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TYPE-DEPENDENT POSITIVE DEFINITE FUNCTIONS ON FREE PRODUCTS OF GROUPS

BY

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1. Introduction. Let $\{G_i\}_{i\in I}$ be a family of nontrivial groups. We shall consider their *free product* $G = \ast_{i\in I} G_i$ in which every element x has a unique representation as a *reduced word*

$$(1)\qquad x = g_1 \dots g_n, \quad \text{where } n \geq 0,\ g_k \in G_{i_k} \setminus \{e\} \text{ and } i_k \neq i_{k+1}\,.$$

For such an x we define its *length* $|x| = n$ and its *type* putting $t(x) = i_1 \dots i_n$ (cf. [6]). A function f on $\ast_{i\in I} G_i$ is said to be *radial* if $f(x)$ depends only on $|x|$, and f is said to be *type-dependent* if $f(x)$ depends only on $t(x)$. In particular, each radial function is type-dependent. Note that each type-dependent function f can be uniquely expressed as the composition $f = f' \circ t$, where f' is a function on the set of all types on G.

Harmonic analysis on the free product of groups was studied in several papers (see [2] and [3] and the references given there). For instance Iozzi and Picardello [3] considered the free product of finite cyclic groups of the same order and convolution algebras of radial functions. Such algebras are commutative and their multiplicative functionals are called spherical functions. In this paper we deal with positive definite type-dependent functions on the free product of any groups and the crucial result is given in Theorem 3.2. Our main tool is the notion of τ-positive definite function and Theorem 3.2 provides a motivation for the study of this notion presented in Section 2. In the next section our main result is stated and proved. Then we use our technique to describe positive definite spherical functions (defined in [2]) on the free product of two cyclic groups and on the free product of cyclic groups of the same order (cf. [3]).

We adopt the following notation: if X is a set then $\mathcal{F}(X)$ denotes the linear space of all finitely supported complex functions on X (i.e. $\mathcal{F}(X)$ consists of all linear combinations $\sum a_i \delta_{x_i}$ where δ_x is the characteristic function of $\{x\}$) and $\mathcal{F}_0(X)$ is the subspace of $\mathcal{F}(X)$ formed by all $f \in \mathcal{F}(X)$ such that $\sum f(x) = 0$. The dual space, i.e. the set of all complex functions on X will be denoted by $\mathcal{F}'(X)$. For any complex functions f, g on X we write $\langle f, g\rangle = \langle f, g\rangle_X = \sum f(x) g(x)$ whenever the sum is finite.

2. τ-positive definite functions on $S(I)$. Let I be any fixed set, let $S(I)$ (or simply S) denote the set of all formal words of the form

$$u = i_1 \ldots i_n, \quad n \geq 0,\ i_k \in I,\ i_k \neq i_{k+1}\,, \tag{2}$$

and denote by e the empty word in S. For such a u we define its length $|u| = n$ and put $u^* = i_n \ldots i_1$. Let $\tau : I \to [0, \infty)$ be a fixed function. We define a complex *-algebra $\mathcal{F}_\tau(S) = \langle \mathcal{F}(S), \underset{\tau}{*}, {}^* \rangle$ by the following relations:

$$\delta_i \underset{\tau}{*} \delta_i = (1 - \tau(i))\delta_i + \tau(i)\delta_e \quad \text{for } i \in I\,, \tag{3.a}$$

$$\delta_{i_1} \underset{\tau}{*} \ldots \underset{\tau}{*} \delta_{i_n} = \delta_u \quad \text{for } u = i_1 \ldots i_n \in S(I)\,, \tag{3.b}$$

$$(\delta_u)^* = \delta_{u^*} \quad \text{for } u \in S(I)\,. \tag{3.c}$$

$\mathcal{F}_\tau(S)$ is in fact the free complex unital algebra generated by $\{\delta_i\}_{i \in I}$ (with unit δ_e) and with the only relations

$$(\delta_i)^* = \delta_i \quad \text{and} \quad \delta_i \underset{\tau}{*} \delta_i = (1 - \tau(i))\delta_i + \tau(i)\delta_e \quad \text{for } i \in I\,.$$

Note that $\mathcal{F}_0(S)$ is an ideal in $\mathcal{F}_\tau(S)$. For $f \in \mathcal{F}(S)$, $\phi \in \mathcal{F}'(S)$ we define their *dual left* and *right τ-convolutions* $f \underset{\tau}{\square} \phi$, $\phi \underset{\tau}{\square} f \in \mathcal{F}'(S)$ putting for any $g \in \mathcal{F}(S)$

$$\langle f \underset{\tau}{\square} \phi, g \rangle = \langle \phi, f^\vee \underset{\tau}{*} g \rangle, \quad \langle \phi \underset{\tau}{\square} f, g \rangle = \langle \phi, g \underset{\tau}{*} f^\vee \rangle\,, \tag{4}$$

where by definition $f^\vee(u) = f(u^*)$. This means that for any $f \in \mathcal{F}(S)$ the map $\mathcal{F}'(S) \ni \phi \mapsto f \underset{\tau}{\square} \phi \in \mathcal{F}'(S)$ (resp. $\mathcal{F}'(S) \ni \phi \mapsto \phi \underset{\tau}{\square} f \in \mathcal{F}'(S)$) is dual to $\mathcal{F}(S) \ni g \mapsto f^\vee \underset{\tau}{*} g \in \mathcal{F}(S)$ (resp. $\mathcal{F}(S) \ni g \mapsto g \underset{\tau}{*} f^\vee \in \mathcal{F}(S)$). Obviously

$$(f_1 + f_2) \underset{\tau}{\square} \phi = f_1 \underset{\tau}{\square} \phi + f_2 \underset{\tau}{\square} \phi\,, \tag{5.a}$$

$$f_1 \underset{\tau}{\square} (f_2 \underset{\tau}{\square} \phi) = (f_1 \underset{\tau}{*} f_2) \underset{\tau}{\square} \phi\,, \tag{5.b}$$

$$(f \underset{\tau}{\square} \phi)^\vee = \phi^\vee \underset{\tau}{\square} f^\vee\,, \tag{5.c}$$

and similarly for the right convolution, where $f, f_1, f_2 \in \mathcal{F}(S)$ and $\phi \in \mathcal{F}'(S)$.

