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ON THE EXPONENTIAL INTEGRABILITY OF FRACTIONAL INTEGRALS ON SPACES OF HOMOGENEOUS TYPE

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In this paper we show that the fractional integral of order α on spaces of homogeneous type embeds $L^{1/\alpha}$ into a certain Orlicz space. This extends results of Trudinger [T], Hedberg [H], and Adams–Bagby [AB].

1. Definitions and statement of results. We will state the main definitions needed in this paper and will refer to [GV] for other definitions and properties. In this paper (X,δ,μ) will denote a space of homogeneous type that is normal and will be referred to as a *normal space*. The property of normality is defined as follows: Let $\mathcal{B}_r(x)$ be the ball of center x and radius r; then there are positive constants A_1 and A_2 such that for all x in X

$$A_1 r \le \mu(\mathcal{B}_r(x)) \quad \text{if } 0 < r < R_x$$

and

$$\mu(\mathcal{B}_r(x)) \le A_2 r \quad \text{if } r \ge r_x,$$

where $r_x = 0$ if $\mu(\{x\}) = 0$, $r_x = \sup\{r > 0 : \mathcal{B}_r(x) = \{x\}\}$ if $\mu(\{x\}) \ne 0$ and $R_x = \infty$ if $\mu(X) = \infty$, $R_x = \inf\{r > 0 : \mathcal{B}_r(x) = X\}$ if $\mu(X) < \infty$. For $1 \le p \le \infty$, $L^p = L^p(X,\delta,\mu)$ has its usual meaning. The space (X,δ,μ) is said to be of *order* γ, $0 < \gamma \le 1$, if there exists a positive constant M such that for every x, y, and z in X,

$$|\delta(x,z) - \delta(y,z)| \le M\delta(x,y)^\gamma(\max\{\delta(x,z),\delta(y,z)\})^{1-\gamma}.$$

In order to define the kernel of the fractional integral without having to distinguish the case when the measure μ has atoms we shall adopt the following abuse of notation: for $0 < \alpha < 1$ we define

$$\frac{1}{\delta(x,y)^{1-\alpha}} = \begin{cases} 1/\delta(x,y)^{1-\alpha} & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

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The *fractional integral of order* α, $0<\alpha<1$, in $L^{1/\alpha}$ is defined by

$$I_\alpha f(x) = \int \frac{f(y)}{\delta(x,y)^{1-\alpha}}\, d\mu(y)$$

if f has bounded support, and otherwise by

$$\widetilde{I}_\alpha f(x) = \int \left\{ \frac{1}{\delta(x,y)^{1-\alpha}} - \frac{\psi_z(y)}{\delta(z,y)^{1-\alpha}} \right\} f(y)\, d\mu(y)$$

where ψ_z is the characteristic function of the complement of the ball $\mathcal{B}_1(z)$, and z is any fixed point in X.

Remark. The convergence a.e. of both integrals and the fact that they are elements of BMO was shown in [GV]. Note that the class of $\widetilde{I}_\alpha f$ in BMO is independent of the choice of z. If f has bounded support then $I_\alpha f$ and $\widetilde{I}_\alpha f$ define the same class in BMO.

THEOREM 1. *Let* (X,δ,μ) *be a normal space*, $0<\alpha<1$, *and let* f *be in* $L^{1/\alpha}$ *with support in a ball* $\mathcal{B}$. *Then there are constants* C_1 *and* c *independent of* $\mathcal{B}$ *and* f *such that*

$$\int_{\mathcal{B}} \exp\left\{ \left(\frac{|I_\alpha f(x)|}{C_1\|f\|_{1/\alpha}} \right)^{1/(1-\alpha)} \right\} d\mu(x) \le c\mu(\mathcal{B})\,.$$

THEOREM 2. *Let* (X,δ,μ) *be a normal space of order* γ, $0<\gamma\le 1$. *Let* $0<\alpha<1$ *and let* f *belong to* $L^{1/\alpha}$. *Then there is a constant* C_2 *independent of* f *such that for every ball* $\mathcal{B}$ *we have*

$$\int_{\mathcal{B}} \left[\exp\left\{ \left(\frac{|\widetilde{I}_\alpha f(x) - m_{\mathcal{B}}(\widetilde{I}_\alpha f)|}{C_2\|f\|_{1/\alpha}} \right)^{1/(1-\alpha)} \right\} - 1 \right] d\mu(x) \le \mu(\mathcal{B})\,.$$

where $m_{\mathcal{B}}(\widetilde{I}_\alpha f) = \mu(\mathcal{B})^{-1}\int_{\mathcal{B}} \widetilde{I}_\alpha f\, d\mu$.

