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CHAIN RULES FOR CANONICAL STATE EXTENSIONS ON VON NEUMANN ALGEBRAS

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In previous papers we introduced and studied the extension of a state defined on a von Neumann subalgebra to the whole of the von Neumann algebra with respect to a given state. This was done by using the standard form of von Neumann algebras. In the case of the existence of a norm one projection from the algebra to the subalgebra preserving the given state our construction is simply equivalent to taking the composition with the norm one projection.

In this paper we study couples of von Neumann subalgebras in connection with the state extension. We establish some results on the ω-conditional expectation and give a necessary and sufficient condition for the chain rule of our state extension to be true.

We extensively use the language of spatial derivatives developed by Connes [5] and summarized also in [2], [3]. Given two normal states φ and ω of a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ we set

$$[\varphi/\omega] = \Delta(\varphi, \omega')^{1/2} \Delta(\omega, \omega')^{-1/2} \tag{1}$$

where $\Delta(\psi, \omega')$ is the spatial derivative with respect to the auxiliary faithful normal state ω' on $\mathcal{M}'$. If $\varphi \le \alpha\omega$ for some $\alpha \in \mathbb{R}^+$ then $[\varphi/\omega]$ has a bounded closure belonging to $\mathcal{M}$ which is the analytic continuation of the Connes cocycle $[D\varphi, D\omega]_t$ at the point $-i/2$ (cf. [4]). (Recall that in the general case $[\varphi/\omega]$ is a nonclosable operator.)

On the von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ we consider a state ω induced by a cyclic and separating vector Ω and we denote by J the modular conjugation. For a subalgebra $\mathcal{M}_i \subset \mathcal{M}$, let $\mathcal{H}_i$ be the closure of $\mathcal{M}_i\Omega$ and let E_i be the corresponding projection. So $E_i\mathcal{M}_i|\mathcal{H}_i$ has Ω as a cyclic and separating vector and J_i is written for the corresponding modular conjugation. (Mostly we identify $\mathcal{M}_i$ with $E_i\mathcal{M}_i|\mathcal{H}_i$.)

Let us recall ([1]) that the *ω-conditional expectation* $\mathcal{E}_\omega^{;i} : \mathcal{M} \to \mathcal{M}_i$ is defined by the formula

$$\mathcal{E}_\omega^{;i}(a) = J_i E_i J a J E_i J_i \quad (a \in \mathcal{M})\,. \tag{2}$$

If $\mathcal{M}_i \subset \mathcal{M}_j \subset \mathcal{M}$ then $\mathcal{E}_\psi^{j,i}$ stands for the ψ-conditional expectation $\mathcal{M}_j \to \mathcal{M}_i$ for a state ψ of $\mathcal{M}_j$.

Given a state φ_0 of $\mathcal{M}_0 \subset \mathcal{M} \subset B(\mathcal{H})$, its *extension* $\varrho_\omega(\varphi_0)$ *to* $\mathcal{M}$ *with respect to* ω is defined by the vector

$$[\varphi_0/\omega_0]\Omega \quad (\omega_0 \equiv \omega|\mathcal{M}_0)\,. \tag{3}$$

It was proved in [2], [3] that

$$\mathcal{E}^{;0}_{\varrho_\omega(\varphi_0)}(x) = \mathcal{E}^{;0}_\omega(v^*xv) \quad (x \in \mathcal{M}) \tag{4}$$

where $Jv'J = v \equiv v(\omega, \varphi_0)$ and

$$v' : a[\varphi_0/\omega_0]\Omega \mapsto a[\varrho_\omega(\varphi_0)/\omega]\Omega \quad (a \in \mathcal{M})$$

is a partial isometry in $\mathcal{M}'$. When $\psi \le \alpha\varrho_\omega(\varphi_0)$ and $\psi|\mathcal{M}_0 = \varphi_0$ then similarly

$$\mathcal{E}^{;0}_\psi(x) = \mathcal{E}^{;0}_\omega(a^*xa) \quad (x \in \mathcal{M}) \tag{5}$$

where $Ja'J = a$ and

$$a' : b[\varphi_0/\omega_0]\Omega \mapsto b[\psi/\omega]\Omega \quad (b \in \mathcal{M}) \tag{6}$$

is a bounded operator in $\mathcal{M}'$.