Definition 2.1. We say that a complex function ϕ on $S(I) = S$ is *τ-positive definite* if $\langle \phi, \alpha^* \underset{\tau}{*} \alpha \rangle \geq 0$ for all $\alpha \in \mathcal{F}(S)$. Similarly, a complex function ψ on S is *τ-negative definite* if $\langle \psi, \beta^* \underset{\tau}{*} \beta \rangle \leq 0$ for all $\beta \in \mathcal{F}_0(S)$. We will denote by $\mathcal{P}_\tau(S)$ and by $\mathcal{N}_\tau(S)$ the convex cones of all τ-positive definite and all τ-negative definite functions on S, respectively.

Notice that in the case $\tau \equiv 1$, $S(I)$ can be regarded as the free product group $\ast_{i \in I} \mathbb{Z}_2$ and in the case $\tau \equiv 0$ as a *-semigroup generated by I in which $i^2 = i^* = i$ for all $i \in I$.

PROPOSITION 2.2. *Let $\sigma, \tau : I \to [0, \infty)$. Then there exists a *-isomorphism $H_{\tau\sigma}$ of $\mathcal{F}_\sigma(S)$ onto $\mathcal{F}_\tau(S)$. Moreover, there exists an automorphism $T_{\tau\sigma}$ of the linear space $\mathcal{F}'(S)$ such that*

(a) $$T_{\tau\sigma}(\mathcal{P}_\sigma(S)) = \mathcal{P}_\tau(S)\,,$$

(b) $$T_{\tau\sigma}(\mathcal{N}_\sigma(S)) = \mathcal{N}_\tau(S)\,,$$

(c) $$T_{\tau\sigma}(f \underset{\sigma}{\square} \phi) = H_{\tau\sigma}(f) \underset{\tau}{\square} T_{\tau\sigma}(\phi)\,,$$

(d) $$T_{\tau\sigma}(\phi \underset{\sigma}{\square} f) = T_{\tau\sigma}(\phi) \underset{\tau}{\square} H_{\tau\sigma}(f)\,,$$

where $f \in \mathcal{F}(S)$, $\phi \in \mathcal{F}'(S)$.

Proof. For any $i \in I$ we put

(6) $$H_{\tau\sigma}(\delta_i) = \frac{1+\sigma(i)}{1+\tau(i)}\delta_i + \frac{\tau(i)-\sigma(i)}{1+\tau(i)}\delta_e\,.$$

Simple calculation shows that $H_{\tau\sigma}(\delta_i \underset{\sigma}{*} \delta_i) = H_{\tau\sigma}(\delta_i) \underset{\tau}{*} H_{\tau\sigma}(\delta_i)$, hence $H_{\tau\sigma}$ extends to a *-homomorphism of $\mathcal{F}_\sigma(S)$ into $\mathcal{F}_\tau(S)$. Moreover, $H_{\upsilon\tau} \circ H_{\tau\sigma} = H_{\upsilon\sigma}$ so $H_{\tau\sigma}$ is an isomorphism. Defining $T_{\tau\sigma} : \mathcal{F}'(S) \to \mathcal{F}'(S)$ to be the dual of $H_{\sigma\tau} : \mathcal{F}(S) \to \mathcal{F}(S)$ we have

(7) $$(T_{\tau\sigma}\phi)(u) = \langle \phi, H_{\sigma\tau}(\delta_u)\rangle \quad \text{and} \quad \langle (T_{\tau\sigma}\phi), g\rangle = \langle \phi, H_{\sigma\tau}(g)\rangle$$

for any $g \in \mathcal{F}(S)$ and the proposition easily follows.

EXAMPLE 2.3. In the examples below we assume $\tau \equiv 0$, so that we treat S as a *-semigroup in which $i^2 = i^* = i$ for each generator $i \in I$ and our notion of positive definiteness coincides with that on *-semigroups (see [1]).

1) Let $A = (a_{ij})$, $i, j \in I \cup \{e\}$, be a positive definite matrix such that $a_{ii} = 1$ for all $i \in I \cup \{e\}$. Define a function ϕ_A on S in the following way: $\phi_A(e) = 1$ and for $u \in S$ as in (2), $n \geq 1$,

$$\phi_A(u) = a_{ei_1}a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ne}\,.$$

Note that for any $u, v \in S$ we have $\phi_A(v^*u) = \overline{\alpha(v)}a_{j_1i_1}\alpha(u)$, where $\alpha(e) = 1$, $\alpha(u) = a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ne}$ for u as in (2), $|u| \geq 1$, and i_1, j_1 are the first letters of u and v respectively; for $u = e$ or $v = e$ we put $i_1 = e$ or $j_1 = e$ respectively. Hence for any complex finitely supported function f on S we have

$$\sum_{u,v\in S(I)} \phi_A(v^*u)f(u)\overline{f(v)} = \sum_{i,j\in I\cup\{e\}} a_{ji}s(i)\overline{s(j)}$$

with $s(e) = f(e)$ and $s(i) = \sum \alpha(u)f(u)$ for $i \in I$, where the summation is over all $u \in S$ as in (2) with $i_1 = i$. This proves that ϕ_A is a 0-positive definite ($\tau \equiv 0$) function on S. Note that if matrices $A = (a_{ij})$, $B = (b_{ij})$, $i, j \in I \cup \{e\}$, satisfy our assumptions then $\phi_A\phi_B = \phi_{A\circ B}$, where $A \circ B$ denotes the Schur product $(a_{ij}b_{ij})$, $i, j \in I \cup \{e\}$.

2) Assume that $r \in [0,1]$, $I = \{1,\ldots,N\}$, $N \ge 2$, and consider the matrix $A = (a_{ij})$, $i,j \in I \cup \{0\}$, given by $a_{ii} = 1$ for $i \ge 0$, $a_{i0} = a_{0i} = \sqrt{r}$ for $i \ge 1$ and $a_{ij} = (Nr-1)/(N-1)$ for $i \ne j$, $i,j \ge 1$. Note that A is positive definite. Indeed, for any complex $z_0, z_1, \ldots, z_N$ we have

$$\sum_{i,j=0}^{N} a_{ij} z_i \overline{z}_j = |z_0 + \sqrt{r}(z_1 + \ldots + z_N)|^2 + \frac{1-r}{N-1} \sum_{1 \le i < j \le N} |z_i - z_j|^2 \ge 0 \, .$$

The corresponding positive definite function ϕ_A is

$$\phi_A(u) = \begin{cases} 1 & \text{if } u = e\,, \\ r\left(\dfrac{Nr-1}{N-1}\right)^{|u|-1} & \text{if } u \ne e\,. \end{cases}$$

3. Type-dependent functions. Let H be a finite group, $\#(H) = k$, and let μ_H (resp. μ) denote the probability measure equidistributed over H (resp. over $H \setminus \{e\}$). Since $\mu_H * \mu_H = \mu_H$ and $\mu_H = (1/k)\delta_e + ((k-1)/k)\mu$ we immediately obtain

$$\mu * \mu = (1-r)\mu + r\delta_e \, , \tag{8}$$

where $r = 1/(k-1)$. Now let $\{G_i\}_{i\in I}$ be a fixed family of discrete groups. We put $\tau(i) = 1/(\#(G_i) - 1)$.

