq m$, if U is upper and lower irreducible on J with respect to (s,m).

We note here that if there exists $t_0 \in J$ such that the constant matrix $U(t_0)$ is irreducible in the classical sense then U is irreducible on J with respect to each pair of indices.

The interval $J = [t_1, t_2]$ will be fixed throughout this paper. For $\tau \in J$, set $J_1 = [t_1, \tau]$ and $J_2 = [\tau, t_2]$. Moreover, we denote by $\mathcal{C}_i$ ($i = 1,2$) the class of all continuous n-vector functions on J_i ($i = 1,2$).

2. Preliminary estimates. This section will be devoted to a variety of estimates concerning partial derivatives (with respect to the initial values) of solutions of (0.1). As will be seen these estimates play an essential role.

In the sequel we make the following assumptions with regard to (0.1).

(A) Both $f = (f_1, \ldots, f_n)$ and its Jacobian matrix $f_x = (f_{ij})$, where $f_{ij} = \partial f_i/\partial x_j$, $i, j = 1, \ldots, n$, are continuous on $J \times \mathbb{R}^n$.

(B) All solutions of all initial value problems for (0.1) extend to J.

It is well known that if the right-hand side of (0.1) is as stated above then for each initial condition $x(\tau) = \alpha$, where $(\tau, \alpha) \in J \times \mathbb{R}^n$, (0.1) has a unique solution x defined on the whole of J. That solution will be denoted by $x(t) = x(t; \tau, \alpha)$. It is also well known that the solution $x(t; \tau, \alpha)$ is continuously differentiable with respect to the initial values $(\tau, \alpha) \in J \times \mathbb{R}^n$.

Fix $(\tau, \alpha) \in J \times \mathbb{R}^n$. The linear differential system $y' = f_x(t, x(t))y$, $t \in J$, is called the *variational equation along* $x(t) = x(t; \tau, \alpha)$. It is known that the matrix $X(t; \tau, \alpha) = (x_{ij}(t; \tau, \alpha))$, where $x_{ij} = \partial x_i/\partial \alpha_j$, $i, j = 1, \ldots, n$, is the fundamental matrix solution of this equation.

Combining these comments with Lemma 2.2 and Corollary 2.1 of [8], we infer the following result.

Lemma 2.1. *Let $\Lambda \in \Delta$ and $\tau \in J$. If the family $\{f_x^\circ(t, z(t)) : z \in \mathcal{C}_2\}$ is contained in $\mathcal{D}_\Lambda^+(J_2)$, then so is $\{X(t; \tau, \alpha) : \alpha \in \mathbb{R}^n\}$. More precisely, for each $\alpha \in \mathbb{R}^n$ the following estimates hold on J_2*:

$$\begin{aligned}
(2.1)\qquad \lambda_{ij} x_{ij}(t; \tau, \alpha) &\geq \delta_{ij} \exp \int_\tau^t f_{ii}(w, x(w))\, dw \\
&+ \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \ldots \int_\tau^{\tau_{l-2}} \bigg(\Big(\exp \int_\tau^{\tau_{l-1}} f_{jj}(w, x(w))\, dw\Big) \\
&\times \prod_{q=0}^{l-2} \lambda_{i_q i_{q+1}} f_{i_q i_{q+1}}(\tau_{q+1}, x(\tau_{q+1})) \exp \int_{\tau_{q+1}}^{\tau_q} f_{i_q i_q}(w, x(w))\, dw\bigg)\, d\tau_{l-1} \\
&\geq \delta_{ij} \exp \int_\tau^t f_{ii}(w, x(w))\, dw \qquad (\tau_0 = t), \qquad i, j = 1, \ldots, n,
\end{aligned}$$

where $x(t) = x(t; \tau, \alpha)$ and $i_0, i_1, \ldots, i_{l-1}$ are arbitrary indices in $\{1, \ldots, n\}$ such that $i_0 = i$, $i_{l-1} = j$, $i_q \neq i_{q+1}$ for $q = 0, 1, \ldots, l-2$, and δ_{ij} is the Kronecker delta.

