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ON EMBEDDABILITY OF CONES IN EUCLIDEAN SPACES

BY

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In 1937 S. Claytor [4] proved

THEOREM. *A locally connected continuum X is embeddable in S^2 if and only if X does not contain any of Kuratowski's curves K_1, K_2, K_3, K_4.*

Denote by CX the space $X \times [0,1]/X \times \{1\}$, called the *cone* of X. We will prove the following

THEOREM 1. *If X is a locally connected continuum and its cone CX is embeddable in $\mathbb{R}^n$ where $n \le 3$, then X is embeddable in S^{n-1}.*

Proof. If CX is embeddable in $\mathbb{R}$, then X is a one-point space.

If CX is embeddable in $\mathbb{R}^2$, then it is clear that X does not contain a triod T (i.e. a set homeomorphic to a cone with a three-point base), because CT contains each of Kuratowski's curves. A non-empty non-degenerate locally connected continuum X which does not contain a triod is an arc or a simply closed curve.

Now, consider the case when CX is embeddable in $\mathbb{R}^3$. The theorem will be proved if we show that the cones CK_1, CK_2, CK_3, CK_4 of Kuratowski's curves are not embeddable in $\mathbb{R}^3$. This will be done in a sequence of lemmas.

First we define Kuratowski's curves.

DEFINITION 1. *Kuratowski's graph* K_1 is a space homeomorphic to the juncture of two three-point sets. It is equivalent to the graph shown in Fig. 1.

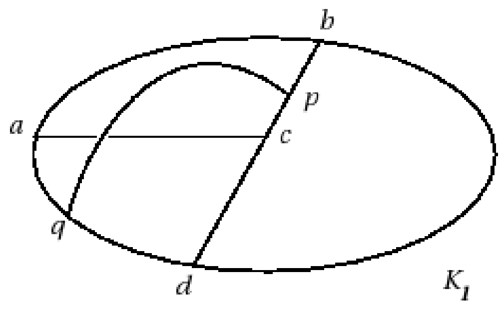

Fig. 1

Definition 2. *Kuratowski's graph* K_2 is a space homeomorphic to the 1-dimensional skeleton of a 4-dimensional simplex. It is equivalent to the graph shown in Fig. 2.

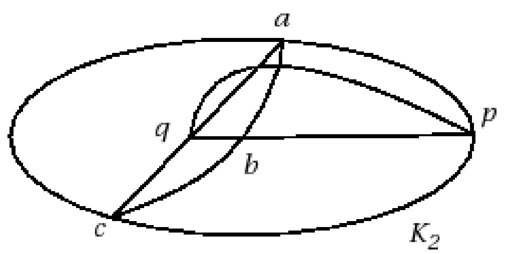

Fig. 2

Definition 3. For each $i \in \mathbb{N}$, let Z_i be a graph as in Fig. 3. Assume that the family of graphs $\{Z_i\}_{i\in\mathbb{N}}$ and the family of open arcs $(p_i q_{i+1})$, where p_i, q_i are as in Fig. 3, have the property that the sets Z_i and $(p_i q_{i+1})$ are pairwise disjoint and their diameters are smaller than 4^{-i}. Let $q_\infty = \lim_{i\to\infty} q_i$ and let $[q_\infty z]$ be a closed arc disjoint from $\bigcup_{i=1}^\infty Z_i \cup \bigcup_{i=1}^\infty (p_i q_{i+1})$. Then *Kuratowski's curve* K_3 is defined by $K_3 = \bigcup_{i=1}^\infty Z_i \cup \bigcup_{i=1}^\infty (p_i q_{i+1}) \cup [q_\infty z]$.

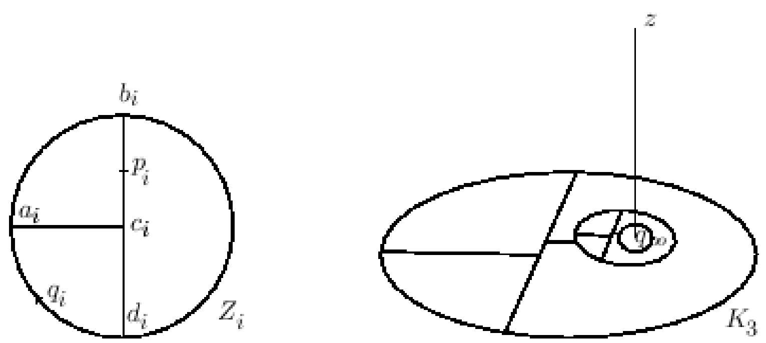

Fig. 3

Definition 4. *Kuratowski's curve* K_4 is defined as in Definition 3 with Z_i replaced by R_i shown in Fig. 4.

