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PRENORMALITY OF IDEALS AND COMPLETENESS OF THEIR QUOTIENT ALGEBRAS

BY

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1. Introduction. It is well known that if a nontrivial ideal $\mathfrak{I}$ on κ is normal, its quotient Boolean algebra $\mathcal{P}(\kappa)/\mathfrak{I}$ is κ^+-complete. It is also known that such completeness of the quotient does not characterize normality, since $\mathcal{P}(\kappa)/\mathfrak{I}$ turns out to be κ^+-complete whenever $\mathfrak{I}$ is *prenormal*, i.e. whenever there exists a minimal $\mathfrak{I}$-measurable function in ${}^\kappa\kappa$. Recently, it has been established by Zrotowski (see [Z1], [CWZ] and [Z2]) that for non-Mahlo κ, not only is the above condition sufficient but also necessary for $\mathcal{P}(\kappa)/\mathfrak{I}$ to be κ^+-complete. In the present note we are going to visualize that Zrotowski's result is a consequence of the Boolean structure of $\mathcal{P}(\kappa)$ exclusively, rather than of its other particular properties.

To this end, we shall outline a theory of functions with values in a (sufficiently complete) Boolean algebra, which comprises that of ordinal-valued ones. The theory is given rise to by the fact that any function $f : X \to \alpha$ from a set into an ordinal might be identified with the well-ordered sequence $\langle f^{-1}\{\xi\} : \xi < \alpha\rangle \in {}^\alpha\mathcal{P}(X)$ of counter-images of singletons, i.e. with a function from the ordinal into the power set. It then seems to be quite natural to replace the power set with an appropriate Boolean algebra and consider functions from an ordinal into the algebra instead of from a set into an ordinal. Many familiar notions would then have their Boolean analogues: For instance, since for $f : X_1 \to \alpha$, $g : X_2 \to \alpha$, $X_1, X_2 \subseteq X$, we have

$$\mathrm{dom}(f) = \bigcup_{\xi<\alpha} f^{-1}\{\xi\}\,,$$

$$\{x \in X : f(x) < g(x)\} = \bigcup_{\xi<\zeta} f^{-1}\{\xi\} \cap g^{-1}\{\zeta\}\,,$$

these might be redefined for Boolean-valued functions. Now, it should be clear that in fact any notion definable in terms of ordinal-valued functions, their counter-images of the singletons and the Boolean theory of the power set would find its way into our theory, and so would any statement expressible and provable in these terms. Below we shall corroborate this assertion by reproving a characterization of prenormality in terms of completeness of both the ideal and its quotient. More precisely, we are going to prove that, under certain cardinality assumptions, an ideal $\mathfrak{I} \subseteq \mathcal{B}$ is prenormal whenever the completeness of the quotient algebra $\mathcal{B}/\mathfrak{I}$ is greater than or equal to that of $\mathfrak{I}$ itself (cf. [CWZ], Theorems 2.4 and 2.5). Incidentally, we shall present a modified proof of the result in which two apparently distinct cases, originally dealt with separately, are treated at once.

We adopt standard set-theoretical terminology and notation (see e.g. [J]). In particular, $\alpha, \beta, \gamma, \ldots$ stand for ordinals, while κ, λ, μ denote cardinals, all infinite, and ω stands for the first of them. $\mathcal{B}$ always denotes a Boolean algebra $\langle \mathcal{B}, \cdot, +, -, \mathbf{1} \rangle$. For $X \subseteq \mathcal{B}$, $\sum(X)$ stands for the least upper bound of X in $\mathcal{B}$ (provided it exists), but when X is an indexed family of elements of $\mathcal{B}$, $X = \{X_i : i \in I\}$, we write $\sum_{i \in I} x_i$ instead. For $X \subseteq \mathcal{B}$, two cardinal coefficients, $\mathrm{cpl}(X)$ and $\mathrm{cov}(X)$, called the *completeness* and *covering* coefficient of X respectively, are defined as follows:

$$\begin{aligned}
\mathrm{cpl}(X) &= \min\{\mathrm{card}(Y) : Y \subseteq X, Y \text{ has no lub in } X\}\,, \\
\mathrm{cov}(X) &= \min\{\mathrm{card}(Y) : Y \subseteq X, \textstyle\sum(Y) = \sum(X)\}\,.
\end{aligned}$$