The fundamental matrix solution X is also defined to the left of τ. A result parallel to Lemma 2.1, concerning the fundamental matrix solution to the left of τ, is useful in later applications. We state it as a lemma, omitting its proof.

Lemma 2.2. *Let $\Lambda \in \Delta$ and $\tau \in J$. If the family $\{f_x^\circ(t, z(t)) : z \in \mathcal{C}_1\}$ is contained in $\mathcal{D}_\Lambda^-(J_1)$, then $\{X(t; \tau, \alpha) : \alpha \in \mathbb{R}^n\}$ is contained in $\mathcal{D}_\Lambda^+(J_1)$, and for each $\alpha \in \mathbb{R}^n$ the estimate* (2.1) *holds on J_1.*

Remark. The assumptions of Lemmas 2.1 and 2.2 yield that the ith coordinate f_i of f is quasi-monotone with respect to x_j, for $i,j=1,\ldots,n$, $i\neq j$. Precisely, f_i is nondecreasing (nonincreasing) with respect to x_j if $\lambda_{ij}=1$ (resp. -1).

COROLLARY 2.1. (i) *In addition to the hypotheses of Lemma* 2.2 *assume that for some* $k\in\{1,\ldots,n\}$ *there exists a function* b_{kk} *continuous on* J_1 *such that*

$$f_{kk}(t,z(t))\le b_{kk}(t)\quad \text{on } J_1 \text{ for every } z\in\mathcal{C}_1\,. \tag{2.2}$$

Then

$$\lambda_{kk}x_{kk}(t;\tau,\alpha)\ge\eta_{kk}(t)\quad \text{on } J_1\times\mathbb{R}^n\,, \tag{2.3}$$

where η_{kk} *is a continuous positive function on* J_1.

(ii) *In addition to the hypotheses of Lemma* 2.1 *assume that for some* $k\in\{1,\ldots,n\}$ *there exists a function* c_{kk} *continuous on* J_2 *such that*

$$f_{kk}(t,z(t))\ge c_{kk}(t)\quad \text{on } J_2 \text{ for every } z\in\mathcal{C}_2\,. \tag{2.4}$$

Then

$$\lambda_{kk}x_{kk}(t;\tau,\alpha)\ge\varrho_{kk}(t)\quad \text{on } J_2\times\mathbb{R}^n\,,$$

where ϱ_{kk} *is a continuous positive function on* J_2.

Proof. This is an immediate consequence of (2.1) with $i=j=k$.

COROLLARY 2.2. (i) *Let the hypotheses of Lemma* 2.2 *be satisfied. Assume that the family* $\{f_x^\circ(t,z(t)) : z\in\mathcal{C}_1\}$ *has a major-matrix which is upper irreducible on* J_1 *with respect to a pair of indices* (s,m), $s\neq m$. *In addition, assume that the family* $\{\operatorname{diag} f_x(t,z(t)) : z\in\mathcal{C}_1\}$ *is uniformly upper bounded on* J_1, *i.e.*, (2.2) *holds for* $k=1,\ldots,n$. *Then*

$$\lambda_{sm}x_{sm}(t;\tau,\alpha)\ge\eta_{sm}(t)\quad \text{on } J_1\times\mathbb{R}^n\,, \tag{2.5}$$

where η_{sm} *is a continuous positive function on* J_1.

(ii) *Let the hypotheses of Lemma* 2.1 *be satisfied. Assume that the family* $\{f_x^\circ(t,z(t)) : z\in\mathcal{C}_2\}$ *has a minor-matrix which is lower irreducible on* J_2 *with respect to a pair of indices* (r,p), $r\neq p$. *In addition, assume that the family* $\{\operatorname{diag} f_x(t,z(t)) : z\in\mathcal{C}_2\}$ *is uniformly lower bounded on* J_2, *i.e.*, (2.4) *holds for* $k=1,\ldots,n$. *Then*

$$\lambda_{rp}x_{rp}(t;\tau,\alpha)\ge\varrho_{rp}(t)\quad \text{on } J_2\times\mathbb{R}^n\,,$$

where ϱ_{rp} *is a continuous positive function on* J_2.

Proof. We only prove (i), the proof of (ii) being quite similar.

