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VECTOR SETS WITH NO REPEATED DIFFERENCES

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We consider the question when a set in a vector space over the rationals, with no differences occurring more than twice, is the union of countably many sets, none containing a difference twice. The answer is "yes" if the set is of size at most $\aleph_2$, "not" if the set is allowed to be of size $(2^{2^{\aleph_0}})^+$. It is consistent that the continuum is large, but the statement still holds for every set smaller than continuum.

Paul Erdős showed in [2] that if $2^\omega > \omega_1$, then there exists a set $S \subset \mathbb{R}$ such that for every $a \in \mathbb{R}$ there can be at most two solutions of the equation $x+y = a$ $(x, y \in S)$, but if S is decomposed into countably many parts, then in some part, for some $a \in \mathbb{R}$, there are two solutions of $x + y = a$. This is not true under the continuum hypothesis, for then there is a decomposition of $\mathbb{R}$ into countably many linearly independent sets (over $\mathbb{Q}$, the rationals). Erdős and P. Zakrzewski asked if a similar result holds for differences as well.

In this paper V is a vector space over $\mathbb{Q}$, and S is a subset of V. If κ is a cardinal (not necessarily infinite), S is *κ-sum-free* iff for any $a \in V$, there are less than κ solutions of the equation $x + y = a$ $(x, y \in S)$. S is *κ-difference-free* iff for every $d \in V$, $d \neq 0$, there are less than κ solutions of the equation $x - y = d$ $(x, y \in S)$. In the former case, we consider the solutions (x, y) and (y, x) identical. In this notation, Erdős asked if every 3-difference-free set is the union of countably many 2-difference-free sets.

In the paper, the word *sum* is reserved to two-term sums. Also, we sometimes use the coloring terminology, i.e. confuse a decomposition into countably many parts with a coloration with countably many colors.

We first consider when the choice $S = V$ works for questions of the given type.

THEOREM 1. (a) *If $|V| \leq \omega_1$, then V is the union of countably many 2-difference-free sets.*

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(b) *If* $|V| \geq \omega_2$, *then* V *is not the union of countably many* ω_2-*difference-free sets.*

Proof. (a) By a well-known theorem of Erdős and Kakutani (see [3]), every vector space of cardinal ω_1 is the union of countably many bases.

(b) Assume that the vectors $\{x_\alpha, y_\beta : \alpha < \omega_2, \beta < \omega_1\}$ are independent. By a theorem of P. Erdős and A. Hajnal (see e.g. [1]), if the vectors $\{x_\alpha + y_\beta : \alpha < \omega_2, \beta < \omega_1\}$ are colored by countably many colors, then there is a set $Z \subset \omega_2$ of size ω_2 and $\beta_1 < \beta_2 < \omega_1$ such that the vectors $\{x_\alpha + y_{\beta_i} : \alpha \in Z,\ i = 1, 2\}$ get the same color. Then the difference $y_{\beta_1} - y_{\beta_2} = (x_\alpha + y_{\beta_1}) - (x_\alpha + y_{\beta_2})$ is expressed in ω_2 many ways in the same part. ∎

The case of sums is different.

THEOREM 2. (a) *If* $|V| \leq 2^\omega$, *then* V *is the union of countably many* ω-*sum-free sets.*

(b) *If* $|V| > 2^\omega$ *then* V *is not the union of countably many* ω_1-*sum-free sets.*

Proof. (a) We can assume that $V = \mathbb{R}$. Let B be a Hamel basis for $\mathbb{R}$. We color $\mathbb{R} - \{0\}$ with countably many colors as follows. We require that from the color of

$$x = \sum_{i=1}^{n} \lambda_i b_i \qquad (b_1 < \ldots < b_n)$$

the ordered sequence (of rationals) $\lambda_1, \ldots, \lambda_n$ should be recovered, and also a sequence of $n-1$ rational numbers, separating $b_1, \ldots, b_n$ from each other. This is possible as there are countably many rational numbers. If x, y get the same color, and a basis element b appears in both, then, by our above coding requirements, b has the same index, say i, in x and y. The corresponding coordinate in the sum is then $2\lambda_i \neq 0$. There are, therefore, only finitely many possibilities to decompose a given vector as $x + y$.

(b) Let $\{b(\alpha) : \alpha < (2^\omega)^+\}$ be independent. By the Erdős–Rado theorem (see [4]), if we color the vectors $\{b(\alpha) - b(\beta) : \alpha < \beta < (2^\omega)^+\}$ with countably many colors, then there is an increasing sequence $\{\alpha_\xi : \xi \leq \omega_1\}$ such that $\{b(\alpha_\xi) - b(\alpha_\zeta) : \xi < \zeta \leq \omega_1\}$ get the same color. But then

$$b(\alpha_0) - b(\alpha_{\omega_1}) = (b(\alpha_0) - b(\alpha_\xi)) + (b(\alpha_\xi) - b(\alpha_{\omega_1}))$$

is the sum of ω_1 monocolored pairs. ∎

We now consider the more general case when S is an arbitrary subset of V.