Evidently, if $\sum(X) \notin X$, then $\mathrm{cpl}(X) \le \mathrm{cov}(X)$. $\mathfrak{I}$ always denotes a proper ideal in $\mathcal{B}$ with $\sum(\mathfrak{I}) = \mathbf{1}$ and $\mathrm{cpl}(\mathfrak{I}) < \mathrm{cpl}(\mathcal{B})$. Lower case a, b usually stand for elements of $\mathcal{B}$ and $[a]_{\mathfrak{I}}, [b]_{\mathfrak{I}}$ for their equivalence classes in the quotient algebra $\mathcal{B}/\mathfrak{I}$, but we shall often use the elements to represent their own classes; thus, we write e.g. $a =_{\mathfrak{I}} b$, $a \le_{\mathfrak{I}} b$ instead of $[a]_{\mathfrak{I}} = [b]_{\mathfrak{I}}$, $[a]_{\mathfrak{I}} \le [b]_{\mathfrak{I}}$. Subscripts are employed whenever helpful.

2. Boolean-valued functions. The main tools used in defining and studying (pre-)normality of ideals in $\mathcal{P}(\kappa)$ have been certain subsets of the set ${}^{\kappa}\alpha$ and ordering relations on this set (see [BTW], [CWZ]). Now, most of these notions appear to be expressible in terms of Boolean operations on $\mathcal{P}(\kappa)$ and counter-images of singletons. Indeed, given an ordinal α and functions $f : X \to \alpha$, $g : Y \to \alpha$, $X, Y \subseteq \kappa$, we have for instance

$$\begin{aligned}
\mathrm{dom}(f) &= \bigcup_{\xi<\alpha} f^{-1}\{\xi\}\,, \\
\{\eta < \kappa : f(\eta) = g(\eta)\} &= \bigcup_{\xi<\alpha} f^{-1}\{\xi\} \cap g^{-1}\{\xi\}\,, \\
\{\eta < \kappa : f(\eta) \le g(\eta)\} &= \bigcup_{\xi\le\zeta<\alpha} f^{-1}\{\xi\} \cap g^{-1}\{\zeta\}\,, \\
\{\eta < \kappa : f(\eta) < g(\eta)\} &= \bigcup_{\xi<\zeta<\alpha} f^{-1}\{\xi\} \cap g^{-1}\{\zeta\}\,.
\end{aligned}$$

Since any $f : X \to \alpha$, $X \subseteq \kappa$, might be identified with the sequence $\langle f^{-1}\{\xi\} : \xi < \alpha\rangle \in {}^{\alpha}\mathcal{P}(\kappa)$, these equalities suggest how to define Boolean analogues of the left-hand side notions above (and, in fact, many others defined similarly) for elements of ${}^{\alpha}\mathcal{B}$: This is achieved by replacing counter-images of singletons with values at them, and unions in $\mathcal{P}(\kappa)$ with suprema in $\mathcal{B}$. Since, however, distinct counter-images are disjoint, some extra care is needed when defining sets of Boolean-valued functions we want to consider.

So, let $\mathcal{B}$ be a Boolean algebra. Given $\alpha < \mathrm{cpl}(\mathcal{B})$, let $\|f\|$, $\|f = g\|$, $\|f \le g\|$ and $\|f < g\|$ denote the following $\mathcal{B}$-valued predicates on ${}^{\alpha}\mathcal{B}$:

$$\|f\| = \sum_{\xi<\alpha} f(\xi)\,, \qquad \|f = g\| = \sum_{\xi<\alpha} f(\xi)\cdot g(\xi)\,,$$
$$\|f \le g\| = \sum_{\xi\le\zeta<\alpha} f(\xi)\cdot g(\zeta)\,, \qquad \|f < g\| = \sum_{\xi<\zeta<\alpha} f(\xi)\cdot g(\zeta)\,.$$

Notice that for any $f, g, h \in {}^{\alpha}\mathcal{B}$ and $a \in \mathcal{B}$,

$$\|f = f\| = \|f\|, \qquad \|f = g\| = \|g = f\|,$$
$$\|f + g = h\| = \|f = h\| + \|g = h\|,$$
$$\|a\cdot f\| = a\cdot\|f\|, \qquad \|a\cdot f \le g\| = a\cdot\|f \le g\|, \quad \text{etc.}$$

Since they all are easily verified by direct calculation, below we shall apply these, and many other equally elementary properties of the predicates, without any further notice.