Let $U=(u_{ij})\in\mathcal{D}_\Lambda^-(J_1)$ be an upper irreducible (with respect to (s,m)) major-matrix of $\{f_x^\circ(t,z(t)) : z\in\mathcal{C}_1\}$. Then it is easy to deduce from (2.1) with $(i,j)=(s,m)$ and from Definitions 1.2 and 1.3 that

$$\lambda_{sm}x_{sm}(t;\tau,\alpha)\geq\int\limits_{\tau}^{t}d\tau_1\int\limits_{\tau}^{\tau_1}d\tau_2\ldots\int\limits_{\tau}^{\tau_{l-2}}\Big(\Big(\exp\int\limits_{\tau}^{\tau_{l-1}}f_{mm}(w,x(w))\,dw\Big)$$
$$\times\prod_{q=0}^{l-2}\lambda_{k_qk_{q+1}}u_{k_qk_{q+1}}(\tau_{q+1})\exp\int\limits_{\tau_{q+1}}^{\tau_q}f_{k_qk_q}(w,x(w))\,dw\Big)\,d\tau_{l-1}$$

for every $(t,\alpha)\in J_1\times\mathbb{R}^n$. Hence, by (2.2) we have

$$\lambda_{sm}x_{sm}(t;\tau,\alpha)\geq\int\limits_{\tau}^{t}d\tau_1\int\limits_{\tau}^{\tau_1}d\tau_2\ldots\int\limits_{\tau}^{\tau_{l-2}}\Big(\Big(\exp\int\limits_{\tau}^{\tau_{l-1}}b_{mm}(w)\,dw\Big)\tag{2.6}$$
$$\times\prod_{q=0}^{l-2}\lambda_{k_qk_{q+1}}u_{k_qk_{q+1}}(\tau_{q+1})\exp\int\limits_{\tau_{q+1}}^{\tau_q}b_{k_qk_q}(w)\,dw\Big)d\tau_{l-1}$$

for every $(t,\alpha)\in J_1\times\mathbb{R}^n$. Therefore (2.5) holds on $J_1\times\mathbb{R}^n$ with η_{sm} defined by the right-hand side of (2.6). The fact that $\eta_{sm}(t)>0$ for $t\in J_1$ follows from Definition 1.3 and the properties of the exponential function. This proves the corollary.

Note that the proof does not require the validity of (2.2) for all $k\in\{1,\ldots,n\}$. The following remark substantiates this observation.

Remark. The assertion of Corollary 2.2(i) remains true if (2.2) only holds for $k=k_0,k_1,\ldots,k_{l-1}$, where the sequence $k_0,k_1,\ldots,k_{l-1}$ occurs in the conditions of Definition 1.3. A similar comment is also valid for part (ii) of Corollary 2.2.

3. The main results. This section is concerned with conditions sufficient to ensure the uniqueness and existence of solutions of the problem (0.1), (0.2). They are natural extensions of some of the results in [5–7].

In the subsequent discussion let $M_0^n=\{(s,m,r,p): s,m,r,p\in\{1,\ldots\ldots,n\}$, and $m<p\}$.

The main result of this paper reads as follows.

THEOREM 3.1. *Let $f:J\times\mathbb{R}^n\to\mathbb{R}^n$ satisfy conditions* (A) *and* (B), *and let $(s,m,r,p)\in M_0^n$ and $t_1<\tau<t_2$. Assume that*

(C_1^-) *there exists $\Lambda_1=(\lambda_{ij}^{(1)})\in\Delta$ such that $\{f_x^\circ(t,z(t)):z\in\mathcal{C}_1\}\subset\mathcal{D}_{\Lambda_1}^-(J_1)$;*

(C_2^-) *if $s\neq m$ then $\{f_x^\circ(t,z(t)):z\in\mathcal{C}_1\}$ has a major-matrix which is upper irreducible on J_1 with respect to (s,m);*

(C_3^-) *$\{\operatorname{diag}f_x(t,z(t)):z\in\mathcal{C}_1\}$ is uniformly upper bounded on J_1;*