Assume for a moment that all G_i are finite and let $\mathcal{F}_t(G)$ denote the linear space of all finitely supported type-dependent complex functions on $G = *_{i\in I} G_i$. Then $\mathcal{F}_t(G)$ consists of all linear combinations of the μ_u, where for $u \in S = S(I)$ as in (2), μ_u denotes the probability measure equidistributed over all elements of type u, i.e. $\mu_u(x) = \tau(i_1)\ldots\tau(i_n)$ if $t(x) = u$ and $\mu_u(x) = 0$ otherwise. By (8) and by the definition of the free product we have

$$\mu_i * \mu_i = (1-\tau(i))\mu_i + \tau(i)\mu_e \quad \text{for } i \in I\,, \tag{9.a}$$

$$\mu_u = \mu_{i_1} * \ldots * \mu_{i_n} \quad \text{for } u = i_1 \ldots i_n \in S\,, \tag{9.b}$$

$$(\mu_u)^* = \mu_{u^*} \quad \text{for } u \in S\,. \tag{9.c}$$

Formulas (3) make it obvious that the *-algebra $\mathcal{F}_\tau(S)$ is isomorphic to $\mathcal{F}_t(G)$ and the isomorphism is given by

$$\mathcal{F}(S) \ni f \mapsto \sum_{u\in S} f(u)\mu_u \, .$$

The following proposition explains the term "dual τ-convolution" introduced in Section 2.

PROPOSITION 3.1. *Under the above assumptions we have*

$$\sum_{u\in S} f(u)\mu_u * (\phi\circ t) = (f \mathop{\square}_{\tau} \phi)\circ t\,,$$
$$(\phi\circ t) * \sum_{u\in S} f(u)\mu_u = (\phi \mathop{\square}_{\tau} f)\circ t$$

for any $\phi\in\mathcal{F}'(S)$, $f\in\mathcal{F}(S)$.

Proof. For any $x\in G$ we have

$$\Big(\sum_{u\in S} f(u)\mu_u * (\phi\circ t)\Big)(x) = \Big\langle \sum_{u\in S} f(u)\mu_u * (\phi\circ t), \delta_x\Big\rangle$$
$$= \Big\langle \sum_{u\in S} f(u)\mu_u * (\phi\circ t), \mu_{t(x)}\Big\rangle = \Big\langle \phi\circ t, \Big(\sum_{u\in S} f(u)\mu_u\Big)^{\vee} * \mu_{t(x)}\Big\rangle$$
$$= \langle \phi, f^{\vee} \mathop{*}_{\tau} \delta_{t(x)}\rangle_S = \langle f \mathop{\square}_{\tau} \phi, \delta_{t(x)}\rangle_S = (f \mathop{\square}_{\tau} \phi)(t(x))\,.$$

We are now able to formulate the main result.

THEOREM 3.2. *Let* $\{G_i\}_{i\in I}$ *be any family of discrete groups and suppose that* ϕ *is a complex function on* $S(I)$. *Then the function* $\phi\circ t$ *is positive* (*resp. negative*) *definite on the free product group* $G = *_{i\in I} G_i$ *if and only if* ϕ *is* τ-*positive* (*resp.* τ-*negative*) *definite on* $S(I)$, *where* $\tau(i) = 1/(\#(G_i)-1)$ *for any* $i\in I$.

Proof. Let ϕ be a τ-positive definite function on $S(I)=S$. Fix f in $\mathcal{F}(G)$ and define $\widetilde{f}\in\mathcal{F}(S)$ by $\widetilde{f}(u)=\sum f(x)$, where the summation is over all $x\in G$ of type u. We are going to show that

$$(f^* * f)^{\sim} = \widetilde{f}^* \mathop{*}_{\tau} \widetilde{f} + R\,, \tag{10}$$

where R is a finite sum of terms of the form $\beta^* \mathop{*}_{\tau} \beta$, $\beta\in\mathcal{F}_0(S)$. We shall do it by induction on n, the maximal length of elements in the support of f.

If $n=0$ we have $f=f(e)\delta_e$ and the formula is obvious ($R=0$). Now take any $f\in\mathcal{F}(G)$. Then f can be expressed as

$$f = f(e)\delta_e + \sum_{x\in G\setminus\{e\}} f(x)\delta_x = f(e)\delta_e + \sum_{i\in I}\sum_{g\in G_i\setminus\{e\}} \delta_g * f_g\,,$$

where all f_g satisfy the induction assumption, i.e.

$$(f_g^* * f_g)^{\sim} = \widetilde{f}_g^* \mathop{*}_{\tau} \widetilde{f}_g + R(g)\,.$$

We use the following simple formulas:

$$(\delta_g * f_g)^{\sim} = \delta_i \mathop{*}_{\tau} \widetilde{f}_g \quad \text{if } g\in G_i\setminus\{e\}\,,$$

$$[(\delta_h * f_h)^* * (\delta_g * f_g)]^\sim = \begin{cases} [f_g^* * f_g]^\sim = \widetilde{f}_g^* \underset{\tau}{*} \widetilde{f}_g + R(g) & \text{if } h = g\,, \\ \widetilde{f}_h^* \underset{\tau}{*} \delta_i \underset{\tau}{*} \widetilde{f}_g & \text{if } h \neq g\,, \\ & \quad h, g \in G_i \setminus \{e\}\,, \\ \widetilde{f}_h^* \underset{\tau}{*} \delta_j \underset{\tau}{*} \delta_i \underset{\tau}{*} \widetilde{f}_g & \text{if } g \in G_i \setminus \{e\}\,, \\ & \quad h \in G_j \setminus \{e\},\ i \neq j. \end{cases}$$