Remark. The expression $I_\alpha f - m_{\mathcal{B}}(I_\alpha f)$ coincides a.e. with $\widetilde{I}_\alpha f - m_{\mathcal{B}}(\widetilde{I}_\alpha f)$ if f has bounded support. Therefore it suffices to state the theorem for $\widetilde{I}_\alpha$.

As mentioned above it was shown in [GV] that for f in $L^{1/\alpha}$, $\widetilde{I}_\alpha f$ is in BMO and $\|\widetilde{I}_\alpha f\|_{\mathrm{BMO}} \le c\|f\|_{1/\alpha}$. This result and the John–Nirenberg theorem [JN], [CW] imply that there are constants K_1 and K_2 such that

$$\int_{\mathcal{B}} \exp\left\{ \frac{|\widetilde{I}_\alpha f - m_{\mathcal{B}}(\widetilde{I}_\alpha f)|}{K_1\|f\|_{1/\alpha}} \right\} d\mu \le K_2\mu(\mathcal{B})$$

for every ball $\mathcal{B}$. But a stronger result is true as stated in Theorem 2. To prove Theorem 2 it is convenient to introduce the related Orlicz space norms. Let ϕ be a convex increasing continuous function on $[0,\infty)$ with $\phi(0)=0$, and $\phi(t)/t\to\infty$ as $t\to\infty$. Let $\mathcal{B}$ be a ball in (X,δ,μ). We say that a measurable function g on $\mathcal{B}$ is in $L_\phi(\mathcal{B})$ if there exists a $\lambda > 0$ such that $\int_{\mathcal{B}} \phi(|g(x)|/\lambda)\, d\mu(x) < \infty$. For $c>0$ we define the norm

$$N_{\mathcal{B},c}(g) = \inf\Big\{\lambda > 0 : \int_{\mathcal{B}} \phi(|g|/\lambda)\, d\mu \le c\mu(\mathcal{B})\Big\}.$$

Then $L_\phi(\mathcal{B})$ is a Banach space with respect to the norm $N_{\mathcal{B},c}$ and these norms are equivalent for different choices of c as shown in Lemma 2.

2. Lemmata and proofs of the theorems

LEMMA 1. *Let (X,δ,μ) be a normal space and $0 < r \le R < \infty$. Then there is a constant B_1 independent of x, r and R such that*

$$\int_{r\le\delta(x,y)\le R} \frac{d\mu(y)}{\delta(x,y)} \le B_1 \log\frac{2R}{r}.$$

Proof. Without loss of generality we can assume that $r_x \le r$. Let K be the smallest positive integer such that $2^{K+1}r > R$. Then using normality we have

$$\begin{aligned}\int_{r\le\delta(x,y)\le R} \frac{d\mu(y)}{\delta(x,y)} &\le \sum_{k=0}^{K} \int_{2^k r\le\delta(x,y)<2^{k+1}r} \frac{d\mu(y)}{\delta(x,y)}\\ &\le \sum_{k=0}^{K} \frac{1}{2^k r} \int_{\delta(x,y)<2^{k+1}r} d\mu(y) \le 2A_2(K+1) \le 4A_2K.\end{aligned}$$

Now observe that $2^{K-1}r \le R$, and that therefore $K \le (1/\log 2)\log(2R/r)$. This proves the lemma with $B_1 = 4A_2/\log 2$.