(C_1^+) *there exists $\Lambda_2=(\lambda_{ij}^{(2)})\in\Delta$ such that $\{f_x^\circ(t,z(t)):z\in\mathcal{C}_2\}\subset\mathcal{D}_{\Lambda_2}^+(J_2)$;*

(C_2^+) *if* $r \neq p$ *then* $\{f_x^\circ(t, z(t)) : z \in \mathcal{C}_2\}$ *has a major-matrix which is lower irreducible on* J_2 *with respect to* (r, p);

(C_3^+) $\{\operatorname{diag} f_x(t, z(t)) : z \in \mathcal{C}_2\}$ *is uniformly lower bounded on* J_2.

Then, if

$$\lambda_{sm}^{(1)}\lambda_{rp}^{(2)} = -\lambda_{sp}^{(1)}\lambda_{rm}^{(2)}, \tag{3.1}$$

the boundary value problem (0.1), (0.2) *has a unique solution for each* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$.

The idea of the proof is the following. Fix $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. With each $(u, v) \in \mathbb{R}^2$ we associate a vector $\alpha_{mp}(u, v) \in \mathbb{R}^n$ as follows:

$$\alpha_{mp}(u, v) = (\alpha_1, \ldots, \alpha_{m-1}, u, \alpha_{m+1}, \ldots, \alpha_{p-1}, v, \alpha_{p+1}, \ldots, \alpha_n).$$

To prove Theorem 3.1 we will show that the system of equations

$$x_s(t_1; \tau, \alpha_{mp}(u, v)) = \alpha_m, \qquad x_r(t_2; \tau, \alpha_{mp}(u, v)) = \alpha_p$$

where x_s and x_r are the sth and rth coordinates of the solution $x(t) = x(t; \tau, \alpha_{mp}(u, v))$ of (0.1), has exactly one solution (u, v).

For this we require the following result.

LEMMA 3.1. *Consider the system of equations*

$$\phi_1(u, v) = a_1, \qquad \phi_2(u, v) = a_2, \tag{3.2}$$

where the functions ϕ_1 *and* ϕ_2 *have continuous partial derivatives throughout* $\mathbb{R}^2$. *Assume that there exist constants* $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$ *and* $\delta_1 > 0$, $\delta_2 > 0$ *such that the following conditions hold on* $\mathbb{R}^2$:

$$\varepsilon_1 \frac{\partial \phi_1}{\partial u} \geq \delta_1, \qquad \varepsilon_2 \frac{\partial \phi_2}{\partial v} \geq \delta_2, \qquad \varepsilon_1 \varepsilon_2 \frac{\partial \phi_1}{\partial v} \frac{\partial \phi_2}{\partial u} \leq 0. \tag{3.3}$$

Then for each pair (a_1, a_2) *of real numbers the system* (3.2) *has a unique solution* (u, v).

Proof. This lemma can be established by repeating the arguments used in the proof of [3, Theorem 8.1(ii)] with appropriate changes.

Remark. It is clear that the assertion of Lemma 3.1 also holds if (3.3) is replaced by

$$\varepsilon_1 \frac{\partial \phi_1}{\partial v} \geq \delta_1, \qquad \varepsilon_2 \frac{\partial \phi_2}{\partial u} \geq \delta_2, \qquad \varepsilon_1 \varepsilon_2 \frac{\partial \phi_1}{\partial u} \frac{\partial \phi_2}{\partial v} \leq 0.$$

We are now ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. In view of Lemma 3.1 it is enough to show that (3.3) holds with ϕ_1 and ϕ_2 replaced by $x_s(t_1; \tau, \alpha_{mp}(u, v))$ and $x_r(t_2; \tau, \alpha_{mp}(u, v))$, respectively. Thus it suffices to verify that

$$\varepsilon_1 x_{sm}(t_1; \tau, \alpha_{mp}(u, v)) \geq \delta_1, \qquad \varepsilon_2 x_{rp}(t_2; \tau, \alpha_{mp}(u, v)) \geq \delta_1, \tag{3.4}$$

$$\varepsilon_1 \varepsilon_2 x_{sp}(t_1; \tau, \alpha_{mp}(u, v)) x_{rm}(t_2; \tau, \alpha_{mp}(u, v)) \leq 0 \tag{3.5}$$

on $\mathbb{R}^2$, where $\delta_1, \delta_2 > 0$.