Although on ${}^{\alpha}\mathcal{B}$ the $\mathcal{B}$-relations $\|f = g\|$, $\|f \le g\|$ and $\|f < g\|$ do not share most of the essential features of their two-valued counterparts, they do behave just as Boolean-valued equality, order and strict order are expected to when restricted to

$$F_\alpha(\mathcal{B}) = \{f \in {}^{\alpha}\mathcal{B} : (\forall \xi < \zeta < \alpha)\ f(\xi)\cdot f(\zeta) = \mathbf{0}\}\,.$$

In particular, for any $f, g, h \in F_\alpha(\mathcal{B})$,

$$\|f = g\|\cdot\|g = h\| = \|f = h\|, \qquad \|f \le f\| = \|f\|,$$
$$\|f \le g\|\cdot\|f \ge g\| = \|f = g\|,$$
$$\|f \le g\|\cdot\|g \le h\| \le \|f \le h\|,$$
$$\|f < f\| = \|f < g\|\cdot\|f > g\| = \mathbf{0},$$
$$\|f < g\|\cdot\|g < h\| \le \|f < h\|\,.$$

Consequently, given an ideal $\mathfrak{I}$ in $\mathcal{B}$, the equality $\|f \le g\| =_{\mathfrak{I}} \mathbf{1}$ defines a (two-valued) pre-order on the set

$$T_\alpha(\mathcal{B}) = \{f \in F_\alpha(\mathcal{B}) : \|f\| = \mathbf{1}\}$$

and $\|f = g\| =_{\mathfrak{I}} \mathbf{1}$ yields the equivalence relation on this set corresponding to that pre-order. By way of abuse, we shall denote these relations on

$T_\alpha(\mathcal{B})$ by $f \le_{\mathfrak{I}} g$ and $f =_{\mathfrak{I}} g$ respectively, which leads to no confusion. Moreover, instead of passing to the quotient of $T_\alpha(\mathcal{B})$ modulo the equivalence relation and to the induced order, we shall just apply, somewhat incorrectly, order terminology to the initial pre-order on $T_\alpha(\mathcal{B})$. For instance, given $f, g \in T_\alpha(\mathcal{B})$, $\|f \le g\| \cdot f + \|f > g\| \cdot g$ is called their minimum since its class is the actual minimum of those of f and g in the quotient.

With this in mind, notice that the set

$$R_\alpha(\mathfrak{I}) = \{f \in T_\alpha(\mathcal{B}) : (\forall \xi < \alpha)\ \ f(\xi) \in \mathfrak{I}\}$$

is downward directed, hence it has a least element provided it has a minimal one. Since, however, $R_\alpha(\mathfrak{I}) \neq \emptyset$ amounts to the existence of a decomposition of $\mathbf{1}$ into $\mathrm{card}(\alpha)$ many elements from $\mathfrak{I}$, the first α for which such an element might at all exist in $R_\alpha(\mathfrak{I})$ is $\mathrm{cov}(\mathfrak{I})$, the covering coefficient of $\mathfrak{I}$.

DEFINITION. An ideal $\mathfrak{I}$ in $\mathcal{B}$ is *prenormal* iff $\mathrm{cov}(\mathfrak{I}) < \mathrm{cpl}(\mathcal{B})$ and there exists a minimal function in $R_{\mathrm{cov}(\mathfrak{I})}(\mathfrak{I})$.

Now, for $\mathcal{B} = \mathcal{P}(\kappa)$, we are back where we started, since for this $\mathcal{B}$, each $f \in F_\alpha(\mathcal{B})$ might be identified with a function h_f from a subset of κ into α such that $h_f^{-1}\{\xi\} = f(\xi)$, $\xi < \alpha$. Then, via this identification, $F_\alpha(\mathcal{B})$ might be viewed as the set of all partial mappings from κ into α; $\|f\|$, as the domain of f; $\|f = g\|$, $\|f \le g\|$ and $\|f < g\|$, as the sets of those arguments for which the respective (two-valued) predicate is satisfied; and $f =_{\mathfrak{I}} g$, $f \le_{\mathfrak{I}} g$, $f <_{\mathfrak{I}} g$, minimality, prenormality and many other concepts could easily be seen to have their usual meaning (cf. e.g. [BTW], [CWZ]).

3. From prenormality to completeness. Let

$$P_\alpha(\mathfrak{I}) = \{f \in F_\alpha(\mathcal{B}) : (\forall \xi < \alpha)\ f(\xi) \in \mathfrak{I}\}\,.$$

Then any function minimal in $R_\alpha(\mathfrak{I})$ proves to be, in a sense, minimal in $P_\alpha(\mathfrak{I})$ too. (Recall that, unless explicitly stated otherwise, $\alpha < \mathrm{cpl}(\mathcal{B})$ is tacitly assumed throughout.)

