COLLOQUIUM MATHEMATICUM

VOL. LXIV 1993 FASC. 1

CHANGE OF VARIABLES FORMULA UNDER MINIMAL ASSUMPTIONS

BY

PIOTR HAJŁASZ (WARSZAWA)

1. Introduction. In the previous papers concerning the change of variables formula (in the form involving the Banach indicatrix) various assumptions were made about the corresponding transformation (see e.g. [BI], [GR], [F], [RR]). The full treatment of the case of continuous transformation is given in [RR]. In [BI] the transformation was assumed to be continuous, a.e. differentiable and with locally integrable Jacobian. In this paper we show that none of these assumptions is necessary (Theorem 2). We only need the a.e. existence of approximate partial derivatives.

In Section 3 we consider the general form of the change of variables formula for Sobolev mappings.

The author wishes to thank Professor Bogdan Bojarski for many stimulating conversations and suggestions.

2. Assumptions and result. We start with recalling the classical result of Whitney [W] on equivalent conditions for a.e. approximate differentiability of a function.

Let u be a real-valued function defined on a subset E of $\mathbb{R}^n$. We say that $L = (L_1, \ldots, L_n)$ is an *approximate total differential* of u at x_0 if for every $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ x : \frac{|u(x) - u(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon \right\}$$

has x_0 as a density point. If this is the case then x_0 is a density point of E and L is uniquely determined. If x_0 is a point of density in the direction of each axis then the L_i are the *approximate partial derivatives* of f at x_0.

THEOREM 1 ([W], Th. 1). *Let $f : E \to \mathbb{R}$ be measurable, $E \subseteq \mathbb{R}^n$. Then the following conditions are equivalent*:

(a) *f is approximately totally differentiable a.e. in E.*

This work was supported by a KBN grant.

(b) *f is approximately derivable with respect to each variable a.e. in E.*

(c) *For each $\varepsilon > 0$ there exists a closed set $F \subseteq E$ and a function $g \in C^1(\mathbb{R}^n)$ such that $|E \setminus F| < \varepsilon$, $f_{|F} = g_{|F}$ (by $|\cdot|$ we denote the Lebesgue measure).*

If f maps E to $\mathbb{R}^n$ and each component of f satisfies the conditions of Theorem 1 (for simplicity, we will say that f itself satisfies them) then we can define the Jacobian J_f in the usual manner.

EXAMPLE. If $f : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ is open, has partial derivatives a.e. then (c) holds. For example, this is the case for $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$.

In the sequel Ω denotes an arbitrary open subset of $\mathbb{R}^n$.

Let $f : \Omega \to \mathbb{R}^n$. We say that f satisfies the *condition N* (*Lusin's condition*) if

$$E \subseteq \Omega,\ |E| = 0 \ \Rightarrow\ |f(E)| = 0\,.$$

Let $f : \Omega \to \mathbb{R}^n$, and $E \subseteq \Omega$. The function $N_f(\cdot, E) : \mathbb{R}^n \to \mathbb{N} \cup \{\infty\}$ defined by

$$N_f(y, E) = \mathrm{card}(f^{-1}(y) \cap E)$$

is called the *Banach indicatrix* of f.

Now we can state our main result.

THEOREM 2. *Let $f : \Omega \to \mathbb{R}^n$ be any mapping, where $\Omega \subseteq \mathbb{R}^n$ is an arbitrary open subset.*

If f satisfies one of the conditions (a), (b), (c), *then we can redefine it on a subset of measure zero in such a way that the new f satisfies the condition N.*

If f satisfies one of the conditions (a), (b), (c) *and the condition N then for every measurable function $u : \mathbb{R}^n \to \mathbb{R}$ and every measurable subset E of Ω the following statements are true*:

1) *The functions $(u \circ f)|J_f|$ and $u(y)N_f(y, E)$ are measurable.*

2) *If moreover $u \ge 0$ then*

$$\int_E (u \circ f)|J_f|\, dx = \int_{\mathbb{R}^n} u(y) N_f(y, E)\, dy\,.$$

3) *If one of the functions $(u \circ f)|J_f|$ and $u(y)N_f(y, E)$ is integrable then so is the other* (*integrability of $(u \circ f)|J_f|$ concerns the set E*) *and the formula of* 2) *holds.*

Remarks. 1) A priori it is not evident that $(u \circ f)|J_f|$ is well defined, because the composition of two mappings, with the left mapping being defined a.e., may be undefined on a set of positive measure. But if we put $(u \circ f)(x)|J_f(x)| = 0$ whenever $|J_f(x)| = 0$ (even if $(u \circ f)(x)$ is not defined) it follows from the proof that the function $(u \circ f)|J_f|$ is well defined a.e.

2) It may happen (see Section 3) that f is continuous and the condition N does not hold, so when redefining f to make it satisfy this condition we may change it to a discontinuous mapping.

Proof of Theorem 2. In the proof we need a classical result:

THEOREM 3. *Special case of Theorem* 2 *when f is a locally Lipschitz mapping* (*in this case the condition N holds*).

A short and nice proof of a slightly different version of Theorem 3 can be found in [BI]. For the sake of completeness we sketch the proof of the above version in the Appendix.

Now we can prove our theorem.

Suppose that f satisfies one of the conditions (a), (b), (c). By Theorem 1(c) there exists a sequence of closed sets $X_k \subseteq \Omega$ and functions $g_k \in C^1(\mathbb{R}^n)$ such that $X_k \subseteq X_{k+1}$, $|\Omega \setminus \bigcup_k X_k| = 0$ and $g_{k|X_k} = f_{|X_k}$. Now we redefine f on the set $Z = \Omega \setminus \bigcup_k X_k$ by sending this set to a point. The new f satisfies the condition N.

Now we prove the second part of the theorem. It is easy to see that it suffices to consider any representative satisfying the condition N; we take the one defined above.

Assume first that $u \geq 0$ is an arbitrary measurable function and $E \subseteq \Omega$ is an arbitrary measurable set.

It follows from Theorem 3 that for $k = 1, 2, \ldots$

$$\int_{E \cap X_k} (u \circ g_k)|J_{g_k}|\,dx = \int_{\mathbb{R}^n} u(y) N_{g_k}(y, E \cap X_k)\,dy\,.$$

Since $f = g_k$ in X_k, it is easy to prove that $J_f = J_{g_k}$ a.e. in X_k. Hence

$$\int_{E \cap X_k} (u \circ f)|J_f|\,dx = \int_{\mathbb{R}^n} u(y) N_f(y, E \cap X_k)\,dy\,. \tag{1}$$

Clearly,

$$(u \circ f)|J_f|\chi_{E \cap X_k} \to (u \circ f)|J_f|\chi_{E \setminus Z} \quad \text{as } k \to \infty\,.$$

Hence passing to the limit on the left hand side of (1) we obtain

$$\int_{E \cap X_k} (u \circ f)|J_f|\,dx \to \int_E (u \circ f)|J_f|\,dx \quad \text{as } k \to \infty \tag{2}$$

(we have used the fact that $|Z| = 0$). Consider the right hand side of (1). It is clear that

$$N_f(y, E \cap X_k) \nearrow N_f\Big(y, E \cap \bigcup_k X_k\Big) \quad \text{for all } y \in \mathbb{R}^n \text{ as } k \to \infty\,.$$

Since $|f(Z)| = 0$, we have $N_f(y, Z) = 0$ for a.e. $y \in \mathbb{R}^n$, and hence

$$N_f(y, E \cap X_k) \nearrow N_f\Big(y, E \cap \bigcup_k X_k\Big) + N_f(y, E \cap Z) = N_f(y, E)$$

for a.e. $y \in \mathbb{R}^n$ as $k \to \infty$. Now passing to the limit in the right hand side of (1) we get

$$\int_{\mathbb{R}^n} u(y) N_f(y, E \cap X_k)\, dy \to \int_{\mathbb{R}^n} u(y) N_f(y, E)\, dy \quad \text{as } k \to \infty\,. \tag{3}$$

Putting together (1)–(3) we obtain the theorem for $u \geq 0$. The general case follows by the decomposition $u = u^+ - u^-$.