By (C_1^-) the hypotheses of Lemma 2.2 hold with $\Lambda = \Lambda_1$. Consequently, choosing (i, j) equal to (s, p) and using (2.1) we deduce that

$$\lambda_{sp}^{(1)} x_{sp}(t_1; \tau, \alpha_{mp}(u, v)) \geq 0 \quad \text{on } \mathbb{R}^2 . \tag{3.6}$$

By (C_1^-) and (C_3^-) the hypotheses of Corollary 2.1(i) hold with $\Lambda = \Lambda_1$. On the other hand, by (C_1^-)–(C_3^-) the hypotheses of Corollary 2.2(i) hold with Λ as above. Therefore the estimate

$$\lambda_{sm}^{(1)} x_{sm}(t_1; \tau, \alpha_{mp}(u, v)) \geq \eta_{sm}(t_1) > 0 \quad \text{on } \mathbb{R}^2 \tag{3.7}$$

follows from (2.3) if $s = m$ and from (2.5) if $s \neq m$.

Further, using (C_1^+)–(C_3^+) in place of (C_1^-)–(C_3^-) and employing Corollaries 2.1(ii) and 2.2(ii), we conclude that

$$\lambda_{rm}^{(2)} x_{rm}(t_2; \tau, \alpha_{mp}(u, v)) \geq 0 \quad \text{on } \mathbb{R}^2 , \tag{3.8}$$

$$\lambda_{rp}^{(2)} x_{rp}(t_2; \tau, \alpha_{mp}(u, v)) \geq \varrho_{rp}(t_2) > 0 \quad \text{on } \mathbb{R}^2 . \tag{3.9}$$

Put $\varepsilon_1 = \lambda_{sm}^{(1)}$ and $\varepsilon_2 = \lambda_{rp}^{(2)}$. So, by (3.7) and (3.9) the inequalities in (3.4) hold with

$$\delta_1 = \eta_{sm}(t_1) > 0 , \qquad \delta_2 = \varrho_{rp}(t_2) > 0 .$$

Finally, (3.5) follows from the definition of ε_1 and ε_2, the estimates (3.6), (3.8) and the condition (3.1). The proof is therefore complete.

Remark. The conclusion of Theorem 3.1 remains true if the condition (C_3^-) is weakened to (2.2) with $k = s$ provided $s = m$, or to the conditions mentioned in the Remark to Corollary 2.2(i) provided $s \neq m$. A similar comment is valid for (C_3^+).

Remark. The assertion of Theorem 3.1 holds if the pairs (s, m) in (C_2^-) and (r, p) in (C_2^+) are replaced by (s, p) and (r, m) respectively. Indeed, we then repeat the argument with Lemma 3.1 replaced by the Remark following it.

We conclude this section with some special cases of (0.1), (0.2). We start with a linear differential equation

$$x' = A(t)x + g(t) , \tag{3.10}$$

where A is an $n \times n$ matrix and g is an n-vector function continuous on J. An application of Theorem 3.1 to (3.10) gives the following result.

Corollary 3.1. *Let A be an $n \times n$ matrix continuous on J, and let $(s, m, r, p) \in M_0^n$ and $\tau \in J$. Suppose further that*

(CL_1^-) *there exists $\Lambda_1 \in \Delta$ such that $A^\circ \in \mathcal{D}_{\Lambda_1}^-(J_1)$;*
(CL_2^-) *if $s \neq m$, then A° is upper irreducible on J_1 with respect to (s, m);*
(CL_1^+) *there exists $\Lambda_2 \in \Delta$ such that $A^\circ \in \mathcal{D}_{\Lambda_2}^+(J_2)$;*

(CL_2^+) *if* $r \neq p$, *then* A° *is lower irreducible on* J_2 *with respect to* (r,p).

Then if condition (3.1) *holds*, *the problem* (3.10), (0, 2) *has a unique solution for each* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$.

