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ON SCHWARTZ'S C-SPACES AND ORLICZ'S O-SPACES

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1. In what follows, $[E, \tau]$ denotes a Hausdorff topological vector space. We start by recalling some different kinds of convergence of series. A series $\sum x_n$ of elements in E is said to be:

(i) *unconditionally Cauchy* if for every 0-neighbourhood U of E there exists $n_0 \in \mathbb{N}$ such that $\sum_{i\in\sigma} x_i \in U$ for all finite $\sigma \subset \mathbb{N}$ with $\inf \sigma > n_0$;

(ii) c_0-*multiplier Cauchy* (*convergent*) if $\sum \alpha_n x_n$ is Cauchy (convergent) for every sequence $(\alpha_n) \in c_0$.

The definitions of C-space and O-space are also linked to series. A survey about these spaces can be found in [10, Section 3.10].

Definition 1. E is a *C-space* if every c_0-multiplier convergent series in E is convergent. C-spaces were originally considered by L. Schwartz who proved [12] that $L^p(\Omega, \Sigma, \mu)$, $0 \le p < \infty$, are C-spaces.

Definition 2. E is an *O-space*, or satisfies condition (O), if every series $\sum x_n$ of elements in E which is perfectly bounded (this means that the set

$$S(x_n) = \Big\{ \sum_{i\in\sigma} x_i : \sigma \text{ is a finite subset of } \mathbb{N} \Big\}$$

is bounded), is convergent. Matuszewska–Orlicz [9] showed that a large class of modular spaces satisfy condition (O).

From the above definitions, we see that these concepts are quite similar. Some authors have studied their relations in the framework of *complete* topological vector spaces (see [2], [7]). In this context, Thomas [13] introduced the notion of Σ-completeness: a space E is said to be Σ-*complete* if every unconditionally Cauchy series in E is convergent. This concept was implicitly considered in a well-known result on Banach spaces due to Bessaga and Pe/lczy/nski [1]. Related to this, Kalton [7] characterized complete C-spaces as those not containing a copy of c_0.

Our aim is to connect these concepts with each other for a class of topological vector spaces that is large enough so as to comprise the main known C-spaces and O-spaces. We also give the corresponding examples to distinguish these concepts.

2. We say that E has *property* $(*)$ if for every perfectly bounded series $\sum x_n$ in E, the set

$$B(x_n)=\Big\{\sum_{i\in\sigma}\alpha_i x_i : |\alpha_i|\le 1,\ \sigma \text{ is a finite subset of } \mathbb{N}\Big\}$$

is bounded. The following spaces have property $(*)$:

(1) [11, proof of Theorem 1] Locally bounded spaces (e.g., L^p, ℓ^p, $0<p<1$), locally convex spaces (e.g., L^p, ℓ^p, $1\le p$) and more generally pseudolocally convex spaces.

(2) [8, Theorem 2, Remark 2] Generalized Orlicz spaces $L^\psi(T,\mathcal{A},\mu;E)$ over finitely additive measures. We note that these spaces are not generally complete and include $L^0(E)$, the space of Bochner measurable functions with values in a Banach space. Drewnowski–Orlicz [5] proved that the modular spaces mentioned before reduce to these spaces in the usual cases.

A subset D of E is called a *disk* if it is bounded and absolutely convex. Let us denote by E_D the linear span of D endowed with the topology defined by the gauge ϱ_D of D. A space E is *locally complete* if for every closed disk D in E, the normed space E_D is a Banach space. For locally convex spaces, this concept was characterized by Dierolf [3] in terms of closed absolutely convex hulls of null sequences. Now, we give another characterization using a lemma which clarifies the relation between c_0-multiplier Cauchy series and perfectly bounded series.

LEMMA 1. (1) *Every c_0-multiplier Cauchy series in E is perfectly bounded.*

(2) *Assume that E has property $(*)$. If $\sum x_n$ is perfectly bounded, then it is c_0-multiplier Cauchy and $\sum \alpha_n x_n$ is unconditionally Cauchy for all $(\alpha_n)\in c_0$.*

Proof. (1) If $\sum x_n$ is not perfectly bounded, then there exists a balanced 0-neighbourhood U such that $S(x_n)\not\subset kU$ for all $k\in\mathbb{N}$. Take a balanced 0-neighbourhood V such that $V+V\subset U$. If we set

$$S_r(x_n)=\Big\{\sum_{i\in\sigma}x_i : \sigma \text{ is a finite subset of } \mathbb{N},\ \inf\sigma>r\Big\},$$

then, for each $r\in\mathbb{N}$, there must exist $p>r$, $p\in\mathbb{N}$, such that $S_r(x_n)\not\subset pV$. Otherwise, if there exists $r\in\mathbb{N}$ with $S_r(x_n)\subset pV$ for all $p>r$ $(p\in\mathbb{N})$, then we have $S(x_n)\subset\lambda U$ for some $\lambda>0$, since $\{\sum_{i\in\sigma}x_i : \sigma\subset\{1,\dots,r\}\}$ is finite, so bounded.