LEMMA 2. *If $0 < c_1 < c_2$, then*

$$N_{\mathcal{B},c_2} \le N_{\mathcal{B},c_1} \le \frac{c_2}{c_1} N_{\mathcal{B},c_2}.$$

Proof. The first inequality is immediate from the definition of $N_{\mathcal{B},c}$. To prove the second inequality let $\lambda > N_{\mathcal{B},c_2}$. Then

$$\int_{\mathcal{B}} \phi(|f|/\lambda)\, d\mu \le c_2\mu(\mathcal{B}).$$

Multiplying this by c_1/c_2 and using the fact that for $0 < \nu < 1$, $\phi(\nu t) \le \nu\phi(t)$, we get

$$\int_{\mathcal{B}} \phi\Big(\frac{|f|}{(c_2/c_1)\lambda}\Big)\, d\mu \le c_1\mu(\mathcal{B}).$$

This implies the second inequality.

Proof of Theorem 1. Let $\mathcal{B} = \mathcal{B}_r(x_0)$. If x_0 is an atom and $r \le r_{x_0}$ then $I_\alpha f(x_0) = 0$ and the estimate is trivial. Let, then, $r > r_{x_0}$, let $x \in \mathcal{B}$ and let $0 < \varrho < 2\kappa r$ where κ is the constant in the "triangle inequality" $\delta(x,y) \le \kappa(\delta(x,z) + \delta(z,y))$. Then

$$|I_\alpha f(x)| \le \int_{\mathcal{B}} \frac{|f(y)|}{\delta(x,y)^{1-\alpha}}\, d\mu(y) \le \int_{\delta(x,y)\le 2\kappa r} \frac{|f(y)|}{\delta(x,y)^{1-\alpha}}\, d\mu(y)$$
$$\le \int_{\delta(x,y)<\varrho} + \int_{\varrho \le \delta(x,y) \le 2\kappa r} = I_1 + I_2 .$$

We first estimate I_1. If x is an atom and $\varrho \le r_x$ then $I_1 = 0$. Let $\varrho > r_x$ and let $\mathcal{K}$ be the set of nonnegative integers k such that $2^{-k}\varrho > r_x$. Denote by Mf the Hardy–Littlewood maximal function of f. Then

$$I_1 = \sum_{k\in\mathcal{K}} \int_{2^{-k-1}\varrho \le \delta(x,y) < 2^{-k}\varrho} \frac{|f(y)|}{\delta(x,y)^{1-\alpha}}\, d\mu(y)$$
$$\le \sum_{k\in\mathcal{K}} \frac{\mu(\mathcal{B}_{2^{-k}\varrho}(x))}{(2^{-k-1}\varrho)^{1-\alpha}} Mf(x)$$
$$\le Mf(x) \sum_{k=0}^{\infty} \frac{A_2 2^{-k}\varrho}{(2^{-k-1})^{1-\alpha}\varrho^{1-\alpha}} = A_\alpha \varrho^\alpha Mf(x) ,$$

with $A_\alpha = A_2 \cdot 2/(2^\alpha - 1)$.

We now estimate I_2. Using Hölder's inequality with $p = 1/\alpha$ and Lemma 1 we have

$$I_2 \le \|f\|_{1/\alpha} \biggl(\int_{\varrho \le \delta(x,y) \le 2\kappa r} \frac{d\mu(y)}{\delta(x,y)} \biggr)^{1-\alpha} \le \|f\|_{1/\alpha} \biggl(B_1 \log \frac{4\kappa r}{\varrho} \biggr)^{1-\alpha} .$$

If $A_\alpha(2\kappa r)^\alpha Mf(x) \le \|f\|_{1/\alpha}$ we set $\varrho = 2\kappa r$, since $\operatorname{supp}(f)$ is contained in $\mathcal{B}$, $I_2 = 0$ and hence

$$|I_\alpha f(x)| \le I_1 \le \|f\|_{1/\alpha} .$$

If, on the other hand, $A_\alpha(2\kappa r)^\alpha Mf(x) > \|f\|_{1/\alpha}$ then there is a unique ϱ in $(0, 2\kappa r)$ for which $A_\alpha \varrho^\alpha Mf(x) = \|f\|_{1/\alpha}$, i.e. $\varrho = [\|f\|_{1/\alpha}/(A_\alpha Mf(x))]^{1/\alpha}$. With this value of ϱ we have

$$|I_\alpha f(x)| \le I_1 + I_2 \le \|f\|_{1/\alpha} \biggl[1 + \biggl(B_1 \log \frac{4\kappa r A_\alpha^{1/\alpha} Mf(x)^{1/\alpha}}{\|f\|_{1/\alpha}^{1/\alpha}} \biggr)^{1-\alpha} \biggr]$$

and hence in both cases

$$\biggl[\frac{I_\alpha f(x)}{C_1 \|f\|_{1/\alpha}} \biggr]^{1/(1-\alpha)} \le 1 + \log^+ \frac{4\kappa r A_\alpha^{1/\alpha} Mf(x)^{1/\alpha}}{\|f\|_{1/\alpha}^{1/\alpha}}$$

where $C_1 = 2^\alpha \max(1, B_1^{1-\alpha})$.