Remark. The above theorem admits some generalizations. For example one can generalize the "*co-area*" formula (see [H]).

3. Change of variables formula for Sobolev mappings. As noticed above, each $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies condition (b) in Theorem 1, and so Theorem 2 holds for such f. This theorem generalizes the change of variables formula for Sobolev mappings (see e.g. [BI], Th. 8.4, [GR], Th. 1.8, Ch. 5) where the attention was restricted to continuous $W^{1,n}$ mappings satisfying the condition N. The latter formula plays an important role in the quasiregular mappings theory, and so it seems that its extension to arbitrary $W^{1,1}_{\text{loc}}$ mappings can also play a role, especially in connection with the recent results extending the quasiconformal theory to $W^{1,p}_{\text{loc}}$ mappings where $p < n$ (see e.g. [IM]).

In this section we obtain another proof of Theorem 2 (avoiding Theorem 1) for $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$. In fact, we obtain a stronger result. Namely, we prove that it suffices to redefine $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ on the set $\{M|\nabla f| = \infty\}$ for the condition N to be satisfied, where Mh denotes the Hardy–Littlewood maximal function and the mapping f coincides everywhere with its canonical representative:

$$f(x) = \limsup_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy\,. \tag{4}$$

Lemma. *Let $f \in W^{1,1}_{\text{loc}}(\Omega)$. Then there exists a sequence of compact sets $X_k \subseteq X_{k+1} \subseteq \Omega$ and Lipschitz functions $g_k \in \operatorname{Lip}(\Omega)$ such that $f_{|X_k} = g_{k|X_k}$, $\Omega \setminus \bigcup_k X_k = \{M|\nabla f| = \infty\}$ and hence $|\Omega \setminus \bigcup_k X_k| = 0$.*

Then we can complete our proof as in Section 1.

The proof of this lemma presented below is due to Professor Bojarski.

Proof. It is enough to consider $\Omega = \mathbb{R}^n$. We need two well known inequalities:

For almost all $y \in Q$,

$$|f(y) - f_Q| \le C_1 \int\limits_Q \frac{|\nabla f(z)|}{|y-z|^{n-1}}\, dz\,.$$

If g *is measurable then for all* $x \in Q$,

$$\int\limits_Q \frac{|g(y)|}{|x-y|^{n-1}}\, dy \le C_2(\operatorname{diam} Q) Mg(x)\,.$$

Here Q denotes a cube and $h_Q = |Q|^{-1}\int_Q h$. The proof of the first inequality can be found in [GT], Lemma 7.16, for the second see [He].

Integrating both sides of the first inequality over a ball (more precisely, taking $|B(x,r)|^{-1}\int_{B(x,r)} \ldots\, dy$) and applying the inequality

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} |y-z|^{1-n}\, dy \le C_3|x-z|^{1-n}$$

we see that the right hand side is estimated by $C_4\int_Q |\nabla f||x-z|^{1-n}\, dz$. Now letting $r \to 0$ we obtain

$$|f(x) - f_Q| \le C_4 \int\limits_Q \frac{|\nabla f(z)|}{|x-z|^{n-1}}\, dz$$

for all x (where $f(x)$ is defined by (4)).