LEMMA 1. *For $f \in R_\alpha(\mathfrak{I})$, f is minimal in $R_\alpha(\mathfrak{I})$ iff for every $g \in P_\alpha(\mathfrak{I})$, $\|g\| =_{\mathfrak{I}} \|f \le g\|$.*

Proof. By the elementary properties of the $\mathcal{B}$-valued predicates, given $f \in R_\alpha(\mathfrak{I})$ and $g \in P_\alpha(\mathfrak{I})$, $h = -\|g\| \cdot f + g$ is in $R_\alpha(\mathfrak{I})$, $\|f \le h\| = -\|g\| + \|f \le g\|$ and $\|f \le g\| \le \|g\|$. Thus, if f is minimal in $R_\alpha(\mathfrak{I})$, $\|g\| =_{\mathfrak{I}} \|f \le g\|$ follows. The other part of the lemma is trivial.

Notice that this lemma implies that for any $\alpha < \mathrm{cpl}(\mathcal{B})$, if f is minimal in $R_\alpha(\mathfrak{I})$, then so are all its restrictions and extensions by $\mathbf{0}$'s in all $R_\beta(\mathfrak{I})$, $\mathrm{cov}(\mathfrak{I}) \le \beta < \mathrm{cpl}(\mathcal{B})$; therefore, by definition, to prove the prenormality of $\mathfrak{I}$ it suffices to exhibit a minimal function in any $R_\alpha(\mathfrak{I})$, $\mathrm{cov}(\mathfrak{I}) \le \alpha < \mathrm{cpl}(\mathcal{B})$.

Now, for $\alpha < \mathrm{cpl}(\mathcal{B})$, $f \in {}^{\alpha}\mathcal{B}$ and $a_\xi \in \mathcal{B}$, $\xi < \alpha$, we define the *α-diagonal union of a_ξ's with respect to f* by the equality

$$\mathop{\nabla^f}_{\xi<\alpha} a_\xi = \sum_{\xi<\alpha}\Big(a_\xi - \sum_{\zeta\le\xi} f(\zeta)\Big)\,.$$

Then, for any $\alpha < \mathrm{cpl}(\mathcal{B})$, $\mathfrak{I}$ turns out to be closed under the α-diagonal unions with respect to functions minimal in $R_\alpha(\mathfrak{I})$, hence such unions determine suprema of families of cardinalities less than or equal to $\mathrm{cpl}(\mathfrak{I})$ in the quotient algebras of certain prenormal ideals.

Lemma 2. *For $f \in R_\alpha(\mathfrak{I})$, f is minimal in $R_\alpha(\mathfrak{I})$ iff for every $a_\xi \in \mathfrak{I}$, $\xi < \alpha$, $\nabla^f_{\xi<\alpha} a_\xi \in \mathfrak{I}$.*

Proof. By virtue of Lemma 1, it is enough to notice first that for $a_\xi \in \mathfrak{I}$, $\xi < \alpha$, if we set $g(\xi) = a_\xi - \sum_{\zeta<\xi} a_\zeta$, $\xi < \alpha$, then $g \in P_\alpha(\mathfrak{I})$ and $\nabla^f_{\xi<\alpha} a_\xi = \nabla^f_{\xi<\alpha} g(\xi)$, and next that for $f \in T_\alpha(\mathcal{B})$ and $g \in {}^{\alpha}\mathcal{B}$, $\nabla^f_{\xi<\alpha} g(\xi) = \|f > g\|$.

Proposition 1. *If $\mathfrak{I}$ is a prenormal ideal in $\mathcal{B}$ such that* $\mathrm{cpl}(\mathfrak{I}) = \mathrm{cov}(\mathfrak{I})$, *then* $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}) > \mathrm{cpl}(\mathfrak{I})$.

Proof. Let $\alpha = \mathrm{cpl}(\mathfrak{I}) = \mathrm{cov}(\mathfrak{I})$ and let f be minimal in $R_\alpha(\mathfrak{I})$. By the preceding lemma, given $a, a_\xi \in \mathcal{B}$, $\xi < \alpha$, such that a is an upper bound of a_ξ in $\mathcal{B}/\mathfrak{I}$, then $\nabla^f_{\xi<\alpha} a_\xi \le_{\mathfrak{I}} a$, hence $[\nabla^f_{\xi<\alpha} a_\xi]_{\mathfrak{I}} = \sum_{\xi<\alpha}[a_\xi]_{\mathfrak{I}}$, as the assumption $\mathrm{cpl}(\mathfrak{I}) = \mathrm{cov}(\mathfrak{I})$ guarantees that $\nabla^f_{\xi<\alpha} a_\xi$ is such an upper bound itself.