THEOREM 3. *If* $|S| \leq \aleph_2$ *is* $\aleph_2$-*difference-free*, *then it is the union of countably many* 2-*difference-free sets.*

P r o o f. We are going to decompose S into the increasing continuous union of sets of size $\aleph_1$, $S = \bigcup\{S_\alpha : \alpha < \omega_2\}$, and again, $S_{\alpha+1} - S_\alpha$ as $\bigcup\{T_{\alpha,\xi} : \xi < \omega_1\}$, the increasing continuous union of countable sets, and then we color the elements in $T_{\alpha,\xi+1} - T_{\alpha,\xi}$ with different colors. We show that if the sets S_α, $T_{\alpha,\xi}$ are sufficiently closed, then no quadruple of the form $\{a, a+x, b, b+x\}$ can get the same color. This suffices, as, by an old observation of R. Rado, every vector space is the union of countably many sets, none containing a three-element arithmetic progression. We require that if a difference $d \neq 0$ occurs as the difference between two elements or two sums in S_α, then all pairs with difference d should be in S_α. Assume that $\{a, a+x, b, b+x\}$ get monocolored, and that $S_{\alpha+1}$ is the first set including all. By the above closure property, at most two of the elements can be in S_α. There are several cases to consider.

C a s e 1: $a, a+x \in S_\alpha,\ b, b+x \in S_{\alpha+1} - S_\alpha$. Impossible, by the closure properties of S_α.

C a s e 2: $a, b \in S_\alpha,\ a+x, b+x \in S_{\alpha+1} - S_\alpha$. Same as Case 1.

C a s e 3: $a, b+x \in S_\alpha,\ a+x, b \in S_{\alpha+1} - S_\alpha$. We show that to any $a+x$ in $S_{\alpha+1} - S_\alpha$ there can only be one b as above. If b is good, then $(a+x)+b = a+(b+x)$ is the sum of two elements in S_α, so if b_1, b_2 are good, then $b_1 - b_2$ is the difference of two sums in S_α, and so $b_1, b_2 \in S_\alpha$, by our assumptions on S_α. Likewise, to every element $b \in S_{\alpha+1} - S_\alpha$ only one good $a+x$ can exist, so if the sets $T_{\alpha,\xi}$ are closed under the $b \mapsto a+x$, $a+x \mapsto b$ functions, then b, $a+x$ appear in the same $T_{\alpha,\xi+1}$, and so they get different colors.

C a s e 4: $a \in S_\alpha,\ a+x, b \in T_{\alpha,\xi},\ b+x \in T_{\alpha,\xi+1} - T_{\alpha,\xi}$. It suffices to show that to a given pair $\{a+x, b\}$ there can correspond at most one $b+x$ as above; then an argument similar to the one given in Case 3 concludes the proof. If $a_1 + x_1 = a_2 + x_2$, $a_1, a_2 \in S_\alpha$, then $a_2 - a_1 = (b+x_1) - (b+x_2)$, so $b+x$ must be in S_α, a contradiction.

C a s e 5: $b \in S_\alpha,\ a, a+x \in T_{\alpha,\xi},\ b+x \in T_{\alpha,\xi+1} - T_{\alpha,\xi}$. Again, it is enough to show that to a given pair $\{a, a+x\}$ there can only be one good $b+x$. Notice that a, $a+x$ already determine x. If b_1+x, b_2+x were good, then their difference $b_1 - b_2$ would occur as the difference of two elements in S_α, so again b_1+x, b_2+x would both be in S_α.

C a s e 6: $a, a+x, b, b+x \in S_{\alpha+1} - S_\alpha$. Assume that $a, a+x, b \in T_{\alpha,\xi}$, $b+x \in T_{\alpha,\xi+1} - T_{\alpha,\xi}$. In this case $b+x = b+(a+x)-a$, so if we make $T_{\alpha,\xi}$ closed under $u+v-w$ for $u, v, w \in T_{\alpha,\xi}$, we see that this case cannot occur. ∎

THEOREM 4. *If* $|V| = (2^{2^\omega})^+$, *then there is a* 3*-difference set* $S \subset V$ *which is not the union of countably many* 2*-difference sets.*