Hence we have

$$\begin{aligned} &[f^* * f]^\sim - \widetilde{f}^* * \widetilde{f} \\ &\quad = \sum_{i,j\in I} \sum_{\substack{g\in G_i\setminus\{e\} \\ h\in G_j\setminus\{e\}}} \{[(\delta_h * f_h)^* * (\delta_g * f_g)]^\sim - \widetilde{f}_h^* \underset{\tau}{*} \delta_j \underset{\tau}{*} \delta_i \underset{\tau}{*} \widetilde{f}_g\} \\ &\quad = \sum_{i\in I} \Big(\sum_{g\in G_i\setminus\{e\}} [\widetilde{f}_g^* \underset{\tau}{*} \widetilde{f}_g + R(g) - \widetilde{f}_g^* \underset{\tau}{*} \delta_i \underset{\tau}{*} \delta_i \underset{\tau}{*} \widetilde{f}_g] \\ &\quad\quad + \sum_{\substack{g,h\in G_i\setminus\{e\} \\ g\neq h}} [\widetilde{f}_h^* \underset{\tau}{*} \delta_i \underset{\tau}{*} \widetilde{f}_g - \widetilde{f}_h^* \underset{\tau}{*} \delta_i \underset{\tau}{*} \delta_i \underset{\tau}{*} \widetilde{f}_g] \Big) \\ &\quad = \sum_{i\in I} \Big(\sum_{g\in G_i\setminus\{e\}} [(1-\tau(i))\widetilde{f}_g^* \underset{\tau}{*} (\delta_e - \delta_i) \underset{\tau}{*} \widetilde{f}_g + R(g)] \\ &\quad\quad + \sum_{\substack{g,h\in G_i\setminus\{e\} \\ g\neq h}} [-\tau(i)\widetilde{f}_h^* \underset{\tau}{*} (\delta_e - \delta_i) \underset{\tau}{*} \widetilde{f}_g] \Big) \\ &\quad = \sum_{i\in I} \Big(R(i) + \sum_{g\in G_i\setminus\{e\}} R(g) \Big)\,, \end{aligned}$$

where

$$R(i) = \sum_{g\in G_i\setminus\{e\}} \widetilde{f}_g^* \underset{\tau}{*} (\delta_e - \delta_i) \underset{\tau}{*} \widetilde{f}_g$$

if $\tau(i) = 0$ (i.e. if G_i is infinite), and

$$R(i) = \frac{1}{2}\tau(i) \sum_{\substack{g,h\in G_i\setminus\{e\} \\ g\neq h}} (\widetilde{f}_g - \widetilde{f}_h)^* \underset{\tau}{*} (\delta_e - \delta_i) \underset{\tau}{*} (\widetilde{f}_g - \widetilde{f}_h)$$

when G_i is finite. Since we have $(\delta_e - \delta_i) \underset{\tau}{*} (\delta_e - \delta_i) = (1+\tau(i))(\delta_e - \delta_i)$ and $(\delta_e - \delta_i) \underset{\tau}{*} \alpha \in \mathcal{F}_0(S)$ for any $\alpha \in \mathcal{F}(S)$, formula (10) is proved.

Next, $\langle \phi \circ t, f_1 \rangle_G = \langle \phi, \widetilde{f}_1 \rangle_S$ for any $f_1 \in \mathcal{F}(G)$ and

$$\langle \phi \circ t, f^* * f \rangle_G = \langle \phi, \widetilde{f}^* \underset{\tau}{*} \widetilde{f} \rangle_S + \langle \phi, R \rangle_S \geq 0$$

so $\phi \circ t$ is a positive definite function on G.

Now, suppose that $\phi\circ t$ is positive definite on G and first suppose that all G_i are finite. Fix $\alpha\in\mathcal{F}(S)$ and let $f=\sum\alpha(u)\mu_u$, $u\in S$. Then, by (9) we have $0\le\langle\phi\circ t,f^**f\rangle_G=\langle\phi,\alpha^*\underset{\tau}{*}\alpha\rangle_S$ so ϕ is τ-positive definite on S. In the general case for any natural n define $\tau_n(i)=\tau(i)$ if $\tau(i)>0$ and $\tau_n(i)=1/n$ if $\tau(i)=0$. Let $G'=\ast_{i\in I}G'_i$, where $G'_i=G_i$ if G_i is finite and $G'_i=\mathbb{Z}_{n+1}$ otherwise. In virtue of Proposition 2 of [5] the function $\phi\circ t'$ is positive definite on G', where t' is the type on G', so ϕ is τ_n-positive definite on S for all $n\in\mathbb{N}$. Hence we have

$$0\le\lim_{n\to\infty}\langle\phi,\alpha^*\underset{\tau_n}{*}\alpha\rangle=\langle\phi,\alpha^*\underset{\tau}{*}\alpha\rangle\,.$$

Since the arguments remain true for negative definite functions, the proof is complete.

COROLLARY 3.3. *Let $\{G_i\}_{i\in I}$ be any family of finite groups, let $G=\ast_{i\in I}G_i$ and consider the natural projection $\mathcal{E}$ from the functions on G onto the type-dependent functions on G defined by $(\mathcal{E}\phi)(x)=\langle\phi,\mu_{t(x)}\rangle$. Then $\mathcal{E}$ maps positive* (*resp. negative*) *definite functions to positive* (*resp. negative*) *definite functions.*

Proof. Let f be any function in $\mathcal{F}_t(G)$ (resp. in $\mathcal{F}_t(G)\cap\mathcal{F}_0(G)$). Then $\langle\mathcal{E}\phi,f^**f\rangle=\langle\phi,\mathcal{E}(f^**f)\rangle=\langle\phi,f^**f\rangle$, which concludes the proof.

COROLLARY 3.4. *Let G be the amalgamated free product $\ast_{A,i\in I}G_i$ and let ϕ be any function on $S(I)$. Then the function $\phi\circ t$ is positive* (*resp. negative*) *definite on G if and only if ϕ is τ-positive* (*resp. τ-negative*) *definite on $S(I)$, where $\tau(i)=((G_i:A)-1)^{-1}$.*