The next proposition, preceded with three auxiliary results, shows that the assumption $\mathrm{cpl}(\mathfrak{I}) = \mathrm{cov}(\mathfrak{I})$ is redundant in Proposition 1. First, recollect that a mapping φ which sends a subset of α into α is called *regressive* on $A \subseteq \alpha$ provided that $\varphi(\xi) < \xi$ for all $\xi \in A$, and that NS_α denotes the set of all non-stationary subsets of α, i.e. of subsets $A \subseteq \alpha$ such that there exists $\varphi : A \to \alpha$ regressive on A with $\mathrm{card}(\varphi^{-1\prime\prime}\{\xi\}) < \alpha$, $\xi < \alpha$ (see e.g. [J]).

Lemma 3. *For $\alpha \le \mathrm{cpl}(\mathfrak{I})$, $A \subseteq \alpha$, $\varphi : A \to \alpha$ regressive on A and $f \in F_\alpha(\mathcal{B})$, if we set $g(\zeta) = \sum_{\xi\in\varphi^{-1\prime\prime}\{\zeta\}} f(\xi)$, $\zeta < \alpha$, then $g \in F_\alpha(\mathcal{B})$ and $\|g < f\| = \sum_{\xi\in A} f(\xi)$.*

Proof. Straightforward, as $\varphi^{-1\prime\prime}\{\zeta\}$, $\zeta < \alpha$, are pairwise disjoint and $\varphi^{-1\prime\prime}\{\zeta\} \subseteq \{\xi < \alpha : \xi > \zeta\}$.

Corollary 1. *For $\alpha \le \mathrm{cpl}(\mathfrak{I})$, $A \in \mathrm{NS}_\alpha$ and $f \in P_\alpha(\mathfrak{I})$, there exists $g \in P_\alpha(\mathfrak{I})$ such that $\|g < f\| = \sum_{\xi\in A} f(\xi)$.*

Next, recall that for $a \in \mathcal{B}$, $\mathfrak{I}(a) = \{b \in \mathcal{B} : a \cdot b \in \mathfrak{I}\}$ is an ideal extending $\mathfrak{I}$ and notice that, by the remark following Lemma 1, $\mathfrak{I}(a)$ is prenormal whenever $\mathfrak{I}$ is so.

LEMMA 4. *Let f be minimal in $R_\alpha(\mathfrak{I})$, $\beta = \min\{\zeta \le \alpha : \sum_{\xi<\zeta} f(\xi) \notin \mathfrak{I}\}$ and $a = \sum_{\xi<\beta} f(\xi)$. Then*

(i) $\mathrm{cpl}(\mathfrak{I}) = \mathrm{cpl}(\mathfrak{I}(a)) = \mathrm{cov}(\mathfrak{I}(a))$,

(ii) $\mathrm{cpl}(\mathfrak{I}) < \mathrm{cpl}(\mathfrak{I}(-a))$.

Proof. Since $\mathrm{cpl}(\mathfrak{I}) \le \mathrm{cpl}(\mathfrak{I}(a)) \le \mathrm{cov}(\mathfrak{I}(a)) \le \beta$ are evident, to prove (i) it suffices to verify that $\beta \le \mathrm{cpl}(\mathfrak{I})$. To this end, note that for given $\gamma < \beta$ and $a_\xi \in \mathfrak{I}$, $\xi < \gamma$, if we set

$$g(\xi) = \begin{cases} a_\xi - \sum_{\zeta<\xi} a_\zeta\,, & \xi < \gamma\,,\\ \mathbf{0}\,, & \gamma < \xi < \alpha\,,\end{cases}$$

then $g \in P_\alpha(\mathfrak{I})$, hence by Lemma 1, $\|g\| =_{\mathfrak{I}} \|f \le g\|$. But as by definitions and Corollary 1 we have $\|g\| = \sum_{\xi<\gamma} a_\xi$, $\|f \le g\| \le \sum_{\xi<\gamma} f(\xi)$ and $\sum_{\xi<\gamma} f(\xi) \in \mathfrak{I}$, thus $\sum_{\xi<\gamma} a_\xi \in \mathfrak{I}$ follows.

For (ii), observe that if $-a \in \mathfrak{I}$, the inequality is obvious by our initial assumption, so we may, and do, assume otherwise. Consequently, β is less than α and since by (i), β is a cardinal, the ordinal $\beta+\beta$ is less than α, too. Thus, given $a_\xi \in \mathfrak{I}(-a)$, $\xi < \mathrm{cpl}(\mathfrak{I})$, we may define $g \in P_\alpha(\mathfrak{I})$ as

$$g(\xi) = \begin{cases} a_\zeta - a - \sum_{\eta<\zeta} a_\eta\,, & \xi = \beta+\zeta,\ \zeta < \beta\,,\\ \mathbf{0}\,, & \text{otherwise}\,.\end{cases}$$

Then, by an argument similar to that for (i), $\sum_{\xi<\mathrm{cpl}(\mathfrak{I})} a_\xi \in \mathfrak{I}(-a)$.