Thus, we can obtain a sequence of finite subsets $\sigma_k \subset \mathbb{N}$ with $\inf \sigma_{k+1} > \sup \sigma_k$ such that $\sum_{i\in\sigma_k} x_i \notin kV$. If we define $\alpha_n = 1/k$ when $n \in \sigma_k$ and 0 otherwise, then $(\alpha_n) \in c_0$, but $\sum \alpha_n x_n$ is not Cauchy.

(2) Suppose that $\sum x_n$ is a perfectly bounded series such that there exist a balanced 0-neighbourhood U in E, $(\alpha_n) \in c_0$ and a sequence of finite subsets $\sigma_k \subset \mathbb{N}$ with $\inf \sigma_{k+1} > \sup \sigma_k$ such that $\sum_{i\in\sigma_k} \alpha_i x_i \notin U$.

Set $\beta_k = \sup\{|\alpha_i| : i \in \sigma_k\} + 1/k$. By property $(*)$, $B(x_n)$ is bounded and since

$$y_k = \sum_{i\in\sigma_k} \frac{\alpha_i}{\beta_k} x_i \in B(x_n)\,,$$

we can obtain $\lambda > 0$ such that $y_k \in \lambda U$ for all $k \in \mathbb{N}$. Bearing in mind that $(\beta_k \lambda)_k$ is a null sequence, we get a contradiction. ∎

THEOREM 1. *If $[E,\tau]$ has property $(*)$, then E is locally complete if and only if every c_0-multiplier τ-Cauchy series is c_0-multiplier τ-convergent.*

Proof. $\Rightarrow$ Let $\sum x_n$ be a c_0-multiplier τ-Cauchy series in E. Thus, by Lemma 1, $S(x_n)$ is bounded and, by property $(*)$, the closure of $B(x_n)$, which we denote by D, is a closed disk in E. Since E is locally complete, $[E_D, \varrho_D]$ is a Banach space. On the other hand, $\varrho_D(a) \le 1$, for all $a \in S(x_n)$, that is, $S(x_n)$ is ϱ_D-bounded, thus, by Lemma 1, $\sum x_n$ is c_0-multiplier ϱ_D-Cauchy and, therefore, c_0-multiplier ϱ_D-convergent. Since the topology generated by ϱ_D on E_D is finer than the topology induced by τ on E_D, we conclude that $\sum x_n$ is c_0-multiplier τ-convergent.

$\Leftarrow$ Let D be a closed disk in E and (x_n) a Cauchy sequence in $[E_D, \varrho_D]$. By induction, we can obtain a strictly increasing sequence of positive integers (n_k) such that

$$\varrho_D(x_{n_{k+1}} - x_{n_k}) \le \frac{1}{2^k k^2} \quad \text{for all } k \in \mathbb{N}\,.$$

Take $y_k = x_{n_{k+1}} - x_{n_k}$ for $k = 1, 2, \ldots$ The last inequality shows that $\sum 2^k y_k$ is c_0-multiplier ϱ_D-Cauchy, thus c_0-multiplier τ-Cauchy. According to the hypothesis, it is also c_0-multiplier τ-convergent. In particular, the series $\sum \frac{1}{2^k} 2^k y_k$ is τ-convergent to $z \in E$. On the other hand,

$$\sum_{j=1}^{r} y_j = x_{n_{r+1}} - x_{n_1} \quad \text{for all } r \in \mathbb{N}\,.$$

This means that (x_{n_k}) τ-converges to $h = z + x_{n_1}$. In fact, h belongs to E_D because (x_n) is a ϱ_D-bounded sequence (i.e., $(x_n) \subset \lambda D$) and D is τ-closed. Since $[E_D, \varrho_D]$ has a 0-neighbourhood basis formed by τ-closed sets, we deduce [6, p. 59] that (x_{n_k}) is also ϱ_D-convergent to h. Summarizing, (x_n) is a ϱ_D-Cauchy sequence having a subsequence (x_{n_k}) ϱ_D-convergent to $h \in E_D$. Therefore, (x_n) is ϱ_D-convergent to h. ∎

Remarks. (a) If E is locally convex, then c_0-multiplier Cauchy series coincide with weakly unconditionally Cauchy series [6, p. 305] and, therefore, Theorem 2 can be seen as an extension of a result for Banach spaces, due to Bessaga and Pe/lczy/nski [1].