Proof. For any $i\in I$ choose a set $S_i\cup\{e\}$ of left coset representatives of G_i modulo A. Then each x in G has a unique representation as a *reduced word*

$$x=s_1\dots s_na,\qquad\text{where } n\ge0,\ a\in A,\ s_k\in S_{i_k},\ i_k\ne i_{k+1}\,,$$

and $t(x)=i_1\dots i_n$ (see [6] for details). For $i\in I$ let $\tau(i)$ be the inverse of $(G_i:A)-1=\#(S_i)$ and let $\{H_i\}_{i\in I}$ be a family of groups such that $\#(H_i)=(G_i:A)$. We show that a type-dependent function $\phi\circ t$ is positive (resp. negative) definite on G if and only if $\phi\circ t$ is positive (resp. negative) definite on $H=\ast_{i\in I}H_i$. To see this take a family of bijections $h_i:H_i\setminus\{e\}\to S_i$ and put $h(g_1\dots g_n)=h_{i_1}(g_1)\dots h_{i_n}(g_n)$, where $g_k\in H_{i_k}\setminus\{e\}$ and $g_1\dots g_n$ is a reduced word in H. It is obvious that $t(h(x_2)^{-1}h(x_1))=t(x_2^{-1}x_1)$ for any $x_1,x_2\in H$. It is enough to note that for any $f\in\mathcal{F}(G)$, $\alpha\in\mathcal{F}(H)$

$$\sum_{y_1,y_2\in G}\phi(t(y_2^{-1}y_1))f(y_1)\overline{f(y_2)}=\sum_{x_1,x_2\in H}\phi(t(x_2^{-1}x_1))f_H(x_1)\overline{f_H(x_2)}\,,$$

$$\sum_{x_1,x_2\in H}\phi(t(x_2^{-1}x_1))\alpha(x_1)\overline{\alpha(x_2)}=\sum_{y_1,y_2\in G}\phi(t(y_2^{-1}y_1))\alpha_G(y_1)\overline{\alpha_G(y_2)}\,,$$

where $f_H(x) = \sum f(h(x)a)$, $a \in A$ and $\alpha_G(y) = \alpha(x)$ if $y = h(x)$ and $\alpha_G(y) = 0$ if $y \notin h(H)$.

4. Free product of two groups. From now on we restrict our attention to the case $\#(I) = 2$, say $I = \{+,-\}$ (cf. [2]). Our aim is to characterize positive definite spherical functions on $G = \mathbb{Z}_r * \mathbb{Z}_s$, $r \ge s$, considered by D. I. Cartwright and P. M. Soardi in [2]. Recall that a type-dependent function ϕ on $G = \mathbb{Z}_r * \mathbb{Z}_s$ is said to be *spherical* if $\phi(e) = 1$, $\phi(x) = \phi(x^{-1})$ for all $x \in G$ and there exists a complex number λ such that $\phi * \chi_1 = \lambda\phi$ (χ_1 denotes the characteristic function of the set $\{x \in G : |x| = 1\}$). Let us also mention that our notion of type-dependent function coincides with the notion of "semiradial function" used in [2]. Since spherical functions are type-dependent Theorem 3.2 allows us to consider $S(I) = S$ instead of $\mathbb{Z}_r * \mathbb{Z}_s$. For simplicity we write $+n$ and $-n$, $n \in \mathbb{N}$, to denote $u = +-+\ldots\pm$ and $u = -+-\ldots\mp \in S$, respectively, $|u| = n$, and 0 to denote the empty word in S. For example $-(2j+1)$ denotes the word $-+\ldots-$ with length $2j+1$.

Let β_+, β_- be any positive numbers and consider the measure $\mu = \beta_+\delta_+ + \beta_-\delta_-$ on S. For any $\tau : I = \{+,-\} \to [0,\infty)$ let $\mathcal{A}(\tau,\mu)$ denote the convolution subalgebra of $\mathcal{F}_\tau(S)$ generated by μ. Note that for any $\varepsilon, \eta \in \{+,-\}$, $n \in \mathbb{N}$ we have

$$\delta_{\varepsilon n} \underset{\tau}{*} \delta_\eta = \begin{cases} \delta_{\varepsilon(n+1)} & \text{if } \varepsilon(-1)^n = \eta\,, \\ (1-\tau_\eta)\delta_{\varepsilon n} + \tau_\eta\delta_{\varepsilon(n-1)} & \text{otherwise}\,, \end{cases} \tag{11}$$

where $\tau_+ = \tau(+)$ and $\tau_- = \tau(-)$. The following property can be proved similarly to Proposition 2 of [2].

PROPOSITION 4.1. *For a complex function f on S, $f \underset{\tau}{*} \mu = \mu \underset{\tau}{*} f$, if and only if $f(+2j) = f(-2j)$ and*

$$\begin{aligned}\beta_+\tau_+ f(+2j+1) &- \beta_-\tau_- f(-(2j+1)) \\ &= [(1-\tau_+)\beta_+ - (1-\tau_-)\beta_-]f(+2j) + \beta_+ f(-(2j-1)) - \beta_- f(+2j-1)\end{aligned}$$

for all $j \ge 1$. If, moreover, f has finite support, then $f \underset{\tau}{*} \mu = \mu \underset{\tau}{*} f$ if and only if $f \in \mathcal{A}(\tau,\mu)$. Thus $\mathcal{A}(\tau,\mu)$ is a maximal abelian subalgebra of $\mathcal{F}_\tau(S)$.

Now we are going to describe multiplicative functionals on $\mathcal{A}(\tau,\mu)$.

PROPOSITION 4.2. *Let $\tau : I = \{+,-\} \to [0,\infty)$, let β_+, β_- be positive numbers, $\mu = \beta_+\delta_+ + \beta_-\delta_-$ and $\lambda \in \mathbb{C}$, $\lambda \ne x_0 = (\beta_+(1-\tau_+) + \beta_-(1-\tau_-))/2$. Then there exists a unique function ϕ_λ on S satisfying*

$$\phi_\lambda \underset{\tau}{\square} \mu = \lambda\phi_\lambda, \quad \phi_\lambda(u) = \phi_\lambda^\vee(u) \quad \textit{for all } u \in S, \quad \phi_\lambda(0) = 1\,.$$

Proof. Let ϕ be any function on S. Since

$$(\phi \underset{\tau}{\square} \mu)(u) = \langle \phi \underset{\tau}{\square} \mu, \delta_u \rangle = \langle \phi, \delta_u \underset{\tau}{*} \mu \rangle = \beta_+ \langle \phi, \delta_u \underset{\tau}{*} \delta_+ \rangle + \beta_- \langle \phi, \delta_u \underset{\tau}{*} \delta_- \rangle$$

and by (11) we must solve the following equations:

$$\phi(+2j) = \phi(-2j)\,, \tag{12.a}$$

$$\lambda = \beta_+ \phi_\lambda(+1) + \beta_- \phi_\lambda(-1)\,, \tag{12.b}$$

$$\lambda\phi_\lambda(+2j) = \beta_-\tau_-\phi_\lambda(+2j-1)+\beta_-(1-\tau_-)\phi_\lambda(+2j)+\beta_+\phi_\lambda(+2j+1)\,, \quad \text{for } j \ge 1\,, \tag{12.c}$$

$$\lambda\phi_\lambda(+2j+1) = \beta_+\tau_+\phi_\lambda(+2j) + \beta_+(1-\tau_+)\phi_\lambda(+2j+1) + \beta_-\phi_\lambda(+2j+2), \quad \text{for } j \ge 0\,, \tag{12.d}$$

$$\lambda\phi_\lambda(-2j) = \beta_+\tau_+\phi_\lambda(-(2j-1)) + \beta_+(1-\tau_+)\phi_\lambda(-2j) + \beta_-\phi_\lambda(-(2j+1)), \quad \text{for } j \ge 1\,, \tag{12.e}$$

$$\lambda\phi_\lambda(-(2j+1)) = \beta_-\tau_-\phi_\lambda(-2j) + \beta_-(1-\tau_-)\phi_\lambda(-(2j+1)) + \beta_+\phi_\lambda(-(2j+2)), \quad \text{for } j \ge 0\,. \tag{12.f}$$