Proof. Let V be the vector space with the basis $\{g(\alpha,\beta) : \alpha < \beta < (2^{2^\omega})^+\}$. For $\alpha < \beta < \gamma$ put $b(\alpha,\beta,\gamma) = g(\alpha,\beta) + g(\beta,\gamma) - g(\alpha,\gamma)$, and let $S = \{b(\alpha,\beta,\gamma) : \alpha < \beta < \gamma < (2^{2^\omega})^+\}$. If S is decomposed as $S = \bigcup\{S_i : i < \omega\}$, then, by the Erdős–Rado theorem (see [4]), there are $i < \omega$, $\alpha < \beta < \gamma < \delta$ with $b(\alpha,\beta,\gamma), b(\alpha,\beta,\delta), b(\alpha,\gamma,\delta), b(\beta,\gamma,\delta) \in S_i$. But then the nonzero distance

$$\begin{aligned} g(\beta,\gamma) - g(\alpha,\gamma) + g(\alpha,\delta) - g(\beta,\delta) &= b(\alpha,\beta,\gamma) - b(\alpha,\beta,\delta) \\ &= b(\beta,\gamma,\delta) - b(\alpha,\gamma,\delta) \end{aligned}$$

occurs twice.

We have to show that S is a 3-difference-free set. If $\alpha < \beta < \gamma < (2^{2^\omega})^+$, $\alpha' < \beta' < \gamma' < (2^{2^\omega})^+$, and there is at most one common element in $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$, then there is no cancellation in $c = b(\alpha,\beta,\gamma) - b(\alpha',\beta',\gamma')$, so the sets can be recovered from c. If the two triplets look like $\{\alpha,\beta,\gamma\}$, $\{\alpha,\gamma,\delta\}$, then

$$b(\alpha,\beta,\gamma) - b(\alpha,\gamma,\delta) = g(\alpha,\beta) + g(\beta,\gamma) - 2g(\alpha,\gamma) + g(\alpha,\delta) - g(\gamma,\delta)\,,$$

the triplets can be reconstructed again. The remaining cases

$$\begin{aligned} b(\alpha,\beta,\delta) - b(\alpha,\gamma,\delta) &= g(\alpha,\beta) + g(\beta,\delta) - g(\alpha,\gamma) - g(\gamma,\delta) \\ &= b(\alpha,\beta,\gamma) - b(\beta,\gamma,\delta) \end{aligned}$$

give the equality of just two vectors. ∎

THEOREM 5. *If V is a vector space and $S \subset V$ is ω_2-difference-free, then S is the union of countably many ω-difference-free sets.*

Proof. We prove the result by induction on $\kappa = |S|$. For $\kappa \le \omega$ the result is obvious. For $\kappa = \omega_1$ we can use the above-mentioned Erdős–Kakutani result that S can be covered by countably many linearly independent sets (see [3]).

If $\kappa > \omega_1$, decompose S as the increasing, continuous union $S = \bigcup\{S_\alpha : \alpha < \kappa\}$ of sets of size smaller than κ such that if a nonzero difference d occurs in S_α, then its all occurrences are in S_α. By the inductive hypothesis, each $S_{\alpha+1} - S_\alpha$ is a union of countably many ω-difference-free sets. We claim that the union of these decompositions is good as well. Assume that the nonzero difference d occurs infinitely many times between points getting the same color t. If d first occurs in $S_{\alpha+1}$, then by the above closure property of our decomposition, each occurrence of d is either in $S_{\alpha+1} - S_\alpha$, or is between S_α and $S_{\alpha+1} - S_\alpha$. By our hypothesis, only finitely many occurrences of the former type get color t, so d occurs infinitely many times as $x - y$ where $x \in S_\alpha$, $y \in S_{\alpha+1} - S_\alpha$ or $x \in S_{\alpha+1} - S_\alpha$, $y \in S_\alpha$. Infinitely many times the same case occurs. If, now, $a, a' \in S_\alpha$, $b, b' \in S_{\alpha+1} - S_\alpha$, and $a - b = a' - b' = d$, then the nonzero difference $a - a' = b - b'$ occurs in S_α, so $b, b' \in S_\alpha$ should hold, a contradiction. ∎

We can slightly extend this result.

THEOREM 6. *If V is a vector space and $S \subset V$ is ω_2-difference-free, then S is the union of countably many ω-difference-free, ω-sum-free sets.*

Proof. By Theorem 5, we can assume that S is ω-difference-free. We again reason by induction on $\kappa = |S|$. The case $\kappa \le \omega$ is again trivial. Assume that $\kappa \ge \omega_1$. Decompose S into the increasing, continuous union of subsets of size $< \kappa$, $S = \bigcup\{S_\alpha : \alpha < \kappa\}$ such that $a + b - c \in S_\alpha$ when $a, b, c \in S_\alpha$, and, of course, $a + b - c \in S$ holds; moreover, if d is either of the form $a - a'$ or $(a + b) - (a' + b')$ for some $a, a', b, b' \in S_\alpha$ then all pairs with difference d occur in S_α. Build an auxiliary graph G_α on $S_{\alpha+1} - S_\alpha$ by joining a, b if the sum $a + b$ occurs among the pairwise sums in S_α.