PROPOSITION 2. *If $\mathfrak{I}$ is a prenormal ideal in $\mathcal{B}$, then* $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}) > \mathrm{cpl}(\mathfrak{I})$.

Proof. Let f, β and a be as in Lemma 4. Then, by its part (i), the remark preceding it and Proposition 1, $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}(a)) > \mathrm{cpl}(\mathfrak{I})$; moreover, by its part (ii), also $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}(-a)) > \mathrm{cpl}(\mathfrak{I})$, hence $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}) = \min\{\mathrm{cpl}(\mathcal{B}/\mathfrak{I}(a)), \mathrm{cpl}(\mathcal{B}/\mathfrak{I}(-a))\} > \mathrm{cpl}(\mathfrak{I})$.

4. From completeness to prenormality. To prove a limited converse of Proposition 2 we shall employ another, local, characterization of prenormality, which appears to be more applicable in practice than the original definition; a technical lemma precedes this characterization (cf. [S], proofs of Lemmas 3 and 4).

LEMMA 5. *For $f_n \in F_\alpha(\mathcal{B})$, $n \in \omega$, $\prod_{n\in\omega} \|f_{n+1} < f_n\| = \mathbf{0}$.*

Proof. Since for $n \in \omega$, $\|f_{n+1} < f_n\| \le \|f_n\|$, thus if we set $a = \prod_{n\in\omega} \|f_{n+1} < f_n\|$ then $a = a \cdot \|f_{n+1} < f_n\| = a \cdot \|f_n\|$, i.e.

$$a = \sum_{\xi<\alpha} a \cdot f_n(\xi) \cdot \sum_{\zeta<\xi} f_{n+1}(\zeta) = \sum_{\xi<\alpha} a \cdot f_n(\xi)\,.$$

Now, $\{a\cdot f_n(\xi)\cdot\sum_{\zeta<\xi} f_{n+1}(\zeta) : \xi<\alpha\}$ and $\{a\cdot f_n(\xi) : \xi<\alpha\}$ are partitions of a such that

$$a\cdot f_n(\xi)\cdot\sum_{\zeta<\xi} f_{n+1}(\zeta)\le a\cdot f_n(\xi)\,,$$

hence $a\cdot f_n(\xi)\le\sum_{\zeta<\xi} a\cdot f_{n+1}(\zeta)$, $\xi<\alpha$, $n\in\omega$. Consequently, were it $a>\mathbf{0}$, then $\{\xi<\alpha : (\exists n\in\omega)\ a\cdot f_n(\xi)>\mathbf{0}\}$ would be nonempty but without a least element.

PROPOSITION 3. *Let $\mathfrak{I}$ be an ideal in $\mathcal{B}$ with* $\mathrm{cpl}(\mathfrak{I})>\omega$ *and let* $\alpha<\mathrm{cpl}(\mathcal{B})$ *be such that $R_\alpha(\mathfrak{I})\ne\emptyset$. Then $\mathfrak{I}$ is prenormal iff for every $f\in P_\alpha(\mathfrak{I})$, there exists $g\in P_\alpha(\mathfrak{I})$ such that for every $h\in P_\alpha(\mathfrak{I})$, $\|h<f\|\le_{\mathfrak{I}}\|g<f\|$.*

Proof. By Lemma 1 and the remark following it, only the "if" part requires a proof. To this end, define a sequence $f_n\in P_\alpha(\mathfrak{I})$, $n\in\omega$, by induction on n as follows:

f_0 is an arbitrary element of $R_\alpha(\mathfrak{I})$; given $f_n\in P_\alpha(\mathfrak{I})$, pick g as in the assumption and set $f_{n+1}=\|g<f_n\|\cdot g$, $n\in\omega$.

Then $f_{n+1}\in P_\alpha(\mathfrak{I})$, $\|f_{n+1}\|=\|f_{n+1}<f_n\|\le\|f_n\|$ and for every $h\in P_\alpha(\mathfrak{I})$, $\|h<f_n\|\le_{\mathfrak{I}}\|f_{n+1}<f_n\|$, $n\in\omega$. Since, moreover, $\|f_0\|=\mathbf{1}$, $\mathrm{cpl}(\mathfrak{I})>\omega$ and, by Lemma 5, $\prod_{n\in\omega}\|f_{n+1}<f_n\|=\mathbf{0}$, thus

$$f=\sum_{n\in\omega}(\|f_n\|-\|f_{n+1}\|)\cdot f_n$$

is easily seen to be minimal in $R_\alpha(\mathfrak{I})$, hence by the same remark as before, $\mathfrak{I}$ is prenormal.