For any $x, y \in \mathbb{R}^n$, we can find a cube Q containing x, y with $\operatorname{diam} Q \le C_5|x-y|$. Then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_Q| + |f(y) - f_Q| \\ &\le C_4\Big(\int\limits_Q \frac{|\nabla f(z)|}{|x-z|^{n-1}}\, dz + \int\limits_Q \frac{|\nabla f(z)|}{|y-z|^{n-1}}\, dz\Big) \\ &\le C_6(\operatorname{diam} Q)(M|\nabla f|(x) + M|\nabla f|(y)) \\ &\le C_7|x-y|(M|\nabla f|(x) + M|\nabla f|(y))\,. \end{aligned} \tag{5}$$

Hence if $A_k = \{x : M|\nabla f|(x) \le k\}$ then we have $f_{|A_k} \in \operatorname{Lip}_{2kC_7}(A_k)$ and $|\mathbb{R}^n \setminus \bigcup_k A_k| = 0$. Now the lemma follows by the Kirszbraun's theorem ([K], [F], Th. 2.10.43, [S], Th. 5.1) stating that each Lipschitz function defined on a subset of a metric space can be extended to a Lipschitz function defined on the whole space with the same Lipschitz constant.

Now, as noted above, the change of variables formula for Sobolev mappings follows by the same calculations as in Section 1.

Note that since the inequality (5) holds for all x and y such that either $Mf(x)$ or $Mf(y)$ is finite (to avoid the case $|\infty - \infty|$ in the left hand side of (5)), we obtain the following well known result as an immediate consequence:

COROLLARY. $W^{1,\infty}(Q) = \mathrm{Lip}(Q)$.

It is well known that if $p > n$ then every $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ mapping is continuous and satisfies the condition N (see e.g. [BI], Lemma 8.1). An important question arises:

Does Theorem 2 *hold without redefining f on any set provided that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is continuous*?

The answer is negative. Indeed, in [P1], [P2] Ponomarev constructed an example of a homeomorphism $f : [0,1]^n \to [0,1]^n$ of class $W^{1,p}$ for all $p < n$ for which the condition N fails. In [R2] Reshetnyak constructed an example of a continuous mapping of class $W^{1,n}(\mathbb{R}^n)$ without the property N when $n = 2$. In [V] Väisälä extended this result to all $n \geq 2$.

Assume that 1)–3) of Theorem 2 hold for a mapping f for which the condition N fails. Then there exists a set E with $|E| = 0$ and $|f(E)| > 0$. We have

$$0 = \int_E |J_f|\, dx = \int_{\mathbb{R}^n} N_f(y, E)\, dy \geq |f(E)| > 0\,.$$

This contradiction completes the proof.

On the other hand, Reshetnyak proved in [R1], [R2] that if $\Omega \subseteq \mathbb{R}^n$ and $f \in W^{1,n}(\Omega)$ is a homeomorphism then f satisfies the condition N.

Other results and references concerning the condition N can be found in [M].

4. Appendix. In this appendix we sketch the proof of Theorem 3.

THEOREM. *Let $f : \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^n$ is an open subset*, *be a locally Lipschitz mapping. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and $E \subseteq \Omega$ a measurable set. Then*

1) *The functions $(u \circ f)|J_f|$ and $u(y)N_f(y,E)$ are measurable.*
2) *If moreover $u \geq 0$ then*

$$\int_E (u \circ f)|J_f|\, dx = \int_{\mathbb{R}^n} u(y) N_f(y, E)\, dy\,.$$

3) *If one of the functions $(u \circ f)|J_f|$ and $u(y)N_f(y,E)$ is integrable then so is the other* (*integrability of $(u \circ f)|J_f|$ concerns the set E*) *and the formula of* 2) *holds.*

Remarks. 1) If f is a locally Lipschitz mapping then by Rademacher's theorem J_f exists almost everywhere and it is locally bounded because the derivatives of f are bounded by the Lipschitz constant.

2) The first remark made after Theorem 2 also applies here, upon using Lemma 2 below.

Sketch of proof.

LEMMA 1. *Under the assumptions of the theorem*,

$$\int_{\Omega} |J_f(x)|\,dx = \int_{\mathbb{R}^n} N_f(y,\Omega)\,dy\,.$$

Proof. This fact is well known. The reader can find its elegant proof in [BI], Th. 8.3.