CLAIM. *G_α consists of independent edges.*

Proof of Claim. Assume that a is joined to b, b', i.e. $a + b$, $a + b'$ both occur among the pairwise sums in S_α. Then $b - b'$ is the difference of two such sums, so $b, b' \in S_\alpha$ by our assumptions on S_α. ∎

G_α is, therefore, a bipartite graph.

By our inductive hypothesis, there is a good coloring of $S_{\alpha+1} - S_\alpha$ such that each color class is ω-sum-free, and we can assume that these classes constitute a good coloring of G_α as well. Take the union of these colorings; we claim that it works.

Assume that the points a_n, b_n get the same color, and $a_n + b_n = c$ ($n = 0, 1, \ldots$). We consider two cases.

Case 1: *For infinitely many n, there is a β_n such that $a_n \in S_{\beta_n}$, $b_n \in S_{\beta_n+1} - S_{\beta_n}$.* If not all β_n's are the same, then we get e.g. $a \in S_\beta$, $b \in S_{\beta+1} - S_\beta$, $a' \in S_{\beta'}$, $b' \in S_{\beta'+1} - S_{\beta'}$, and $\beta < \beta'$. But then $a, b, a' \in S_{\beta'}$ and $b' = a + b - a' \notin S_{\beta'}$, a contradiction.

If, however, $\beta_n = \beta_m$, i.e. $a, a' \in S_\beta$, $b, b' \in S_{\beta+1} - S_\beta$, then $a - a' = b' - b$, so $b, b' \in S_\beta$ again should hold.

Case 2: *For infinitely many n, there is a β_n such that $a_n, b_n \in S_{\beta_n+1} - S_{\beta_n}$.* Not all the β_n's are the same, as the coloring on $S_{\beta+1} - S_\beta$ is supposed to be good. We get, therefore, elements of the following type: $a + b = a' + b'$, $a, b \in S_\beta$, $a', b' \in S_{\beta+1} - S_\beta$, i.e. the sum $a' + b'$ occurs as a sum in S_β, so a', b' are joined in G_α, so they get different colors. ∎

We now show that it is consistent that 2^ω is arbitrarily high, and Theorem 3 can be extended to all cardinals $< 2^\omega$. For the different notions concerning Martin's axiom, and several applications, we recommend [5].

THEOREM 7. *If MA_κ holds and $|S| \le \kappa$ is ω_2-difference-free, then S is the union of countably many 2-difference-free sets.*

P r o o f. By the previous theorem, we can assume that S is ω-difference-free and ω-sum-free. Let $p = (s, f) \in P$ be a condition, where $s \subseteq S$ is finite, and $f : s \to \omega$ is a good coloring, i.e. $f^{-1}(i)$ is 2-difference-free for every $i < \omega$. Put $(s', f') \le (s, f)$ iff $s' \supseteq s$, $f' \supseteq f$. It is obvious that for any $x \in S$, the set $\{(s, f) : x \in s\}$ is dense, and if $G \subseteq P$ is a generic set meeting all these dense sets, then $\bigcup\{f : (s, f) \in G\}$ is a good coloring of S. The only thing we have to prove is that $(P, \le)$ is ccc, i.e. that among any collection of uncountably many elements in P, some two are compatible. Assume that $p_\alpha \in P$ $(\alpha < \omega_1)$ are given. Using the pigeon-hole principle and the Δ-system lemma, we can assume that $p_\alpha = (s \cup s_\alpha, f_\alpha)$ where the sets $\{s, s_\alpha : \alpha < \omega_1\}$ are disjoint, and the functions f_α have identical restrictions to s. As S is ω-difference-free and ω-sum-free, if $\alpha < \omega_1$, then every difference/sum occurring in $s \cup s_\alpha$ which does not occur in s, occurs only in finitely many other $s \cup s_\beta$. By Hajnal's set mapping theorem (see [5]), we can find an uncountable index set in which for $\alpha \ne \beta$, no nonzero difference or sum occurs both in s_α and s_β, except of course the differences and sums in s. We claim that now p_α, p_β are compatible. Assume, towards a contradiction, that the function $f_\alpha \cup f_\beta$ is not a good coloring of $s \cup s_\alpha \cup s_\beta$. Then some $d \ne 0$ occurs twice as a difference, $d = a - b = a' - b'$, and either $a, a' \in s_\alpha$, $b, b' \in s_\beta$ or $a, b' \in s_\alpha$, $a', b \in s_\beta$. In the former case $b - a = b' - a'$ occurs both in s_α and s_β, which is impossible by our assumptions. In the latter case $a + b' = a' + b$, a contradiction again. ∎

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