Definition 5. We say that a set $D \subset \mathbb{R}^3$ *locally splits* the space $\mathbb{R}^3$ at a point x_0 into n components if for sufficiently small $\varepsilon > 0$ the set $B(x_0;\varepsilon) - D$ has exactly n components $A_1, \ldots, A_n$ such that $x_0 \in \overline{A}_i$ for all $i = 1, \ldots, n$. ($B(x_0;\varepsilon)$ denotes the ball with center x_0 and radius ε.)

Lemma 1. *A homeomorphic image of a disk locally splits* $\mathbb{R}^3$ *at any point of its interior into two components.*

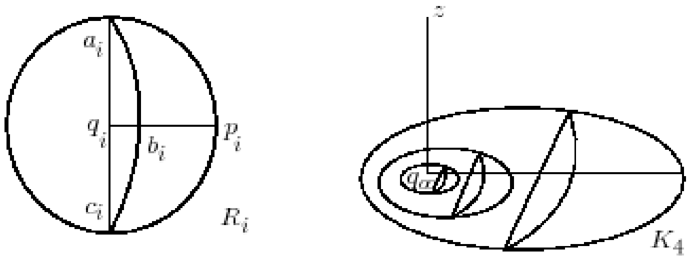

Fig. 4

P r o o f. Let D be a homeomorphic image of a disk. Choose $\varepsilon > 0$ smaller than the distance between x_0 and the boundary of D. Then the component D_0 of $B(x_0;\varepsilon) \cap D$ such that $x_0 \in D_0$ is an open orientable 2-manifold.

If X is closed in $\mathbb{R}^n$ then, by Alexander duality (see [5], VIII, 8.18), $\widetilde{H}_{i-1}(\mathbb{R}^n - X) \approx \check{H}_c^{n-i}(X)$, where $\widetilde{H}_*$ denotes reduced homology and $\check{H}_c^*$ denotes Čech cohomology with compact supports. Therefore, $\widetilde{H}_0(B(x_0;\varepsilon) - D_0) \approx \check{H}_c^2(D_0)$.

On the other hand, if $L \subset K \subset X$ are topological spaces such that L is closed in K, $K - L$ is closed in $X - L$ and $X - L$ is an n-manifold orientable along $K - L$, then $\check{H}_c^i(K, L) \approx H_{n-i}(X - L, X - K)$ (see [5], VIII, 7.14). So, if $L = \emptyset$ and $K = X$, then $\check{H}_c^i(K) \approx H_{n-i}(K)$ (Poincaré duality). Therefore, $\check{H}_c^2(D_0) \approx H_0(D_0) \approx \mathbb{Z}$. Hence, $H_0(B(x_0;\varepsilon) - D_0) \approx \mathbb{Z} \oplus \mathbb{Z}$ and $B(x_0;\varepsilon) - D_0$ has two components.

For arbitrarily small $\delta \in (0, \varepsilon)$, $\widetilde{H}_0(B(x_0;\varepsilon) - (D_0 - B(x_0;\delta))) \approx \check{H}_c^2(D_0 - B(x_0;\delta)) \approx 0$. Hence, $B(x_0;\varepsilon) - (D_0 - B(x_0;\delta))$ is connected. So, x_0 belongs to the closures of both components of $B(x_0;\varepsilon) - D_0$.

LEMMA 2. *If I_i, $i = 1, \ldots, n$, are arcs with common end-points and pairwise disjoint interiors and the map $h : C(\bigcup_{i=1}^n I_i) \to \mathbb{R}^3$ is a homeomorphic embedding, then $C_n = h(C(\bigcup_{i=1}^n I_i))$ locally splits $\mathbb{R}^3$ at its vertex x_0 into n components.*

P r o o f. If $n = 1$, then $C_1 \cap B(x_0;\varepsilon)$ is a 2-manifold with boundary. Therefore, $B(x_0;\varepsilon) - C_1$ is connected. If $n = 2$, C_2 locally splits $\mathbb{R}^3$ at x_0 into 2 components by Lemma 1.

Assume that the lemma holds for $n - 1$. Let $y_0 = h^{-1}(x_0)$ and let $\delta > 0$ be so small that $C = h(C(\bigcup_{i=1}^n I_i) \cap B(y_0;\delta)) \subset B(x_0;\varepsilon)$, where $\varepsilon > 0$ is smaller than the distance between x_0 and the image of the base of the cone. There exists an open connected set U in $\mathbb{R}^3$ such that $C = U \cap C_n$. The set C is homeomorphic to C_n.