In particular,

$$\beta_+\phi_\lambda(+1) + \beta_-\phi_\lambda(-1) = \lambda\,, \tag{13.a}$$

$$\beta_+[\beta_+\tau_+ + (\beta_+(1-\tau_+) - \lambda)\phi_\lambda(+1)] = \beta_-[\beta_-\tau_- + (\beta_-(1-\tau_-) - \lambda)\phi_\lambda(-1)]\,. \tag{13.b}$$

The last two equations have a unique solution $(\phi_\lambda(+1), \phi_\lambda(-1))$ if $\lambda \neq x_0$, namely

$$\phi_\lambda(+1) = \frac{\lambda^2 - \beta_-(1-\tau_-)\lambda + \beta_+^2\tau_+ - \beta_-^2\tau_-}{\beta_+[2\lambda - \beta_+(1-\tau_+) - \beta_-(1-\tau_-)]}\,, \tag{14.a}$$

$$\phi_\lambda(-1) = \frac{\lambda^2 - \beta_+(1-\tau_+)\lambda - \beta_+^2\tau_+ + \beta_-^2\tau_-}{\beta_-[2\lambda - \beta_+(1-\tau_+) - \beta_-(1-\tau_-)]} \tag{14.b}$$

and the rest of the proof is as in [2].

Remark 4.3. a) For any polynomial P we have

$$\langle \phi_\lambda, P(\mu) \rangle = \langle \phi_\lambda \underset{\tau}{\square} P(\mu), \delta_0 \rangle = P(\lambda)\,,$$

hence $g \mapsto \langle \phi_\lambda, g \rangle$ is the unique multiplicative functional on $\mathcal{A}(\tau, \mu)$ taking the value λ at μ. The function ϕ_λ is called the (τ, μ)-*spherical function*. It is worth pointing out that this notion coincides with the notion of spherical function defined in [2] when $\tau_+ = 1/(r-1)$, $\tau_- = 1/(s-1)$, $\beta_+ = r-1$, $\beta_- = s-1$.

b) It is easy to check that if $\beta_-(1+\tau_-) = \beta_+(1+\tau_+)$ and $\lambda = x_0$ then the equations (13) are linearly dependent. Moreover, for $\lambda \neq x_0$

$$\phi_\lambda(+1) = \frac{2\lambda - \beta_-(1-\tau_-) + \beta_+(1-\tau_+)}{4\beta_+},$$

$$\phi_\lambda(-1) = \frac{2\lambda - \beta_+(1-\tau_+) + \beta_-(1-\tau_-)}{4\beta_-}.$$

Therefore if $\beta_-(1+\tau_-) = \beta_+(1+\tau_+)$ then it is natural to define ϕ_{x_0} to be the pointwise limit of ϕ_λ as $\lambda \to x_0$. On the other hand, if $\beta_-(1+\tau_-) \neq \beta_+(1+\tau_+)$ then the equations (13) are inconsistent for $\lambda = x_0$ and ϕ_{x_0} does not exist.

In case $\tau_+ = \tau_- = 0$ the formulas (12) easily yield

COROLLARY 4.4. *Let β_+, β_- be positive numbers, $\mu = \beta_+\delta_+ + \beta_-\delta_-$, and assume that $\tau_+ = \tau_- = 0$. Then for $\lambda \neq (\beta_+ + \beta_-)/2$ the (τ,μ)-spherical function ϕ_λ is given by*

$$\phi_\lambda(0) = 1,$$

$$\phi_\lambda(+-+\ldots\pm) = \frac{\lambda}{2\lambda - \beta_- - \beta_+}\lambda_+\lambda_-\lambda_+\ldots\lambda_\pm,$$

$$\phi_\lambda(-+-\ldots\mp) = \frac{\lambda}{2\lambda - \beta_- - \beta_+}\lambda_-\lambda_+\lambda_-\ldots\lambda_\mp,$$

where $\lambda_+ = (\lambda - \beta_-)/\beta_+$ and $\lambda_- = (\lambda - \beta_+)/\beta_-$.

Now we are in a position to present the main theorem of this section.

THEOREM 4.5. *Let $\tau_+,\tau_- \geq 0$, $\beta_+,\beta_- > 0$ and let ϕ_λ be the (τ,μ)-spherical function. Assume that $\beta_-(1+\tau_-) \leq \beta_+(1+\tau_+)$. Then the following conditions are equivalent*:

(a) *ϕ_λ is τ-positive definite on S,*
(b) *$-\tau_+ \leq \phi_\lambda(+1) \leq 1$ and $-\tau_- \leq \phi_\lambda(-1) \leq 1$,*
(c) *$\lambda \in [-\beta_+\tau_+ - \beta_-\tau_-, -\beta_+\tau_+ + \beta_-] \cup [-\beta_-\tau_- + \beta_+, \beta_+ + \beta_-]$.*

Proof. To show (a)$\Rightarrow$(b) it is sufficient to observe that for any τ-positive definite function ϕ on $S(I)$ and $i \in I$, $-\tau(i)\phi(e) \leq \phi(i) \leq \phi(e)$. Moreover, routine calculations based on (14) yield (b)$\Leftrightarrow$(c).