In the papers [5–7] the problem (3.10), (0.2) was studied for $(s,m,r,p) = (1,1,n,n)$, i.e., when the boundary condition (0.2) has the form

$$x_1(t_1) = \alpha_1\,, \quad x_j(\tau) = \alpha_j \quad (j = 2, \ldots, n-1)\,, \quad x_n(t_2) = \alpha_n\,. \tag{3.11}$$

The results of the above-mentioned papers can be obtained as special cases of Corollary 3.1. This is illustrated by the following example.

Example 3.1 ([5, Theorem 1]). Assume that the entries a_{ij}, $i,j = 1,\ldots,n$, of A satisfy the following conditions.

(a_1) $a_{ij}(t) = 0$ on J for $i,j = 1,\ldots,n$, $i \geq j$, except $a_{n1}(t)$;

(b_1) $a_{in}(t)$, $a_{lk}(t)\begin{cases} \leq 0\,, & t \in J_1\,, \\ \geq 0\,, & t \in J_2\,, \end{cases}$ if i and $l+k$ are even,

and $a_{n1}(t) \geq 0$, $a_{in}(t) \geq 0$, $a_{lk}(t) \geq 0$ on J if i and $l+k$ are odd, for $l = 1,\ldots,n-2$ and $k = 2,\ldots,n-1$.

Then for each $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$ there exists a unique solution of (3.10), (3.11).

Indeed, we note that for the boundary condition (3.11) the hypotheses (CL_2^-) and (CL_2^+) are vacuous. On the other hand, it is straightforward to verify that (a_1), (b_1) imply (CL_1^-), (CL_1^+) with Λ_1 and Λ_2 defined by the vectors $\lambda^{(1)} = (1,-1,\ldots,(-1)^n,-1) \in \mathbb{R}^n$ and $\lambda^{(2)} = (1,\ldots,1) \in \mathbb{R}^n$, respectively. Moreover, (3.1) holds, too. As the hypotheses of Corollary 3.1 are satisfied, the desired results follows.

The above example demonstrates that Corollary 3.1 is an improvement on [5, Theorem 1]. In a similar way it is easy to verify that Corollary 3.1 improves [6, Theorem 2; 7, Theorem 2].

We conclude this section with some facts concerning two-point boundary value problems as special cases of (0.1), (0.2). In particular, if $\tau = t_2$ then (0.2) just reduces to the two-point boundary condition of the form

$$x_s(t_1) = \alpha_m\,, \quad x_j(t_2) = \alpha_j \quad (j = 1,\ldots,n,\ j \neq m)\,, \tag{3.12}$$

where $s, m \in \{1,\ldots,n\}$. On the other hand, if $\tau = t_1$ then (0.2) reduces to

$$x_j(t_1) = \alpha_j\,, \quad x_r(t_2) = \alpha_p \quad (j = 1,\ldots,n,\ j \neq p)\,, \tag{3.13}$$

where $r,p \in \{1,\ldots,n\}$. A detailed analysis of the proof of Theorem 3.1 gives the following result for the problems (0.1), (3.12) and (0.1), (3.13).

Corollary 3.2. (i) *Let* $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfy conditions* (A) *and* (B) *and let* $s, m \in \{1,\ldots,n\}$. *Assume that conditions* (C_1^-)–(C_3^-) *hold. Then* *for each* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ *there exists a unique solution of the problem* (0.1), (3.12).

(ii) *Let* $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfy conditions* (A) *and* (B) *and let* $r, p \in \{1, \ldots, n\}$. *Assume that conditions* (C_1^+)–(C_3^+) *hold. Then for each* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ *there exists a unique solution of the problem* (0.1), (3.13).

Remark. In the special case of two-point boundary value problems of the form (0.1), (3.12) and (0.1), (3.13), Corollary 3.2 is a refinement of Theorem 4.1 in [9].

