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A FIXED POINT THEOREM FOR ASYMPTOTICALLY REGULAR MAPPINGS

BY

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1. Introduction. The concept of asymptotic regularity is due to F. E. Browder and W. V. Petryshyn and in metric notation can be stated as follows: a mapping $T : M \to M$ of a metric space (M, d) to itself is *asymptotically regular* if for any $x \in M$, $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$. It is known (see [1]) that if T is a nonexpansive map of a Banach space, then $T_\lambda = \lambda \mathrm{Id} + (1-\lambda)T$ is asymptotically regular for all $0 < \lambda < 1$.

Recently, in 1987, M. Krüppel [2] proved the following result: Denote by $|||T|||$ the Lipschitz norm of T. Let E be a uniformly convex Banach space and C a closed, convex, bounded subset in E, and let T be a mapping from C into itself. If T is asymptotically regular and $\liminf_{n\to\infty} |||T^n||| \le 1$ then T has a fixed point in C.

At the same time, P. K. Lin [4] constructed an asymptotically regular mapping acting on a weakly compact subset of the Hilbert space l^2 with no fixed point. So the following question is natural: when does an asymptotically regular mapping have a fixed point? In this note we give a sufficient condition for existence of a fixed point generalizing Krüppel's theorem.

2. Main result. Recall that E. A. Lifshitz [3] associated with each metric space (M, d) a constant $\kappa(M)$ defined as follows: denote by $\overline{B}(x, r)$ the closed ball of radius r centered at x. Then

$$\kappa(M) = \sup\{b > 0 : \exists_{a>1} \forall_{x,y\in M} \forall_{r>0} [d(x, y) > r \Rightarrow \exists_{z\in M} \overline{B}(x, br) \cap \overline{B}(y, ar) \subset \overline{B}(z, r)]\}\,.$$

It is immediate that $\kappa(M) \ge 1$ for any metric space. For strictly convex spaces $\kappa(M) > 1$, and it is not difficult to verify that $\kappa(H) = \sqrt{2}$ if H is a Hilbert space.

THEOREM. *Let (M, d) be a complete metric space and T be a mapping from M to M. If T is asymptotically regular*, $\liminf_{n\to\infty} |||T^n||| < \kappa(M)$ *and for some $x \in M$ the sequence $\{T^n x\}$ is bounded then T has a fixed point in C.*

P r o o f. Let $\{n_i\}$ be a sequence of natural numbers such that $\liminf_{n\to\infty} |||T^n||| = \lim_{i\to\infty} |||T^{n_i}||| = k < \kappa(M)$. For any $y \in M$, let

$$r(y) = \inf\{R > 0 : \exists_{x\in M} \limsup_{i\to\infty} d(y, T^{n_i}x) \le R\}.$$

Observe that r is a lower semicontinuous function, and $r(y) = 0$ implies $Ty = y$.

If $\kappa(M) = 1$ then $k < 1$ and the Banach Contraction Principle implies that T has a fixed point. Thus we assume that $k \ge 1$. For $b \in (k, \kappa(M))$ there exists $a > 1$ such that

$$\forall_{u,v\in M}\forall_{r>0}[d(u,v) > r \Rightarrow \exists_{w\in M}\overline{B}(u, br) \cap \overline{B}(v, ar) \subset \overline{B}(w, r)]. \tag{1}$$

Take $\lambda \in (0,1)$ such that $\gamma = \min\{\lambda a, \lambda b/k\} > 1$. We claim that there exists a sequence $\{y_s\} \subset M$ having the property

$$\forall_{s\in\mathbb{N}}[r(y_{s+1}) \le \lambda r(y_s) \text{ and } d(y_{s+1}, y_s) \le (\lambda+\gamma)r(y_s)]. \tag{2}$$

Indeed, take y_1 to be an arbitrary point in M and assume $y_1, \ldots, y_s$ are given. We now construct y_{s+1}. If $r(y_s) = 0$ then $y_{s+1} = y_s$. If $r(y_s) > 0$ then there exists $n_j \in \mathbb{N}$ such that $d(T^{n_j}y_s, y_s) > \lambda r(y_s)$ and $|||T^{n_j}||| \le k\gamma$. From the definition of $r(y_s)$ there exists $x \in M$ for which $\limsup_{i\to\infty} d(T^{n_j}x, y_s) \le r(y_s) < \gamma r(y_s)$. Hence

$$\begin{aligned} d(T^{n_i}x, T^{n_j}y_s) &\le d(T^{n_i}x, T^{n_i+n_j}x) + d(T^{n_i+n_j}x, T^{n_j}y_s) \\ &\le \sum_{q=0}^{n_j-1} d(T^{n_i+q+1}x, T^{n_i+q}x) + |||T^{n_j}|||d(T^{n_i}x, y_s) \end{aligned}$$

and by asymptotic regularity of T, $\limsup_{i\to\infty} d(T^{n_i}x, T^{n_j}y_s) \le k\gamma r(y_s)$.

Since

$$\overline{B}(y_s, \gamma r(y_s)) \cap \overline{B}(T^{n_j}y_s, k\gamma r(y_s)) \subset \overline{B}(y_s, a\lambda r(y_s)) \cap \overline{B}(T^{n_j}y_s, b\lambda r(y_s)) = D$$

the set D is contained in a closed ball centered at w with radius $\lambda r(y_s)$ (condition (1)). Thus $\limsup_{i\to\infty} d(T^{n_i}x, w) \le \lambda r(y_s)$. Take $y_{s+1} = w$. It follows from the above that the sequence $\{y_s\}$ satisfies condition (2). Since $\lambda < 1$, $\{y_s\}$ converges to $y \in M$. But since $r(y) = 0$, y is a fixed point of T.

Added in proof. For the lower bound for $\kappa(L^p)$, $1 < p < +\infty$, see: J. R. L. Webb and W. Zhao, *On connections between set and ball measures of noncompactness*, Bull. London Math. Soc. 22 (1990), 471–477.

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