Now we prove (c)$\Rightarrow$(a) in the case $\tau \equiv 0$ (i.e. $\tau_+ = \tau_- = 0$). Suppose that $\lambda \in [0,\beta_-] \cup [\beta_+, \beta_+ + \beta_-]$ and define

$$w = \left(\frac{(\lambda-\beta_+)(\lambda-\beta_-)}{\beta_+\beta_-}\right)^{1/2},$$

$$\alpha_+ = \left(\frac{\lambda(\lambda-\beta_-)}{\beta_+(2\lambda-\beta_+-\beta_-)}\right)^{1/2}, \qquad \alpha_- = \varepsilon\left(\frac{\lambda(\lambda-\beta_+)}{\beta_-(2\lambda-\beta_+-\beta_-)}\right)^{1/2},$$

where $\varepsilon = -1$ if $\lambda \in [0, \beta_-]$ and $\varepsilon = 1$ if $\lambda \in [\beta_+, \beta_+ + \beta_-]$. According to Corollary 4.4 we have $\phi_\lambda(u) = \alpha_+ w^{n-1}\alpha_\pm$ if $u = + - + \ldots \pm$, $|u| = n$, and $\phi_\lambda(u) = \alpha_- w^{n-1}\alpha_\mp$ if $u = - + - \ldots \mp$, $|u| = n$, $n \geq 1$, and $0 \leq w \leq 1$, $0 \leq \alpha_+ \leq 1$, $-1 \leq \alpha_- \leq 1$. Consider the matrix

$$\begin{pmatrix} 1 & \alpha_+ & \alpha_- \\ \alpha_+ & 1 & w \\ \alpha_- & w & 1 \end{pmatrix}.$$

Its determinant is 0, so the matrix is positive definite. Hence ϕ_λ is positive definite in view of Example 2.3.1. This proves (c)⇒(a) for $\tau \equiv 0$.

We now turn to the general case. Let λ be as in (c) and let ϕ_λ be the (τ, μ)-spherical function. Then

$$\lambda T_{0\tau}(\phi_\lambda) = T_{0\tau}(\phi_\lambda \underset{\tau}{\square} \mu) = T_{0\tau}(\phi_\lambda) \underset{0}{\square} H_{0\tau}(\mu)$$

and, by (6),

$$\begin{aligned} H_{0\tau}(\mu) &= \beta_+[(1+\tau_+)\delta_+ - \tau_+\delta_0] + \beta_-[(1+\tau_-)\delta_- - \tau_-\delta_0] \\ &= \beta_+(1+\tau_+)\delta_+ + \beta_-(1+\tau_-)\delta_- - (\beta_+\tau_+ + \beta_-\tau_-)\delta_0, \end{aligned}$$

hence

$$T_{0\tau}(\phi_\lambda) \underset{0}{\square} \nu = \gamma T_{0\tau}(\phi_\lambda),$$

where

$$\nu = \beta_+(1+\tau_+)\delta_+ + \beta_-(1+\tau_-)\delta_-, \qquad \gamma = \lambda + (\beta_+\tau_+ + \beta_-\tau_-),$$

so $T_{0\tau}(\phi_\lambda)$ is the $(0,\nu)$-spherical function with eigenvalue γ. It remains to observe that if

$$\lambda \in [-\beta_+\tau_+ - \beta_-\tau_-, -\beta_+\tau_+ + \beta_-] \cup [-\beta_-\tau_- + \beta_+, \beta_+ + \beta_-]$$

then

$$\gamma \in [0, \beta_-(1+\tau_-)] \cup [\beta_+(1+\tau_+), \beta_+(1+\tau_+) + \beta_-(1+\tau_-)],$$

so $T_{0\tau}(\phi_\lambda)$ is 0-positive definite. Consequently, $\phi_\lambda = T_{\tau 0}T_{0\tau}(\phi_\lambda)$ is τ-positive definite on S, which completes the proof.

Remark 4.6. Using Theorem 3.2 we can easily apply the last theorem to spherical functions on $G = \mathbb{Z}_r * \mathbb{Z}_s$, $r \geq s$, defined in [2]. In this case ϕ_λ is positive definite on G if and only if

$$\lambda \in [-2, s-2] \cup [r-2, r+s-2].$$

5. Spherical functions on $G_{k,N} = *_{i=1}^N \mathbb{Z}_k$. In this part we indicate how our technique may be used in the case of spherical functions on $G_{k,N}$, the free product of N cyclic groups of order k (investigated in [3–5]). Our purpose is to provide a new proof of [4, Theorem 3]. Let $\mathcal{R}$ denote the class of all complex finitely supported radial functions on $G_{k,N}$, $k = 2, 3, \ldots$, and let μ_n be the probability measure equidistributed over the set of words of length n in $G_{k,N}$. By [3, Corollary 1], $\mathcal{R}$ forms a commutative algebra, with identity $\mu_0 = \delta_e$, generated by μ_1.

DEFINITION 5.1. A radial function ϕ on $G_{k,N}$ is called *spherical* if the functional $f \mapsto \langle f, \phi\rangle$ is multiplicative on $\mathcal{R}$. For any complex z we denote by ϕ_z the unique spherical function on $G_{k,N}$ such that $\phi_z(x) = z$ if $|x| = 1$ (cf. [3]).

Now, let $I = \{1, \ldots, N\}$ and let $\widetilde{\mathcal{R}}$ denote the class of all complex finitely supported radial functions on $S(I) = S$. Then $\widetilde{\mathcal{R}}$ is a linear space with basis $\{\widetilde{\mu}_n\}_{n=0}^{\infty}$, where $\widetilde{\mu}_n$ denotes the probability measure equidistributed over the set W_n of words in S of length n. Note that $\#(W_0) = 1$ and for $n \geq 1$, $\#(W_n) = N(N-1)^{n-1}$. For any $r \in [0, \infty)$ we shall write $\underset{r}{*}$ to denote $\underset{\tau}{*}$, where $\tau \equiv r$.

LEMMA 5.2 (cf. [2, Lemma 1]). *Let $r \in [0, \infty)$. Then for $n \geq 1$*

$$\widetilde{\mu}_1 \underset{r}{*} \widetilde{\mu}_n = \frac{r}{N}\widetilde{\mu}_{n-1} + \frac{1-r}{N}\widetilde{\mu}_n + \frac{N-1}{N}\widetilde{\mu}_{n+1}\,. \tag{16}$$

In particular, $\widetilde{\mathcal{R}}$ *is a commutative algebra* (*with respect to* $\underset{r}{*}$) *generated by* $\widetilde{\mu}_1$.

DEFINITION 5.3. A radial function ϕ on S is called *r-spherical* if the functional $f \mapsto \langle f, \phi\rangle$ is multiplicative on $\widetilde{\mathcal{R}}$ with respect to $\underset{r}{*}$.