Consider the exact sequence of the pair $(U, U - C)$:

$$\ldots \to H_1(U) \to H_0(U, U - C) \to H_0(U - C) \to H_0(U) \to 0\,.$$

Since U is an open 3-manifold, $H_1(U) \approx \check{H}^2_c(U)$ by Poincaré duality. Also $H_0(U, U - C) \approx \check{H}^2_c(C)$ (see [5], VIII, 7.14, where $L = \emptyset$, $K = C$ and $X = U$). Therefore, we can consider an exact sequence

$$\ldots \to \check{H}^2_c(U) \to \check{H}^2_c(C) \to H_0(U - C) \to H_0(U) \to 0\,.$$

Now, we show by induction that the map $\check{H}^2_c(U) \to \check{H}^2_c(C)$ is trivial. If $n = 2$, then C is a disk. Then $H_0(U - C) \approx \mathbb{Z}^2$ by Lemma 1. Since $\check{H}^2_c(C) \approx \mathbb{Z}$ and $H_0(U) \approx \mathbb{Z}$, we obtain an exact sequence $\check{H}^2_c(U) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$. Hence, the map is trivial.

Since $\check{H}^2_c(C_2) \approx \mathbb{Z}$, we obtain by induction $H^2(C) \approx H^2(C_{n-1}) \oplus H^2(C'_2) \approx \mathbb{Z}^{n-2} \oplus \mathbb{Z}$, where $C'_2 = h(C(I_1 \cup I_n))$. The map $\check{H}^2_c(U) \to \check{H}^2_c(C) \approx \check{H}^2_c(C_{n-1}) \oplus \check{H}^2_c(C'_2)$ is trivial because both its coordinates are trivial by the induction hypothesis.

Therefore, the sequence $0 \to \check{H}^2_c(C) \to H_0(U - C) \to H_0(U) \to 0$ is exact. So the sequence $0 \to \mathbb{Z}^{n-1} \to H_0(U - C) \to \mathbb{Z} \to 0$ is also exact. Hence, $H_0(U - C) \approx \mathbb{Z}^n$ and $U - C$ has n components.

The point x_0 belongs to the closure of each of them because if X is C with a small neighborhood of x_0 removed, then $\check{H}^2_c(X) \approx 0$ and $0 \to H_0(U - X) \to H_0(U) \to 0$ is exact, so $H_0(U - X) \approx \mathbb{Z}$.

Therefore, if $B(x_0; \varepsilon') \subset U$, then $B(x_0; \varepsilon') - C_n$ has at least n components such that x_0 belongs to their closures. Now, take $\delta > 0$ so small that $U \subset B(x_0; \varepsilon')$. Then $U - C_n$ has exactly n components. Therefore, $B(x_0; \varepsilon) - C_n$ has exactly n components such that x_0 belongs to their closures.

Remark. Below we often encounter the following situation. The disks C_i locally split $\mathbb{R}^3$ at a point x_0 into two components, and the ε of Definition 5 is common for $i = 1, 2, 3$. We then always call the components A_i and B_i. Let $C = C_1 \cup C_2 \cup C_3$ and K be the component of $C \cap B(x_0; \varepsilon)$ such that $x_0 \in K$. If $K \subset \overline{A}_1$ we relabel the components A_2, B_2 and A_3, B_3 if necessary to have $A_2 \subset A_1$ and $A_3 \subset A_1$. Then C locally splits $\mathbb{R}^3$ at x_0 into components A_2, A_3, B_1.

LEMMA 3. *The cone CK_1 is not embeddable in $\mathbb{R}^3$.*

Proof. Suppose that $h : CK_1 \to \mathbb{R}^3$ is a topological embedding and set

$$K = h(C((ca] \cup [ab))), \quad L = h(C((cp] \cup [pb))), \quad M = h(C((cd] \cup [db)))\,,$$
$$C_1 = \overline{K} \cup \overline{M}, \quad C_2 = \overline{K} \cup \overline{L}, \quad C_3 = \overline{L} \cup \overline{M}, \quad C = \overline{K} \cup \overline{L} \cup \overline{M}\,,$$

where $(xy]$ denotes a "right-closed" arc with end-points x and y, and $\overline{X}$ is the closure of X. The points a, b etc. are as in Definition 1.

Let x_0 be the vertex of CK_1. Choose $\varepsilon > 0$ smaller than the distance from $h(x_0)$ to the image of the base of $h(CK_1)$, and t_0 such that $h(K_1 \times \{t\}) \subset B(h(x_0); \varepsilon)$ for $t \geq t_0$.