4. Applications of the main theorem. In the rest of this paper we shall analyse the result of the previous section in the case that (0.1) is replaced by an nth order differential equation of the form

$$x^{(n)} = g(t, x, x', \ldots, x^{(n-1)}) \tag{4.1}$$

and consequently, the condition (0.2) by

$$x^{(s-1)}(t_1) = \alpha_m\,, \qquad x^{(j-1)}(\tau) = \alpha_j\,, \qquad x^{(r-1)}(t_2) = \alpha_p \qquad (j = 1, \ldots, n,\ j \neq m, p,\ m \neq p)\,, \tag{4.2}$$

where $s, m, r, p \in \{1, \ldots, n\}$.

Corresponding to hypotheses (A) and (B) it will be assumed that $g(t, x_1, \ldots, x_n)$ satisfies the following conditions:

(A′) g and its partial derivatives $\partial g/\partial x_i$, $i = 1, \ldots, n$, are continuous on $J \times \mathbb{R}^n$.

(B′) All solutions of all initial value problems for (4.1) extend to J.

We say that a quadruple (s, m, r, p) belongs to the class:

(i) M_1^n if $(s, m, r, p) \in M_0^n$, $m + p$ is odd and either $s \le m$ and $r \le p$, or $s \le p$ and $r \le m$;

(ii) M_2^n if $(s, m, r, p) \in M_1^n$ and $p < n$.

We can derive some concrete results about the existence and uniqueness of solutions to the problem (4.1), (4.2) as an application of Theorem 3.1.

THEOREM 4.1. *Let the function* g *in* (4.1) *satisfy* (A′) *and* (B′) *and let* $t_1 < \tau < t_2$. *Assume that*

(CE_1^-) $(-1)^{n-i}\partial g(t, z(t))/\partial x_i \le 0$ *on* J_1, $i = 1, \ldots, n-1$, *for every* $z \in \mathcal{C}_1$;

(CE_1^+) $\partial g(t, z(t))/\partial x_i \ge 0$ *on* J_2, $i = 1, \ldots, n-1$, *for every* $z \in \mathcal{C}_2$.

Then for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ *and* $(s, m, r, p) \in M_2^n$ *there exists a unique solution of* (4.1), (4.2).

Proof. Fix $(s, m, r, p) \in M_2^n$. The problem (4.1), (4.2) is equivalent to (0.1), (0.2) with $f(t, x_1, \ldots, x_n) = (x_2, \ldots, x_n, g(t, x_1, \ldots, x_n))$. To conclude the proof it suffices to verify the assumptions of Theorem 3.1.

First of all note that for the Jacobian matrix $f_x(t,x)$ of $f=(f_1,\ldots,f_n)$ defined as above we have $\partial f_i/\partial x_{i+1}=1$, $i=1,\ldots,n-1$, $\partial f_n/\partial x_i=\partial g/\partial x_i$, $i=1,\ldots,n$, and the other entries are zero.

Let Λ_1 be the matrix determined by the n-vector $\lambda_1^{(1)}=(1,-1,\ldots\ldots,(-1)^{n+1})$ and let Λ_2 be the same as in Example 3.1. Conditions (C_1^-), (C_1^+) of Theorem 3.1 are then immediate consequences of assumptions (CE_1^-), (CE_1^+), respectively.

Let $U=(u_{ij})$ denote the matrix such that $u_{i,i+1}=1$ for $i=1,\ldots,n-1$ and $u_{ij}=0$ otherwise. It is easy to see that $U\in\mathcal{D}^-_{\Lambda_1}(J_1)\cap\mathcal{D}^+_{\Lambda_2}(J_2)$. Next, the definition of U together with (CE_1^-) and (CE_1^+) implies that U is a major-matrix of $\{f_x^\circ(t,z(t)):z\in\mathcal{C}_1\}$ and a minor-matrix of $\{f_x^\circ(t,z(t)):z\in\mathcal{C}_2\}$. Moreover, note that U is irreducible on J_1 and J_2 with respect to each pair of indices (i,j) with $j>i$. This together with the definition of M_1^n $(\supset M_2^n)$ and the second remark to Theorem 3.1 implies that conditions (C_2^-) and (C_2^+) of Theorem 3.1 hold.

Now, it is easy to verify that the conditions mentioned in the Remark after Theorem 3.1 (replacing (C_3^-) and (C_3^+)) hold, since $(s,m,r,p)\in M_2^n$.