(b) We note that the right-left implication in the last theorem also proves that every Σ-complete space is locally complete. We only have to change the following: By construction $\sum y_k$ is unconditionally ϱ_D-Cauchy, so unconditionally Cauchy in E and since E is Σ-complete, $\sum y_k$ converges.

The next theorem improves results from [2], [7], [13] and determines completely the connexion between C-spaces and O-spaces.

THEOREM 2. *If E has property $(*)$, then the following are equivalent*:

(1) *E is a C-space and is locally complete.*
(2) *E is an O-space.*
(3) *E is Σ-complete and contains no copy of c_0.*

If E is locally convex, then the above conditions are equivalent to

(4) *$[E, \sigma(E, E')]$ is Σ-complete.*

Proof. (1)⇒(2). This follows from Lemma 1, Theorem 1 and the definition of C-space.

(2)⇒(1). This follows from Lemma 1, Theorem 1 and the definition of O-space. We note that if $\sum x_n$ is c_0-multiplier Cauchy, then $\sum \alpha_n x_n$ is again c_0-multiplier Cauchy, for every $(\alpha_n) \in c_0$.

(2)⇒(3). Let $\sum x_n$ be an unconditionally Cauchy series in E. By [6, p. 305], $S(x_n)$ is precompact and so bounded, and since E is an O-space, we deduce that $\sum x_n$ is convergent. On the other hand, suppose that there exists an isomorphism $T : c_0 \to E$ onto its image. It is clear that $\sum T(e_n)$ is a perfectly bounded series in E and not even Cauchy (e_n denotes as usual the sequence with 1 in the nth place and 0 elsewhere).

(3)⇒(2). Let $\sum x_n$ be a perfectly bounded series in E. Since E is Σ-complete, we can assume that $\sum x_n$ is not unconditionally Cauchy. Therefore, there exists a 0-neighbourhood U in E and a sequence of finite subsets σ_k in $\mathbb{N}$ with $\inf \sigma_{k+1} > \sup \sigma_k$ such that $y_k = \sum_{i\in\sigma_k} x_i \notin U$.

Let (α_n) be a null sequence. Since $S(y_n)$ is bounded, $\sum \alpha_n y_n$ is unconditionally Cauchy by Lemma 1, and because E is Σ-complete, $\sum \alpha_n y_n$ converges. Thus, we can define the following linear mapping:

$$T : c_0 \to E\,, \quad (\alpha_n) \mapsto T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n y_n\,.$$

Let us see that T is continuous. Given a closed balanced 0-neighbourhood V of E, there exists $\lambda > 0$ such that $B(y_n) \subset \lambda V$. If B denotes the closed unit ball of c_0, then it is easy to check that $T(\frac{1}{\lambda}B) \subset V$.

Since E contains no copy of c_0, we deduce from [7] that $(T(e_n) = y_n)_n$ tends to zero and we get a contradiction.

(2)⇒(4). Let $\sum x_n$ be an unconditionally Cauchy series in $[E, \sigma(E, E')]$. By [6, p. 305], $S(x_n)$ is weakly precompact, thus bounded and, by (2), $\sum x_n$ converges.

(4)⇒(2). Let $\sum x_n$ be a perfectly bounded series in E and (α_n) a sequence of 0's and 1's. Then $S(\alpha_n x_n)$ is weakly bounded and so weakly precompact and, therefore, $\sum \alpha_n x_n$ is weakly unconditionally Cauchy. Since E is weakly Σ-complete, $\sum \alpha_n x_n$ is weakly convergent. So, we have proved that $\sum x_n$ is weakly subseries convergent, and by the Orlicz–Pettis theorem, we conclude that $\sum x_n$ is convergent. ∎

Remarks. (a) According to the theorem, if E is a Banach space with a copy of c_0, then $[E, \sigma(E, E')]$ is a non-Σ-complete locally complete space.

(b) The argument from (2)⇒(3) shows that, if E is a C-space, then E contains no copy of c_0.

Theorem 2 allows us to rewrite some results of Thomas (see [14] for definitions and notations) on Radon measures just replacing weak Σ-completeness by containing no copy of c_0. In fact, Theorem 3 answers a question posed by Thomas [15, p. 20].

THEOREM 3. *Let E be a quasicomplete locally convex space. Then every linear continuous mapping from any $C(K)$ to E is a Radon measure if and only if E contains no copy of c_0.*

THEOREM 4. *Let μ be a Radon measure with values in a quasicomplete locally convex space E. If E contains no copy of c_0, then for each scalarly μ-integrable mapping f in the sense of Bourbaki, we have $\int f\, d\mu \in E$.*

3. In our last section, we provide some examples to distinguish the concepts which appear in Theorem 2. Namely:

(1) c_0 is a Banach space, thus Σ-complete. Of course, it contains a copy of c_0 so it is neither a C-space nor an O-space.