Every set C_i, $i = 1, 2, 3$, locally splits $\mathbb{R}^3$ at $h(x_0)$ into two components A_i and B_i. Let $p' = h(p, t_0) \in A_1$. Then we can assume that C locally splits $\mathbb{R}^3$ at $h(x_0)$ into three components A_2, A_3, B_1. Observe that p' and $q' = h(q, t_0)$ are in the same component, because the arc $H = h((pq) \times \{t_0\})$ is contained in $B(h(x_0); \varepsilon) - C$. Hence either $q' \in A_2$ or $q' \in A_3$. So the arc $I = h(((aq] \cup [qd)) \times \{t_0\})$ is contained either in A_2 or in A_3. But $a' = h(a, t_0) \notin \overline{A}_3$ so $I \not\subset A_3$ and $d' = h(d, t_0) \notin \overline{A}_2$ so $I \not\subset A_2$.

Remark. We have obtained a contradiction because the points p' and q' belong to the same component A_1.

LEMMA 4. *The cone CK_2 is not embeddable in $\mathbb{R}^3$.*

Proof. Assume $h : CK_2 \to \mathbb{R}^3$ is a topological embedding. Let x_0, y_0, ε, t_0 be defined as in the previous proof, with K_1 replaced by K_2. Define:

$$K = h(C((ac))), \qquad L = h(C((aq] \cup [qc)))\,,$$
$$M = h(C((ab] \cup [bc))), \qquad N = h(C((ap] \cup [pc)))\,,$$
$$C_1 = \overline{K} \cup \overline{M}, \quad C_2 = \overline{K} \cup \overline{L}\,, \quad C_3 = \overline{L} \cup \overline{M}, \quad C_4 = \overline{L} \cup \overline{N}\,,$$
$$C_5 = \overline{L} \cup \overline{N}, \qquad C_6 = \overline{N} \cup \overline{K}, \qquad C = \overline{K} \cup \overline{L} \cup \overline{M} \cup \overline{N}\,.$$

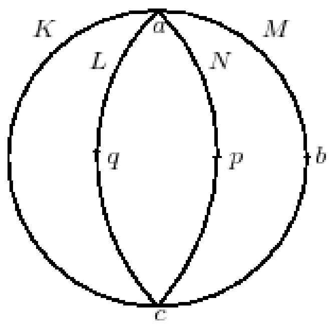

Fig. 5

Every set C_i locally splits $\mathbb{R}^3$ at y_0 into components A_i and B_i. Put $x' = h(x, t_0)$ for any $x \in K_2$ and $H = h((pq) \times \{t_0\})$, $I = h((qb) \times \{t_0\})$, $J = h((bp) \times \{t_0\})$.

Observe that p' and q' belong to the same component, A_1 or B_1, because they are the end-points of the arc H which is contained in $B(y_0; \varepsilon) - C_1$.

Assume that $q' \in A_1$. Then $\overline{K} \cup \overline{L} \cup \overline{M}$ locally splits $\mathbb{R}^3$ at y_0 into components A_2, A_3 and B_1. The point b' belongs to M, so $b' \notin \overline{A}_2$ and $I \subset A_3$. Since $q' \in A_1$ we have either $J \cup H \subset A_2$ or $J \cup H \subset A_3$. But $b' \notin \overline{A}_2$, so $J \cup H \not\subset A_2$. Hence $J \cup H \subset A_3$ and $N \cap A_3 \neq \emptyset$. We can assume $A_4 \subset A_3$ and $A_5 \subset A_3$. Then the cone C locally splits $\mathbb{R}^3$ at y_0 into components A_2, A_4, A_5 and B_1. The arc I is contained either in A_4 or in A_5 because $I \subset A_3$.

But $I \not\subset A_4$ because $b' \notin \overline{A}_4$ and $I \not\subset A_5$ because $q' \notin \overline{A}_5$.

LEMMA 5. *The cone CK_3 is not embeddable in $\mathbb{R}^3$.*

Proof. Assume $h : CK_3 \to \mathbb{R}^3$ is a topological embedding. Let x_0, y_0, ε, t_0 be defined as previously.

The set $C\{q_\infty\}$ is an interval. Put $X = h(C\{q_\infty\})$. There exists $\delta > 0$ such that $h(z, t_0) \notin B(X; \delta) = \{x \in \mathbb{R}^3 : \mathrm{dist}(X, x) < \delta\}$ and $h(Z_1 \times \{t_0\}) \cap B(X; \delta) = \emptyset$, because the distance between disjoint compact sets is positive. By uniform continuity of h there exists i_0 such that $h(CZ_{i_0}) \subset B(X; \delta)$.