We consider the von Neumann algebra $\mathcal{M}_2 \subset \mathcal{M}_1 \subset \mathcal{M}$ and a faithful normal state φ_2 on $\mathcal{M}_2$. Denote by $\tau(\varphi_2; \mathcal{M}_1)(\omega)$ the vector state on $\mathcal{M}$ for the vector

$$[\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega\,. \tag{7}$$

LEMMA 1. *In the above context*

$$[\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega = J_1 v(\omega_1, \varphi_2) J_1 \Omega\,. \tag{8}$$

P r o o f. It follows from the definition of $v(\omega_1, \varphi_2)$ that

$$[\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega = [\omega_2/\varphi_2] J_1 v(\omega_1, \varphi_2) J_1 [\varphi_2/\omega_2]\Omega\,.$$

Here the right hand side may be transformed as follows:

$$\begin{aligned}[\omega_2/\varphi_2] J_1 v(\omega_1, \varphi_2) J_1 [\varphi_2/\omega_2]\Omega &= J_1 v(\omega_1, \varphi_2) J_1 [\omega_2/\varphi_2][\varphi_2/\omega_2]\Omega \\ &= J_1 v(\omega_1, \varphi_2) J_1 \Omega\,,\end{aligned}$$

and the proof is complete. ∎

LEMMA 2. *With the above notation*

$$\tau(\varphi_2; \mathcal{M}_1)(\omega)|\mathcal{M}_1 = \omega|\mathcal{M}_1\,.$$

P r o o f. This is a consequence of the previous lemma because $v(\omega_1, \varphi_2) \in \mathcal{M}_1$ and $v^*(\omega_1, \varphi_2) v(\omega_1, \varphi_2)\Omega = \Omega$. ∎

We may regard the state $\tau(\varphi_2; \mathcal{M}_1)(\omega)$ as a kind of perturbation of ω by φ_2 and $\mathcal{M}_1$. If the objects φ_2 and $\mathcal{M}_1$ fit well to the given state ω then $\tau(\varphi_2; \mathcal{M}_1)(\omega)$ and ω coincide. (Due to Lemma 2, on the subalgebra $\mathcal{M}_1$

they always coincide.) The next result concerns the majorization relation of these two states.

PROPOSITION 3. *There is a constant* $\alpha > 0$ *such that* $\tau(\varphi_2; \mathcal{M}_1)(\omega) \le \alpha\omega$ *if and only if there exists* $a \in \mathcal{M}$ *such that*

$$\mathcal{E}_\omega^{;1}(a) = v(\omega_1, \varphi_2)(a) \quad \textit{and} \quad \mathcal{E}_\omega^{;1}(a^*a) = \mathcal{E}_\omega^{;1}(a)^*\mathcal{E}_\omega^{;1}(a)\,.$$

Proof. The condition $\tau(\varphi_1; \mathcal{M}_1)(\omega) \le \alpha\omega$ is equivalent to the existence of an $a \in \mathcal{M}$ such that

$$JaJ\Omega = [\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega\,.$$

Lemma 1 tells us that the right hand side is $J_1 v(\omega_1, \varphi_2)\Omega$ and our statement may be verified by computation. The converse is based on the fact that

$$\mathcal{E}_\omega^{;1}(a^*a) = \mathcal{E}_\omega^{;1}(a)^*\mathcal{E}_\omega^{;1}(a)$$

is equivalent to $E_1 Ja\Omega = Ja\Omega$ and the above argument may be reversed. ∎

The next result gives conditions for the chain rule of state extension to hold.

THEOREM 4. *In the above setting the following conditions are equivalent.*

(i) $\tau(\varphi_2; \mathcal{M}_1)(\omega) = \omega$.
(ii) $\varrho_\omega \varrho_{\omega_1}(\varphi_2) = \varrho_\omega(\varphi_2)$.
(iii) $\mathcal{E}_\omega^{;1}(v(\varrho_\omega(\varphi_2), \omega_1)) = v(\omega_1, \varphi_2)$.

Proof. As in the previous proof condition (i) is equivalent to

$$JvJ\Omega = [\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega$$

where v is now a partial isometry, $v^*v = I$. This is again equivalent to

$$JvJ[\varphi_2/\omega_2]\Omega = [\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega\,.$$

Here the left hand side is a representative of $\varrho_\omega(\varphi_2)$ and the right hand side is that of $\varrho_\omega\varrho_{\omega_1}(\varphi_2)$. In this way we arrive at the equivalence of (i) and (ii). The proof of the equivalence of condition (iii) follows from the previous proposition. ∎

As a sample of results which can be reached by the previously developed techniques we shall prove the following.

