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ARITHMETIC OF AN ELLIPTIC CURVE AND CIRCLE ACTIONS ON FOUR-MANIFOLDS

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The manifolds we consider here are smooth, orientable, of dimension four. The actions of the circle are smooth. We prove

THEOREM 1. *An S^1 action on a four-manifold of nonzero signature has at least three fixed points.*

An easy consequence of the method is

PROPOSITION 2. *If M^4 carries an S^1 action with two fixed points, then $\sigma(M)=0$ and the representations at the tangent spaces at the fixed points are equivalent.*

Both these results are known, but it is the method we employ here that is of interest: it is to use two residue formulae for the signature and reduce the problem to a diophantine question. The diophantine result from which Theorem 1 follows is

THEOREM 3. *The only rational points on the elliptic curve $(x+y)(z^2+xy)=6xyz$ are $(0,0,1)$, $(0,1,0)$, $(1,0,0)$ and $(-1,1,0)$.*

1. Residue formula. Take a smooth S^1 action on an oriented four-manifold M. Then the characteristic numbers of M are expressed by the fixed point data. First, the normal bundles to the components of the fixed point set are given a complex structure and thus orientation by the representations of S^1 in the fibers. Thus the components of the fixed point set are oriented by the difference orientations and we have the following formula for the signature:

$$(*)\qquad \sigma(M)=\sigma(\mathrm{fix})\,.$$

Second, the Atiyah–Bott–Lefschetz fixed point theorem specializes in dimension four to

$$(**)\qquad \sigma(M)=\frac{1}{3}\Big\{\sum_i\Big(\frac{a_i}{b_i}+\frac{b_i}{a_i}\Big)+\sum_j\int\limits_{F_j}\mathrm{eu}(\nu_j)\Big\}.$$

Here the integers a_i, b_i are the integral weights of the representations at the isolated fixed points and $\mathrm{eu}(\nu_j)$ are the Euler classes of the bundles normal to the two-dimensional components of the fixed point set F_j. The sum is taken over all the components of the fixed point set.

Notice at this point that, by performing equivariant surgery, we can, without changing the signature, always get rid of two-dimensional components of the fixed point set, possibly introducing some new isolated fixed points. This follows essentially from the fact that a circle bundle over a surface is cobordant to a disjoint sum of Hopf fibrations. We are interested only in actions with isolated fixed points.

Remark. In dimension four both of these formulas are elementary. The first follows quickly from the cobordism invariance of signature, while an elementary argument yielding the second can be found in [L].

There is certain tension between these two formulae, and we will take advantage of it.

2. Preliminary considerations. Let us consider the case when we have a single fixed point. Then $(*)$ says $\sigma(M) = 1$ (after changing orientation if necessary). But then $(**)$ reads $a/b+b/a = 3$. However, this equation has no integral solutions—$\sqrt{5}$ is irrational! Hence there is no action with a single fixed point.

If there are two fixed points, the signature is either zero or two. In the first case we have

$$\frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}+\frac{x}{y}+\frac{y}{x}\right) = 0$$

and this implies that either $a/b{=}-x/y$ or $a/b{=}-y/x$. Since the action is effective, this means that the representations at the fixed points are equivalent (the minus sign comes from the different orientations at the fixed points).

In the second case $a/b+b/a+x/y+y/x = 6$, or equivalently $(z+t)(1+zt) = 6zt$. Now both Theorem 1 and Proposition 2 follow from what has already been said and Theorem 3.

3. The proof of Theorem 3. First, putting $z = 0$ we get the three points at infinity $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. All of them are rational.

Now put $z = 1$. The equation takes the form

$$6xyab = (x^2+y^2)ab + (a^2+b^2)xy \tag{0}$$

where a,b,x,y are integers, and $(x,y) = (a,b) = 1$. Then it follows that $xy = ab$ or $xy = -ab$, but since the system

$$6ab = a^2+b^2+x^2+y^2\,, \qquad ab = xy$$

has no solution mod 3, we can assume $xy = -ab$.