(2) $[c_0, \sigma(c_0, \ell_1)]$ is locally complete but not Σ-complete by Theorem 2. Moreover it contains no copy of c_0, since the weak topology cannot be normed except in the finite-dimensional case. Looking at the series $\sum e_n$, we conclude that it is not a C-space.

(3) Let Ω be a nonempty set, $\mathcal{A}$ an infinite Boolean ring of subsets of Ω and E a locally convex Hausdorff space. An $\mathcal{A}$-*simple function* $\varphi : \Omega \to E$ is one with a finite number of nonzero values, each of them taken on a set of $\mathcal{A}$. We denote by $S(\mathcal{A}, E)$ the vector space of all $\mathcal{A}$-simple functions defined on Ω with values in E, endowed with the uniform convergence topology.

The following lemma is essentially contained in the proof of [4, Thm. 4(b)]. It establishes that the terms of a subseries summable sequence in $S(\mathcal{A}, E)$ must be supported on the same sets.

LEMMA 2. *Let* (φ_n) *be a subseries summable sequence in* $S(\mathcal{A}, E)$. *Then there exist nonvoid pairwise disjoint sets* $A_1, \ldots, A_k \in \mathcal{A}$ *such that for all* $n \in \mathbb{N}$ *we can write*

$$\varphi_n = \sum_{i=1}^{k} \chi_{A_i} z_i(n)$$

for some $z_i(n) \in E$ $(i = 1, \ldots, k)$. *As usual* χ_A *denotes the characteristic function of the set* A.

THEOREM 5. (1) *$S(\mathcal{A}, E)$ is a C-space if and only if E is a C-space.*

(2) *Assume that E is complete. Then $S(\mathcal{A}, E)$ is a C-space if and only if E contains no copy of c_0.*

Proof. $\Rightarrow$ Let $\sum x_n$ be a c_0-multiplier convergent series and take $A \in \mathcal{A}$ $(A \neq \emptyset)$. It is clear that $\sum \chi_A x_n$ is c_0-multiplier convergent in $S(\mathcal{A}, E)$ and thus convergent to $\chi_A x$ $(x \in E)$, and we conclude that $\sum x_n$ is convergent to x.

$\Leftarrow$ Let $\sum \varphi_n$ be a c_0-multiplier convergent series in $S(\mathcal{A}, E)$. Then $(\frac{1}{n}\varphi_n)$ is a subseries summable sequence in $S(\mathcal{A}, E)$, thus according to Lemma 2, there exist nonvoid pairwise disjoint sets $A_1, \ldots, A_k \in \mathcal{A}$ such that for all $n \in \mathbb{N}$ we can write

$$\frac{1}{n}\varphi_n = \sum_{i=1}^{k} \chi_{A_i} z_i(n)$$

for some $z_i(n) \in E$. Set $w_i(n) = n z_i(n)$ for all $n \in \mathbb{N}$ and $i = 1, \ldots, k$.

Therefore, $\sum_n w_i(n)$ is c_0-multiplier convergent in E for $i = 1, \ldots, k$. Since E is a C-space, $\sum_n w_i(n)$ is convergent to some $w_i \in E$ for $i = 1, \ldots, k$. Finally, we see that $\sum_n \varphi_n$ is convergent to $\sum_{i=1}^{k} \chi_{A_i} w_i$ in $S(\mathcal{A}, E)$, and this shows that $S(\mathcal{A}, E)$ is a C-space. ■

Since $\mathcal{A}$ is infinite, we can take a nonvoid pairwise disjoint sequence $(A_n) \subset \mathcal{A}$. Consider the sequence of $\mathcal{A}$-simple functions

$$\varphi_n = \chi_{A_n} x\,, \qquad n \in \mathbb{N}\ (x \neq 0)\,.$$

The series $\sum \varphi_n$ is c_0-multiplier Cauchy in $S(\mathcal{A}, E)$, but it is not c_0-multiplier convergent. Therefore, by Theorem 1, $S(\mathcal{A}, E)$ is never locally complete. This means that if E is a Banach space containing no copy of c_0, then $S(\mathcal{A}, E)$ provides examples of C-spaces which are not O-spaces.

(4) Finally, the topological direct sum $[c_0, \sigma(c_0, \ell_1)] \oplus c_0$ is non-Σ-complete but locally complete and, obviously, it contains a copy of c_0.

On the other hand, $[c_0, \sigma(c_0, \ell_1)] \oplus S(\mathcal{P}(\mathbb{N}), \mathbb{R})$ is neither locally complete nor a C-space ($\mathcal{P}(\mathbb{N})$ denotes as usual the algebra of all subsets of $\mathbb{N}$). Bearing in mind [7], and since neither summand contains a copy of c_0, the topological direct sum does not either.

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