Now, in a series of lemmas, we shall describe a construction that unifies those due to Zrotowski (cf. [Z1], [Z2]), though, in fact, it dates back to Ulam (see [U]); under some additional assumptions on $\mathrm{cpl}(\mathfrak{I})$, for a given $f\in P_\alpha(\mathfrak{I})$, it yields a $g\in P_\alpha(\mathfrak{I})$ as in the conclusion of Proposition 3.

LEMMA 6. *For $\beta<\mathrm{cpl}(\mathfrak{I})$ and $g_\xi\in P_\alpha(\mathfrak{I})$, $\xi<\beta$, there exists $g\in P_\alpha(\mathfrak{I})$ such that for every $h\in P_\alpha(\mathfrak{I})$, $\|g<h\|=\sum_{\xi<\beta}\|g_\xi<h\|$.*

Proof. First, set $G(\zeta)=\sum_{\xi<\beta} g_\xi(\zeta)$, and then $g(\zeta)=G(\zeta)-\sum_{\eta<\zeta}G(\eta)$, $\zeta<\alpha$, and see that this g works.

LEMMA 7. *Let $\mathfrak{I}$ be an ideal in $\mathcal{B}$ such that $\mathrm{cpl}(\mathcal{B}/\mathfrak{I})>\alpha=\mathrm{cov}(\mathfrak{I})=\mathrm{cpl}(\mathfrak{I})>\omega$; let, moreover, $\mu<\alpha$ be a regular infinite cardinal, $f\in P_\alpha(\mathfrak{I})$, $A\subseteq\mathrm{cf}^{-1}{}''\{\mu\}\cap\alpha$ and let $a=\sum_{\xi\in A}f(\xi)$. Then there exists $g\in P_\alpha(\mathfrak{I})$ such that for every $h\in P_\alpha(\mathfrak{I})$, $a\cdot\|h<f\|\le_{\mathfrak{I}} a\cdot\|g<f\|$.*

Proof. For every ordinal $\xi\in A$, fix a μ-sequence unbounded in ξ, and for $\eta<\mu$, let $\varphi_\eta(\xi)$ be its ηth element; then define $f_\eta\in F_\alpha(\mathcal{B})$ by the equality

$$f_\eta(\zeta) = \sum_{\xi \in \varphi^{-1\prime\prime}\{\zeta\}} f(\xi)\,, \quad \zeta < \alpha\,,\ \eta < \mu\,.$$

Obviously, $\|f_\eta\| = a$ and since $\xi \in \varphi_\eta^{-1\prime\prime}\{\zeta\}$ implies $\zeta < \xi$, also $\|f_\eta < f\| = a$, $\eta < \mu$; since, moreover, for $\zeta < \alpha$, $\{\xi \in A : \zeta < \xi\} = \bigcup_{\eta<\mu} \bigcup_{\zeta<\xi} \varphi_\eta^{-1\prime\prime}\{\xi\}$, it follows that for every $h \in {}^\alpha\mathcal{B}$, $a \cdot \|h < f\| = \sum_{\eta<\mu} \|h < f_\eta\|$.

Now, $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}) > \alpha$, so for each $\eta < \mu$, pick $a_\eta \le a$ with $\sum_{\zeta<\alpha}[f_\eta(\zeta)]_\mathfrak{I} = [a_\eta]_\mathfrak{I}$, and set

$$g_\eta = (a - a_\eta) \cdot f_\eta\,.$$

Then, by the definitions of f_η and a_η, $g_\eta \in P_\alpha(\mathfrak{I})$ and $\|g_\eta\| = \|g_\eta < f\| = a - a_\eta$, $\eta < \mu$.

Finally, apply the preceding lemma to these g_η's to get $g \in P_\alpha(\mathfrak{I})$ such that $\|g < f\| = \sum_{\eta<\mu} \|g_\eta < f\| = a - \prod_{\eta<\mu} a_\eta$ and see that it is as needed. Indeed, since $\alpha = \mathrm{cpl}(\mathfrak{I})$, therefore for any $h \in P_\alpha(\mathfrak{I})$, $\eta < \mu$ and $\zeta < \alpha$, $\|h < f_\eta\| \cdot f_\eta(\zeta) \in \mathfrak{I}$, and hence, by the very definition of supremum in $\mathcal{B}/\mathfrak{I}$, $\|h < f_\eta\| \cdot a_\eta \in \mathfrak{I}$; consequently, since $\|h < f\| = \sum_{\eta<\mu} \|h < f_\eta\|$ and $\mu < \alpha$, we also have $a \cdot \|h < f\| - \|g < f\| = a \cdot \|h < f\| \cdot \prod_{\eta<\mu} a_\eta \in \mathfrak{I}$, and $a \cdot \|h < f\| \le_\mathfrak{I} a \cdot \|g < f\|$ follows.