Reducing the resulting system mod 4, we see that one of the numbers, say x, is even. We can write $x = pq,\ y = rs,\ a = -pr,\ b = qs$, with p, q, r, s natural and pairwise coprime. Then (0) takes the form

$$6pqrs = (q^2 - r^2)(s^2 - p^2)\,, \tag{1}$$

and thus ps divides $q^2 - r^2$, qr divides $s^2 - p^2$, $q^2 - r^2$ divides $6ps$ and $s^2 - p^2$ divides $6qr$; hence $q^2 - r^2 = kps$, $s^2 - p^2 = lqr$ where k, l are integers with $kl = 6$, the cases $k = 0$ and $l = 0$ being uninteresting.

Using this we have

$$(p^2 - s^2)^2 + (3ps)^2 = \left(\frac{3}{k}(q^2 + r^2)\right)^2.$$

We can assume here that 2 divides p, since 2 divides x and p, q play symmetric roles. Thus $\frac{3}{k}(q^2 + r^2)$ is an integer.

Since the cases $ps = 0$ and $p^2 - s^2 = 0$ are uninteresting, we have to prove

LEMMA. *The equation* $(x^2 - y^2)^2 + (3xy)^2 = z^2$ *has no solutions with* x *even*, $xy \neq 0 \neq x^2 - y^2$.

Proof. Take a solution (x, y, z) with x, y coprime and xy smallest possible. Since $3xy$ is even and $x^2 - y^2$ odd we have

$$3xy = 2mnd\,, \qquad x^2 - y^2 = (m^2 - n^2)d\,. \tag{2}$$

Here the common divisor d is 1 or 3. Consider the case $d = 3$ first. Then (2) becomes

$$xy = 2mn\,, \qquad x^2 - y^2 = 3(m^2 - n^2)\,. \tag{3}$$

We find mutually coprime a, A, b, B such that $x = 2aA$, $y = bB$, $m = ab$, $n = AB$. Now (3) reads

$$A^2(4a^2 + 3B^2) = b^2(3a^2 + B^2)\,. \tag{4}$$

Furthermore, since 5 dividing $3a^2 + B^2$ implies that 5 divides (a, B), we know that $(4a^2 + 3B^2, 3a^2 + B^2) = (-5a^2, 3a^2 + B^2) = 1$. Thus (4) becomes

$$A^2 = 3a^2 + B^2\,, \qquad b^2 = 4a^2 + 3B^2\,. \tag{5}$$

Since b, B are odd, reduction mod 8 gives a contradiction and the first case is proved.

Thus we are left with the case $d = 1$ and now (2) reads

$$3xy = 2mn\,, \qquad x^2 - y^2 = m^2 - n^2\,. \tag{6}$$

Again find mutually coprime a, A, b, B such that $x = 2aA$, $y = bB$ but now for n, m there are two possibilities:

$$m = 3ab\,, \qquad n = AB\,, \tag{X}$$

$$m = ab\,, \qquad n = 3AB\,. \tag{Y}$$

Consider the X case; then (6) is

$$A^2(4a^2+B^2)=b^2(9a^2+B^2)\,. \tag{6X}$$

Now the common divisor $(4a^2+B^2, 9a^2+B^2)=(4a^2+B^2, 5a^2)$ is 1 or 5. Thus we have two subcases X1 and X5.

For X1, (6X) gives

$$9a^2+B^2=5A^2\,,\qquad 4a^2+B^2=5b^2\,. \tag{6X1}$$

Since b, B were odd, considerations mod 8 show that there is no solution.

For X5, (6) becomes

$$9a^2+B^2=5A^2\,,\qquad 4a^2+B^2=b^2\,. \tag{6X5}$$

The second equation yields that $a=wv$, $B=v^2-w^2$, the first equation that a is even; thus one of v, w is even and we have

$$(w^2-v^2)+(3wv)^2=A^2\,. \tag{7}$$

Moreover, 2 does not divide w (say), and w, v is a nontrivial ($wv\neq 0\neq w^2-v^2$) solution. Since xy was the smallest possible, $xy\le wv$, i.e. $2abAB\le a$, a contradiction.