Finally, using $\|Mf\|_{1/\alpha} \le c_1\|f\|_{1/\alpha}$ and normality we have

$$\int_{\mathcal{B}} \exp\left(\left|\frac{I_\alpha f(x)}{C_1\|f\|_{1/\alpha}}\right|^{1/(1-\alpha)}\right) d\mu(x)$$

$$\le e\left(\mu(\mathcal{B}) + \frac{A_\alpha^{1/\alpha} 4\kappa r}{\|f\|_{1/\alpha}^{1/\alpha}} \int_X Mf(x)^{1/\alpha}\, d\mu(x)\right)$$

$$\le e\left(1 + \frac{A_\alpha^{1/\alpha} 4\kappa c_1^{1/\alpha}}{A_1}\right)\mu(\mathcal{B}) = c\mu(\mathcal{B}).$$

This concludes the proof of the theorem with $C_1 = 2^\alpha \max(1, B_1^{1-\alpha})$ and $c = e(1 + A_\alpha^{1/\alpha} 4\kappa c_1^{1/\alpha}/A_1)$.

Proof of Theorem 2. We consider a ball $\mathcal{B} = \mathcal{B}_r(x_0)$ and the Orlicz norm $N_{\mathcal{B},1}$ defined with $\phi(t) = e^{t^{1/(1-\alpha)}} - 1$. For $f \in L^{1/\alpha}(X)$ we write

$$\begin{aligned}
&\widetilde{I}_\alpha f(x) - m_{\mathcal{B}}(\widetilde{I}_\alpha f)\\
&\quad= \int_X \left[\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{\psi_z(y)}{\delta(z,y)^{1-\alpha}}\right] f(y)\, d\mu(y)\\
&\qquad - \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}}\int_X \left[\frac{1}{\delta(t,y)^{1-\alpha}} - \frac{\psi_z(y)}{\delta(z,y)^{1-\alpha}}\right] f(y)\, d\mu(y)\, d\mu(t)\\
&\quad= \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}}\int_X \left[\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{1}{\delta(t,y)^{1-\alpha}}\right] f(y)\, d\mu(y)\, d\mu(t).
\end{aligned}$$

Decompose $X = \widetilde{\mathcal{B}} \cup \widetilde{\mathcal{B}}^{\mathrm{c}}$ where $\widetilde{\mathcal{B}} = \mathcal{B}_{4\kappa^2 r}(x_0)$. The last expression can be written as

$$\begin{aligned}
&\int_{\widetilde{\mathcal{B}}} \frac{1}{\delta(x,y)^{1-\alpha}} f(y)\, d\mu(y) - \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}}\int_{\widetilde{\mathcal{B}}} \frac{1}{\delta(t,y)^{1-\alpha}} f(y)\, d\mu(y)\, d\mu(t)\\
&\quad + \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}}\int_{\widetilde{\mathcal{B}}^{\mathrm{c}}} \left[\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{1}{\delta(t,y)^{1-\alpha}}\right] f(y)\, d\mu(y)\, d\mu(t)\\
&\qquad = J_1 - J_2 + J_3.
\end{aligned}$$