LEMMA 8. *Let $f \in P_\alpha(\mathfrak{I})$, let $\{a_\xi : \xi < \alpha\}$ be an antichain in $\mathcal{B}$, and for every $\xi < \alpha$, let $g_\xi \in P_\alpha(\mathfrak{I})$ be such that for any $h \in P_\alpha(\mathfrak{I})$, $a_\xi \cdot \|h < f\| \le_\mathfrak{I} a_\xi \cdot \|g_\xi < f\|$; further, let $a = \sum_{\xi<\alpha} a_\xi$ and let $\underline{a} \le a$ satisfy $[\underline{a}]_\mathfrak{I} = \sum_{\xi<\alpha}[a_\xi]_\mathfrak{I}$. Then there exists $g \in P_\alpha(\mathfrak{I})$ such that for every $h \in P_\alpha(\mathfrak{I})$, $\underline{a} \cdot \|h < f\| \le_\mathfrak{I} \underline{a} \cdot \|g < f\|$.*

If, moreover, $a_\xi = \sum_{\zeta\in\varphi^{-1}\{\xi\}} f(\zeta)$, $\xi < \alpha$, for some $\varphi : A \to \alpha$ regressive on $A \subseteq \alpha$, then there exists $g \in P_\alpha(\mathfrak{I})$ such that for every $h \in P_\alpha(\mathfrak{I})$, $a \cdot \|h < f\| \le_\mathfrak{I} a \cdot \|g < f\|$.

Proof. By definition of $\underline{a}$ as the $\mathcal{B}/\mathfrak{I}$-supremum of a_ζ's, in the former case it suffices to find $g \in P_\alpha(\mathfrak{I})$ such that for every $h \in P_\alpha(\mathfrak{I})$ and $\zeta < \alpha$, $a_\zeta \cdot \|h < f\| - a_\zeta \cdot \|g < f\| \in \mathfrak{I}$. Now, it is a matter of routine to check that $g(\xi) = \sum_{\zeta<\alpha} \underline{a} \cdot a_\zeta \cdot g_\zeta(\xi)$, $\xi < \alpha$, is as needed. Also, by the definition of a_ζ's and Lemma 3, $g'(\xi) = a_\xi - \underline{a} + g(\xi)$, $\xi < \alpha$, is seen to work in the latter case.

PROPOSITION 4. *If $\mathfrak{I}$ is an ideal in $\mathcal{B}$ such that* $\mathrm{cpl}(\mathfrak{I})$ *is not Mahlo and* $\mathrm{cpl}(\mathcal{B}/\mathfrak{I}) > \mathrm{cov}(\mathfrak{I}) = \mathrm{cpl}(\mathfrak{I}) > \omega$, *then $\mathfrak{I}$ is prenormal.*

Proof. Let $\alpha = \mathrm{cpl}(\mathfrak{I})$ and $A = \{\xi < \alpha : (\exists\beta < \alpha)(\xi = \beta + 1) \vee \mathrm{cf}(\xi) = \xi\}$. Then $A \in \mathrm{NS}_\alpha$ by assumption, and cf is regressive on $\alpha\backslash A$. Thus, given $f \in P_\alpha(\mathfrak{I})$, successive application of Corollary 1, Lemmas 7 and 8 yields $g \in P_\alpha(\mathfrak{I})$ as required by Proposition 3 for $\mathfrak{I}$ to be prenormal.

To close the paper, let us summarize Propositions 2 and 4 as follows.

THEOREM. *For every Boolean algebra* $\mathcal{B}$ *and every ideal* $\mathfrak{I} \subseteq \mathcal{B}$ *such that* $\mathrm{cpl}(\mathfrak{I})$ *is not Mahlo and* $\mathrm{cov}(\mathfrak{I}) = \mathrm{cpl}(\mathfrak{I}) > \omega$,

$$\mathfrak{I} \text{ is prenormal} \quad \text{iff} \quad \mathrm{cpl}(\mathcal{B}/\mathfrak{I}) > \mathrm{cpl}(\mathfrak{I}) .$$

PROBLEM. Are all the assumptions on $\mathrm{cpl}(\mathfrak{I})$ necessary in the above theorem?

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