Finally, we note that Λ_1 and Λ_2 satisfy (3.1) since by assumption $m+p$ is odd. Hence the assertion of the theorem is a consequence of Theorem 3.1. The proof is complete.

THEOREM 4.2. *Let the hypotheses of Theorem* 4.1 *be satisfied. In addition*, *assume that*

(CE_2^-) $\partial g(t,z(t))/\partial x_n\le b_n(t)$ *on* J_1 *for every* $z\in\mathcal{C}_1$, *where* b_n *is a function continuous on* J_1;

(CE_2^+) $\partial g(t,z(t))/\partial x_n\ge c_n(t)$ *on* J_2 *for every* $z\in\mathcal{C}_2$, *where* c_n *is a function continuous on* J_2.

Then for each $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$ *and* $(s,m,r,p)\in M_1^n$ *there exists a unique solution of* (4.1), (4.2).

Proof. The proof can be carried out along the same lines as for Theorem 4.1. Here assumptions (C_3^-) and (C_3^+) follow from (CE_2^-) and (CE_2^+) respectively.

Remark. The results of this part of the present section are related to some results found in [1, 2, 4, 9]. In particular, [4, Theorem 2] follows from Theorem 4.2. On the other hand, for g satisfying condition (A′), Theorem 4.2 generalizes and improves [1, Theorem 3.5], [2, Theorem 4.1] and [9, Theorems 4.6–4.8]. Moreover, Theorem 4.1 is a generalization of [9, Theorem 3.3].

THEOREM 4.3. *Let the hypotheses of Theorem* 4.2 *be satisfied. In addition*, *assume that*

(CE_3^-) $(-1)^{n-1}\partial g(t,z(t))/\partial x_1 \le b_1(t)$ *on* J_1 *for every* $z \in \mathcal{C}_1$, *where* $b_1 \le 0$ *is a nontrivial function continuous on* J_1;

(CE_3^+) $\partial g(t,z(t))/\partial x_1 \ge c_1(t)$ *on* J_2 *for every* $z \in \mathcal{C}_2$, *where* $c_1 \ge 0$ *is a nontrivial function continuous on* J_2.

Then for each $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ *and* $(s,m,r,p) \in M_0^n$ *there exists a unique solution of* (4.1), (4.2).

Proof. Let U be as in the proof of Theorem 4.1. Put $U_1(t) = U$ on J_1 ($U_2(t) = U$ on J_2), except the $(n,1)$ entry which equals b_1 (resp. c_1).

The above definition together with (CE_3^-) (resp. (CE_3^+)) implies that U_1 is a major-matrix of the family $\{f_x^\circ(t,z(t)) : z \in \mathcal{C}_1\}$ (resp. U_2 is a minor-matrix of $\{f_x^\circ(t,z(t)) : z \in \mathcal{C}_2\}$) defined in the proof of Theorem 4.1. Moreover, U_1 (resp. U_2) is irreducible on J_1 (resp. J_2) with respect to each pair of indices (i,j), $i \ne j$.

Therefore the proof of the theorem can be carried out along the same lines as for Theorem 4.1, but with U_1 on J_1 and U_2 on J_2 instead of U.

We close this paper with a simple example considered in [1, 2].

Example 4.1. Consider the third-order problems of the type

$$x''' = tx + x' + x'' \quad (= g(t,x,x',x'')) , \tag{4.3}$$

$$x(t_1) = \alpha_2 , \qquad x(0) = \alpha_1 , \qquad x(t_2) = \alpha_3 . \tag{4.4}$$

It is easy to verify that the function $g(t,x_1,x_2,x_3)$ in (4.3) satisfies the assumptions of Theorem 4.2 for $n = 3$ and an arbitrary interval $J = [t_1,t_2]$ ($t_1 < 0 < t_2$). Moreover, the condition (4.4) is determined by the quadruple $(1,2,1,3) \in M_1^3$. Therefore a straightforward application of Theorem 4.2 ensures the existence and uniqueness of the solution of (4.3), (4.4).

Note that from the results in [1, 2] it only follows that the problem (4.3), (4.4) has a unique solution provided that the length of J satisfies a certain estimate.

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