Since $\|\widetilde{I}_\alpha f - m_{\mathcal{B}}(\widetilde{I}_\alpha f)\|_{\mathcal{B},1} \le \|J_1\|_{\mathcal{B},1} + \|J_2\|_{\mathcal{B},1} + \|J_3\|_{\mathcal{B},1}$ it is enough to show that $\|J_i\|_{\mathcal{B},1} \le M_i\|f\|_{1/\alpha}$, $1 \le i \le 3$, with M_i independent of f. Since $J_1(x) = I_\alpha(f\chi_{\widetilde{\mathcal{B}}})$ we can use Theorem 1 and normality to obtain

$$\begin{aligned}
\int_{\mathcal{B}} \phi\left(\frac{|J_1|}{c_1\|f\|_{1/\alpha}}\right)^{1/(1-\alpha)} d\mu &\le \int_{\mathcal{B}} \phi\left(\frac{|I_\alpha(f\chi_{\widetilde{\mathcal{B}}})|}{c_1\|f\chi_{\widetilde{\mathcal{B}}}\|_{1/\alpha}}\right)^{1/(1-\alpha)} d\mu\\
&\le c\mu(\widetilde{\mathcal{B}}) \le c\mu(\mathcal{B}).
\end{aligned}$$

From the definition of $\|\ \|_{\mathcal{B},c}$ and Lemma 2 it follows that

$$\|J_1\|_{\mathcal{B},1} \le M_1 \|f\|_{1/\alpha}\,.$$

To estimate J_2 we use Jensen's inequality and the estimate above to obtain

$$\int\limits_{\mathcal{B}} \phi\bigg(\frac{J_2}{c_1\|f\|_{1/\alpha}}\bigg)\,d\mu \le \frac{1}{\mu(\mathcal{B})} \int\limits_{\mathcal{B}}\int\limits_{\mathcal{B}} \phi\bigg(\frac{|I_\alpha(f\chi_{\widetilde{\mathcal{B}}})|}{c_1\|f\|_{1/\alpha}}\bigg)\,d\mu(x)\,d\mu(t) \le c\mu(\mathcal{B})\,.$$

As before, from the definition of $\|\ \|_{\mathcal{B},c}$ and Lemma 2 it follows that $\|J_2\|_{\mathcal{B},1} \le M_2\|F\|_{1/\alpha}$.

Finally, for J_3 we will first show that

$$H_f(x,t) = \int\limits_{\widetilde{\mathcal{B}}^c} \bigg[\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{1}{\delta(t,y)^{1-\alpha}}\bigg] f(y)\,d\mu(y)$$

is bounded and $\|H_f\|_\infty \le c\|f\|_{1/\alpha}$.

Since x and t are in $\mathcal{B}$, and y in $\widetilde{\mathcal{B}}^c$, and the space has order γ, Lemma II.3 of [GV] states that

$$\bigg|\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{1}{\delta(t,y)^{1-\alpha}}\bigg| \le B_2\delta(x,t)^\gamma \delta(x,y)^{\alpha-\gamma-1}\,.$$

Using this lemma and Hölder's inequality with $p = 1/\alpha$ we obtain

$$|H_f(x,t)| \le B_2\delta(x,t)^\gamma\Big(\int\limits_{\widetilde{\mathcal{B}}^c} \delta(x,y)^{-1-\gamma/(1-\alpha)}\,d\mu(y)\Big)^{1-\alpha}\Big(\int |f|^{1/\alpha}\,d\mu\Big)^{\alpha}\,.$$

Using inequality II.2 of [GV]:

$$\int\limits_{\widetilde{\mathcal{B}}^c} \delta(x,y)^{-1-\gamma/(1-\alpha)}\,d\mu(y) \le cr^{-\gamma/(1-\alpha)}\,,$$

and $\delta(x,t) \le r$ we get the desired estimate for $\|H_f\|_\infty$.

Therefore $\|J_3\|_\infty \le c\|f\|_{1/\alpha}$. On the other hand, it is easy to show that $\|J_3\|_{\mathcal{B},1} \le c\|J_3\|_\infty$, and hence $\|J_3\|_{\mathcal{B},1} \le M_3\|f\|_{1/\alpha}$. This concludes the proof of Theorem 2.

REFERENCES

[AB] D. R. Adams and R. J. Bagby, *Translation-dilation invariant estimates for Riesz potentials*, Indiana Univ. Math. J. 23 (1974), 1051–1067.

[CW] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[GV] A. E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, in: Analysis and Partial Differential Equations, C. Sadosky (ed.), Dekker, New York 1990, 171–216.

[H] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. 36 (1972), 505–510.
[JN] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1964), 415–426.
[T] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. 17 (1967), 473–483.
[Z] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge 1959.

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