Similarly for the Y case ($m=ab$, $n=3AB$), (6) is

$$A^2(4a^2+9B^2)=b^2(a^2+B^2) \tag{8}$$

and $(4a^2+9B^2, a^2+B^2)=(5B^2, a^2+B^2)=5$ or 1.

In the first subcase again reduction mod 8 gives a contradiction. In the second subcase

$$4a^2+9B^2=(2a)^2+(3B)^2=b^2\,,\qquad a^2+B^2=A^2\,. \tag{8a}$$

Note that $2a, 3B$ are coprime, thus $2a=2wv$, $3B=w^2-v^2$ and the second equation reads

$$(3wv)^2+(w^2-v^2)=(3A)^2\,. \tag{9}$$

Since 2 divides wv and one of w, v is odd, by the minimality of xy we have $xy\le wv$, i.e. $2abAB\le a$. This contradiction concludes the proof of the Lemma, and thus Theorem 3 is proved.

4. Additional comments. The reason why the method is interesting is this. First, one can study actions with more complicated fixed point structure. This is related to the study of the rational points of the hypersurface of degree n in $\mathbb{C}P^{n-1}$ given by the equation $\sum_{i=1}^{n}(x_i+1/x_i)=k$, where n is the number of the fixed points, in particular we can try to use topology to construct rational points.

Second, the same reasoning that we employ here applies in a more general situation of a so-called singular Riemannian flow in the sense of

Molino's book [Mo], or one-dimensional "polarizations-with-singularities" of t-structures in the sense of Cheeger–Gromov [C–G].

Let us briefly describe the latter. Following Cheeger–Gromov, consider a pure t-structure on a manifold. We will say that it admits a *one-dimensional polarization-with-singularities* if the holonomy of the t-structure has a one-dimensional invariant subspace. This means that near each point we have a Killing vector field, and moreover that the Killing fields coming from different points can only differ by a constant scaling factor.

We can use these fields to localize the integrals of characteristic forms to the (well defined!) zero set of the polarization. The result is formally the same as the localization formula for a Killing vector field. One of (at least two) reasons for this is that—at least for the pure substructure obtained by taking closures of the orbits of the family of flows generated by Killing fields—there is no holonomy around loops contained in the zero set.

The conclusion is that the formula $(*)$ looks exactly the same, and $(**)$ is slightly different due to the fact that the weight of the Killing field need not be rational. So $(**)$ reads

$$\sigma(M) = \frac{1}{3}\Big\{ \sum_i \Big(x_i + \frac{1}{x_i} \Big) + \sum_j \int_{F_j} \mathrm{eu}(\nu_j) \Big\}.$$

There is, however, an additional piece of information here, namely that the polarization is an invariant subspace of the holonomy. This implies that the x_i are algebraic integers. Moreover, in dimension four only rank 2 pure t-structures are interesting (one-dimensional ones coincide essentially with circle actions, and three-dimensional ones force the manifold to be an affine T^3 bundle over S^1). Thus in that case the x_i are quadratic irrationals.

The short argument at the beginning of Section 2 proves

PROPOSITION 4. *Suppose that a one-dimensional polarization-with-singularities on a four-manifold M has just one fixed point. Then the holonomy is diagonalizable over* $\mathbb{Z}[(1+\sqrt{5})/2]$.

One can produce examples of such polarizations by doing a surgery on $\mathbb{C}P^2$ (see [H–J] for related constructions).

REFERENCES

[C–G] J. Cheeger and M. Gromov, *Collapsing riemannian manifolds while keeping their curvature bounded. I*, J. Differential Geometry 23 (1986), 309–346.

[H–J] B. Hajduk and T. Januszkiewicz, *Cobordism of polarized t-structures*, Topology 31 (1992), 559–570.

[L] M. Lewkowicz, *The limit η-invariant of polarized torus actions*, Bull. Polish Acad. Sci. 33 (1985), 395–401.

[Mo] P. Molino, *Riemannian Foliations*, Birkhäuser, Boston 1988, Chapter 6.

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