THEOREM 5. *Let* $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$ *be a von Neumann algebra with the inclusions* $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_4$ *and* $\mathcal{A}_1 \supset \mathcal{A}_3 \supset \mathcal{A}_4$. *Let* ω_1 *be a faithful normal state on* $\mathcal{A}_1$ *with restriction* ω_i *to* $\mathcal{A}_i$ $(i = 2, 3, 4)$ *and let* φ_2 *be a faithful normal state on* $\mathcal{A}_2$ *with restriction* φ_4 *to* $\mathcal{A}_4$. *Then any couple of the following conditions implies the third.*

(i) $\mathcal{E}_\omega^{1,3}(v(\omega_1, \varphi_2)) = v(\omega_3, \varphi_4)$,
$\mathcal{E}_\omega^{1,3}(v(\omega_1, \varphi_2)^* v(\omega_1, \varphi_2)) = |\mathcal{E}_\omega^{1,3}(v(\omega_1, \varphi_2))|^2$.

(ii) $\varphi_2 = \varphi_4 \circ \mathcal{E}^{2,4}_{\omega_2}$.

(iii) $\varrho_{\omega_1}(\varphi_2) = \varrho_{\omega_3}(\varphi_4) \circ \mathcal{E}^{1,3}_{\omega_1}$.

Proof. First we formulate all the conditions in terms of vectors. We claim that (i) is equivalent to

(i)′ $[\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega = [\omega_4/\varphi_4][\varrho_{\omega_3}(\varphi_4)/\omega_3]\Omega$.

Here the left hand side is in fact $J_1 v(\omega_1, \varphi_2)\Omega$ and the right hand side is $J_3 v(\omega_3, \varphi_4)\Omega$. So the equivalence (i)⇔(i)′ follows as in Proposition 3.

Next, consider

(ii)′ $[\varphi_2/\omega_2]\Omega = [\varphi_4/\omega_4]\Omega$.

The left hand side is a vector representative of φ_2 from the natural positive cone for Ω and $\mathcal{A}_2$, and similarly the right hand side is a vector representative of φ_4 in the cone for Ω and $\mathcal{A}_4$. By [2], (ii)′ is equivalent to saying that $\varphi_2 = \varrho_{\omega_2}(\varphi_4)$ and $\mathcal{E}^{2,4}_{\varphi_2} = \mathcal{E}^{2,4}_{\omega_2}$. Now by [6] these latter conditions are equivalent to (ii).

Finally, let

(iii)′ $[\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega = [\varrho_{\omega_3}(\varphi_4)/\omega_3]\Omega$.

The equivalence of (iii) and (iii)′ is essentially the same as that of (ii) and (ii)′.

Assume now (i)′ and (ii)′. Since the state induced by the vector (i)′ on $\mathcal{A}_2$ is ω_2, there is a partial isometry $v_2' \in \mathcal{A}_2'$ such that

$$[\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega = v_2'\Omega\,.$$

Then

$$\begin{aligned}[\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega &= [\varphi_2/\omega_2]v_2'\Omega = v_2'[\varphi_2/\omega_2]\Omega \\ &= v_2'[\varphi_4/\omega_4]\Omega = [\varphi_4/\omega_4]v_2' = [\varrho_{\omega_3}(\varphi_4)/\omega_3]\Omega\,.\end{aligned}$$

Now assume (i)′ and (iii)′. Let $v_2' \in \mathcal{A}_2'$ be as earlier. Then we have

$$\begin{aligned}[\varphi_2/\omega_2]\Omega &= v_2'^* v_2'[\varphi_2/\omega_2]\Omega = v_2'^*[\varphi_2/\omega_2][\omega_2/\varphi_2][\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega \\ &= v_2'^*[\varrho_{\omega_1}(\varphi_2)/\omega_1]\Omega = v_2'^*[\varrho_{\omega_3}(\varphi_4)/\omega_3]\Omega \\ &= v_2'^*[\varphi_4/\omega_4][\omega_4/\varphi_4][\varrho_{\omega_3}(\varphi_4)/\omega_3]\Omega \\ &= v_2'^*[\varphi_4/\omega_4]v_2'\Omega = v_2'^* v_2'[\varphi_4/\omega_4]\Omega = [\varphi_4/\omega_4]\Omega\,.\end{aligned}$$

Finally, the proof of (ii)′&(iii)′⇒(i)′ follows the same lines. ■

Further theorems comparing the partial isometries connecting different conditional expectations can be obtained by the same technique.

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