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A SHORT PROOF OF KRULL'S INTERSECTION THEOREM

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Let R denote a ring with an identity element, N a Noetherian left R-module, B a submodule of N and A an ideal contained in the centre of R (i.e. A is a central ideal). The Intersection Theorem of Krull states that an element c of N belongs to $\bigcap_{n=1}^{\infty} A^n N$ if and only if $c = ac$ for some element a of A. Originally formulated for ideals in a commutative Noetherian ring (Krull [1]; Northcott [2], p. 49), the Theorem was subsequently extended to Noetherian modules over a commutative ring ([3], Theorem 18, p. 206). Proofs were based on the theory of primary decompositions of ideals, or modules, and latterly on the Artin–Rees Lemma. In the generalisation of the Theorem to Noetherian modules over a non-commutative ring R ([3], Theorem 2, p. 293), the theory of primary decompositions is no longer available and so the Artin–Rees Lemma, in its generalised form ([3], Theorem 1, p. 292), is presently used to support the proof. A concise direct proof of the Theorem is given below, which avoids the theory of graded rings and polynomials (e.g. the Hilbert Basis Theorem) required in the standard proofs of the Artin–Rees Lemma. When D is a left R-module and E a central ideal of R, the left R-module $D_N : E = \{c \in N \mid ec \in D \text{ for every } e \in E\}$ will be written simply as $D : E$.

LEMMA 1. *There is an integer $m \geq 1$ such that $A^m N \cap B \subseteq AB$.*

P r o o f. The following argument shows that the ideal A may be assumed to be finitely generated (see [3], p. 292). Let $A_0 \subseteq A$, where A_0 is an ideal finitely generated by central elements of R. Then $A_0 N$ is a submodule of N and A_0 can be chosen so that $A_0 N$ is a maximal member of the set of all submodules which arise in this way. Let $a \in A$. Then $aN \subseteq (A_0 + Ra)N = A_0 N$ implying $AN = A_0 N$ and hence $A^n N = A_0^n N$ for all $n \geq 0$. If m is a positive integer such that $A_0^m N \cap B \subseteq A_0 B$, then $A^m N \cap B = A_0^m N \cap B \subseteq A_0 B \subseteq AB$.

Let $a_1, \ldots, a_r$ be a generating set of central elements for A and write $C = AB$. Since N satisfies the ascending chain condition on left R-submodules, positive integers p_j $(1 \le j \le r)$ exist such that for each $j = 0, 1, \ldots r-1$,

$$(1)\quad (C + a_1^{p_1}N + \ldots + a_j^{p_j}N) : a_{j+1}^{p_{j+1}} = (C + a_1^{p_1}N + \ldots + a_j^{p_j}N) : a_{j+1}^{p_{j+1}+1}.$$

We let $b_j = a_j^{p_j}$ and prove $(b_1N + \ldots + b_jN) \cap (C : b_1R + \ldots + b_jR) \subseteq C$ when $1 \le j \le r$. The case $j = 1$ is true, since $b_1N \cap (C : b_1R) = b_1(C : b_1^2R)$ $= b_1(C : b_1R) \subseteq C$ from equation (1) when $j = 0$. Assuming the inequality true for $j \le s-1$, we have

$$\begin{aligned}
&(b_1N + \ldots + b_sN) \cap (C : b_1R + \ldots + b_sR)\\
&\qquad = (b_1N + \ldots + b_sN) \cap (C : b_sR) \cap (C : b_1R + \ldots + b_{s-1}R)\\
&\qquad \subseteq (C + b_1N + \ldots + b_{s-1}N) \cap (C : b_1R + \ldots + b_{s-1}R)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{using (1) when } j = s-1,\\
&\qquad = C + (b_1N + \ldots + b_{s-1}N) \cap (C : b_1R + \ldots + b_{s-1}R) = C
\end{aligned}$$

using the inductive hypothesis. We now put $m = rp$, where $p = \max\{p_1, \ldots, p_r\}$ and deduce that $A^mN \cap B \subseteq A^mN \cap (C : A) \subseteq (b_1N + \ldots + b_rN) \cap (C : b_1R + \ldots + b_rR) \subseteq C = AB$.

The proof of the Intersection Theorem follows at once from Lemma 1 by putting $B = Rc$ where $c \in \bigcap_{n=1}^{\infty} A^nN$. Then, $Rc = A^mN \cap Rc \subseteq A(Rc)$, implying $c = ac$ for some $a \in A$.

REFERENCES

[1] W. Krull, *Primidealketten in allgemeinen Ringbereichen*, S. B. Heidelberg. Akad. Weiss. 1928, 7. Abh.

[2] D. G. Northcott, *Ideal Theory*, Cambridge University Press, 1965.

[3] —, *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, 1968.

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