Following (16) we note that, as on $G_{k,N}$, for any complex z we have a unique r-spherical function $\phi_{z,r}$ such that $\phi_{z,r}(i) = z$ for $i \in I$ (cf. [3]). In particular, we have

COROLLARY 5.4. *For any $z \in \mathbb{C}$*

$$\phi_{z,0}(u) = \begin{cases} 1 & \textit{if } \ u = e\,, \\ z\left(\dfrac{Nz-1}{N-1}\right)^{|u|-1} & \textit{if } \ u \neq e\,. \end{cases}$$

Proof. Denote by $\phi_{z,0}(n)$ the value of $\phi_{z,0}$ at any word of length n. By (16) we have

$$\begin{aligned}\phi_{z,0}(1)\phi_{z,0}(n) &= \langle \phi_{z,0}, \widetilde{\mu}_1\rangle\langle \phi_{z,0}, \widetilde{\mu}_n\rangle = \langle \phi_{z,0}, \widetilde{\mu}_1 \underset{0}{*} \widetilde{\mu}_n\rangle \\ &= \frac{1}{N}\langle \phi_{z,0}, \widetilde{\mu}_n\rangle + \frac{N-1}{N}\langle \phi_{z,0}, \widetilde{\mu}_{n+1}\rangle = \frac{1}{N}\phi_{z,0}(n) + \frac{N-1}{N}\phi_{z,0}(n+1)\,.\end{aligned}$$

Remark 5.5. The function $\phi_{z,0} \circ t$ can be regarded as a spherical function on the free group $G_{\infty,N}$ (cf. [5], [7] and [8]).

Let $\widetilde{\mathcal{E}}$ be the projection of $\mathcal{F}'(S)$ onto the radial functions on S defined by $(\widetilde{\mathcal{E}}f)(u) = \langle f, \widetilde{\mu}_n\rangle$, where $n = |u|$. Observe that $\widetilde{\mathcal{E}}$ is an expectation, i.e.

$\langle\widetilde{\mathcal{E}}f_1, f_2\rangle = \langle f_1, \widetilde{\mathcal{E}}f_2\rangle = \langle\widetilde{\mathcal{E}}f_1, \widetilde{\mathcal{E}}f_2\rangle$ for all functions f_1, f_2 on S. Combining Theorem 3.2 and [4, Theorem 2] we get

COROLLARY 5.6. *For $r = 1/k$, $k = 1, 2, \ldots$, $\widetilde{\mathcal{E}}$ maps r-positive definite functions to r-positive definite functions.*

For any $r, s \in [0, \infty)$ we denote by H_{sr} (resp. T_{sr}) the map $H_{\sigma\tau}$ (resp. $T_{\sigma\tau}$) defined by (6) (resp. (7)), where $\sigma \equiv s$, $\tau \equiv r$.

LEMMA 5.7. *For any $r \in [0, \infty)$, $z \in \mathbb{C}$*

$$\widetilde{\mathcal{E}}(T_{r0}\phi_{z,0}) = \phi_{w,r}, \qquad \textit{where } w = (1+r)z - r\,.$$

Proof. Observe that $H_{0r}(\widetilde{\mu}_1) = (1+r)\widetilde{\mu}_1 - r\widetilde{\mu}_0$. Consequently, $H_{0r}(\widetilde{\mathcal{R}}) = \widetilde{\mathcal{R}}$ and thus for $f, g \in \widetilde{\mathcal{R}}$

$$\begin{aligned}\langle\widetilde{\mathcal{E}}(T_{r0}\phi_{z,0}), f \mathbin{\underset{r}{*}} g\rangle &= \langle T_{r0}\phi_{z,0}, f \mathbin{\underset{r}{*}} g\rangle = \langle\phi_{z,0}, H_{0r}(f \mathbin{\underset{r}{*}} g)\rangle \\ &= \langle\phi_{z,0}, H_{0r}(f) \mathbin{\underset{0}{*}} H_{0r}(g)\rangle = \langle\phi_{z,0}, H_{0r}(f)\rangle\langle\phi_{z,0}, H_{0r}(g)\rangle \\ &= \langle T_{r0}\phi_{z,0}, f\rangle\langle T_{r0}\phi_{z,0}, g\rangle = \langle\widetilde{\mathcal{E}}(T_{r0}\phi_{z,0}), f\rangle\langle\widetilde{\mathcal{E}}(T_{r0}\phi_{z,0}), g\rangle\,,\end{aligned}$$

and $\langle\widetilde{\mathcal{E}}(T_{r0}\phi_{z,0}), \mu_1\rangle = \langle\phi_{z,0}, H_{0r}(\mu_1)\rangle = (1+r)z - r$.

Now we present a new proof of [4, Theorem 3].

THEOREM 5.8. *Let $k \in \{\infty, 2, 3, \ldots\}$. The spherical function ϕ_z on $G_{k,N}$ is positive definite if and only if $z \in [-1/(k-1), 1]$.*

Proof. First observe that $\phi_w = \phi_{w,r} \circ t$, where $r = 1/(k-1)$. It was already noted in the proof of Theorem 4.5 that for any τ-positive definite function on $S(I)$ and $i \in I, -\tau(i)\phi(e) \le \phi(i) \le \phi(e)$. Therefore we only have to prove that $\phi_{w,r}$ is r-positive definite for $w \in [-r, 1]$. But $\phi_{w,r} = \widetilde{\mathcal{E}}(T_{r0}\phi_{z,0})$ by Lemma 5.7, where $z = (w+r)/(1+r) \in [0, 1]$. Therefore $\phi_{w,r}$ is r-positive definite by Example 2.3.2, Proposition 2.2 and Corollary 5.6.

PROBLEM. We do not know whether Corollary 5.6 remains true for all positive r. The affirmative solution would allow one to formulate the last theorem for a continuous parameter r.

REFERENCES

[1] C. Berg, J. P. R. Cristensen and P. Ressel, *Harmonic Analysis on Semigroups*, Springer, 1984.

[2] D. I. Cartwright and P. M. Soardi, *Harmonic analysis on the free product of two cyclic groups*, J. Funct. Anal. 65 (2) (1986), 147–171.

[3] A. Iozzi and M. A. Picardello, *Spherical functions on symmetric graphs*, in: Harmonic Analysis, Proc. Cortona 1982, Lecture Notes in Math. 992, Springer, 1983, 344–387.

[4] W. Młotkowski, *Positive definite radial functions on free product of groups*, Boll. Un. Mat. Ital. (7) 2-B (1988), 53–66.
[5] —, *Positive definite functions on free product of groups*, ibid. (7) 3-B (1989), 343–355.
[6] J.-P. Serre, *Trees*, Springer, Berlin 1980.
[7] R. Szwarc, *Matrix coefficients of irreducible representations of free products of groups*, Studia Math. 94 (1989), 179–185.
[8] J. Wysoczański, Ph.D. Thesis, University of Wrocław, Wrocław 1990.

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