Observe that Z_i is homeomorphic to the graph K_1 with the arc (qp) removed.

Define

$$\begin{aligned} K &= h(C((c_{i_0}a_{i_0}] \cup [a_{i_0}b_{i_0}))) , \\ L &= h(C((c_{i_0}p_{i_0}] \cup [p_{i_0}b_{i_0}))) , \\ M &= h(C((c_{i_0}d_{i_0}] \cup [d_{i_0}b_{i_0}))) , \end{aligned}$$

and, as in Lemma 3, $C_1 = \overline{K} \cup \overline{M}$, $C_2 = \overline{K} \cup \overline{L}$, $C_3 = \overline{L} \cup \overline{M}$, $C = \overline{K} \cup \overline{M} \cup \overline{L}$.

The set C_1 locally splits $\mathbb{R}^3$ at y_0 into two components. The points $p' = h(p_{i_0}, t_0)$ and $q' = h(q_{i_0}, t_0)$ lie in the same component because there exist arcs in $h(CK_3 - CZ_{i_0})$ joining p' to $h(z, t_0)$ and q' to $h(Z_1 \times \{t_0\})$. The rest of the proof is the same as for Lemma 3.

LEMMA 6. *The cone CK_4 is not embeddable in $\mathbb{R}^3$.*

Proof. Observe that the set R_i is homeomorphic to the curve K_2 with the arc (qp) removed. So the proof is similar to the proof of Lemma 5, except that after proving that the points q' and p' belong to the same component A_1 or B_1, we will need the proof of Lemma 4 rather than that of Lemma 3.

The proofs of Lemmas 1–6 complete the proof of Theorem 1.

COROLLARY 1. *If the suspension SX of a locally connected continuum X is embeddable in $\mathbb{R}^n$ where $n \le 3$, then X is embeddable in $\mathbb{R}^{n-1}$.*

COROLLARY 2. *If X is a locally connected continuum and CX is embeddable in an n-manifold where $n \le 3$, then X is embeddable in S^{n-1}.*

Proof. If there exists a topological embedding of CX in an n-manifold, then a neighborhood of the image of the vertex of CX is homeomorphic to $\mathbb{R}^n$, so CX is embeddable in $\mathbb{R}^n$.

THEOREM 2. *For each $n \ge 3$ there exists a locally conected continuum X_n such that X_n is not embeddable in $\mathbb{R}^n$ but CX_n is embeddable in $\mathbb{R}^{n+1}$.*

Proof. Consider Blankinship's wild arc J_n lying in the interior of an n-dimensional ball B_n (see [2]). Define X_n to be the quotient space B_n/J_n. It is obvious that X_n is a locally connected continuum.

Suppose that there exists a topological embedding $h : X_n \to \mathbb{R}^n$. Since $h(\partial B_n)$ is homeomorphic to S^{n-1}, it splits $\mathbb{R}^n$ into two components. It is easy to see that the closure of the bounded part U of $\mathbb{R}^n - h(\partial B_n)$ is equal to $h(X_n)$. So $h([J_n]) \in U$ has a neighborhood in U homeomorphic to an open ball. But no neighborhood of $[J_n]$ in X_n is homeomorphic to an open ball because the group $\pi(V - J_n)$ is non-trivial for every neighborhood V of J_n in B_n. So X_n is not embeddable in $\mathbb{R}^n$.

The space $X_n \times (-1,1)$ is homeomorphic to $B_n \times (-1,1)$. In 1962 J. J. Andrews and M. L. Curtis [1] proved that if J is an arc, then $(\mathbb{R}^n/J) \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^{n+1}$. The proof that $X_n \times (-1,1)$ is homeomorphic to $B_n \times (-1,1)$ is the same. The suspension SX_n of X_n is a two-point compactification of $X_n \times (-1,1)$, hence a two-point compactification of $B_n \times (-1,1)$, and this is equal to B_{n+1}, embeddable in $\mathbb{R}^{n+1}$.

So CX_n is embeddable in $\mathbb{R}^{n+1}$, because $CX_n \subset SX_n$.

Remark. SX_n is embeddable in $\mathbb{R}^{n+1}$.

The proof of Theorem 1 uses methods similar to those used in [6]. The results of [6] were generalized by R. Cauty in [3]. The question arises whether a similar generalization is true for the result of this paper.

PROBLEM. *Let X be a locally connected continuum. Supose that $C^n X$ is embeddable in $\mathbb{R}^{n+2}$. Is it true that X is embeddable in S^2?*

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