LEMMA 2. *Let f satisfy the above assumptions. Let $E = \{x : J_f(x) = 0\}$. Then*

$$A \subset \mathbb{R}^n,\ |A| = 0 \ \Rightarrow\ |f^{-1}(A) \setminus E| = 0\,.$$

Proof. If $\Omega' \Subset \Omega$ then by Lemma 1, the function $N_f(\cdot,\Omega')$ is integrable. Taking a sequence $\Omega_k \Subset \Omega_{k+1}$ with $\bigcup_k \Omega_k = \Omega$ we get the general case, so we can restrict our attention to the case when $N_f(\cdot,\Omega)$ is integrable.

Let $A \subset \mathbb{R}^n$, $|A| = 0$. Then for each $\varepsilon > 0$ there exists an open set $U \subseteq \mathbb{R}^n$ such that $A \subseteq U$, $|U| < \varepsilon$. Then $f^{-1}(A) \subseteq f^{-1}(U)$, hence

$$\begin{aligned}\int_{f^{-1}(A)} |J_f| \le \int_{f^{-1}(U)} |J_f| &= \int_{\mathbb{R}^n} N_f(y, f^{-1}(U))\,dy \\ &= \int_U N_f(y, f^{-1}(U))\,dy = \int_U N_f(y,\Omega)\,dy\,.\end{aligned}$$

The function $N_f(\cdot,\Omega)$ is integrable and U is arbitrarily small, hence $\int_{f^{-1}(A)} |J_f| = 0$ by absolute continuity of the integral; but now $|J_f| > 0$ on $f^{-1}(A) \setminus E$, hence $|f^{-1}(A) \setminus E| = 0$.

Now we can divide the proof of the theorem into six steps in a standard manner. Except for Steps 1 and 3 we omit the simple proofs.

Step 1: $E = \Omega$, u a simple function constant on open sets. Let $V \subseteq \mathbb{R}^n$ be an open set. We have

$$\begin{aligned}\int_{\Omega} \chi_V(f(x))|J_f(x)|\,dx = \int_{f^{-1}(V)} |J_f(x)|\,dx &= \int_V N_f(y, f^{-1}(V))\,dy \\ &= \int_V N_f(y,\Omega)\,dy = \int_{\mathbb{R}^n} \chi_V(y) N_f(y,\Omega)\,dy\,.\end{aligned}$$

Now it suffices to take a linear combination of characteristic functions.

Step 2: E a compact subset of Ω, u a simple function constant on open sets.

Step 3: E a compact subset of Ω, u an arbitrary simple function. It suffices to assume that u is a characteristic function of an arbitrary measurable set. Now there exists a nonincreasing sequence u_k of characteristic functions of open sets tending to u a.e. Then:

(6) $$u_k(y)N_f(y,E) \to u(y)N_f(y,E) \quad \text{for almost all } y \in \mathbb{R}^n\,,$$

(7) $$(u_k \circ f)|J_f(x)| \to (u \circ f)(x)|J_f(x)| \quad \text{for almost all } x \in \Omega\,.$$

The convergence (6) is obvious. To prove (7) notice that we have equality of both sides of (7) on the set $\{x : J_f(x) = 0\}$, and the convergence on the complement of that set is a direct consequence of Lemma 2. Now Step 3 follows by passing to the limits (6) and (7) under the integral sign.

Step 5: E a compact subset of Ω, $u \geq 0$ an arbitrary measurable function.

Step 5: E an arbitrary measurable subset of Ω, $u \geq 0$ an arbitrary measurable function.

Step 6: The general case.

REFERENCES

[BI] B. Bojarski and T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in R^n*, Ann. Acad. Sci. Fenn. Ser. AI Math. 8 (1983), 257–324.

[F] H. Federer, *Geometric Measure Theory*, Springer, 1969.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York 1983.

[GR] V. M. Goldshteĭn and Yu. G. Reshetnyak, *Introduction to the Theory of Functions with Generalized Derivatives and Quasiconformal Mappings*, Nauka, Moscow 1983 (in Russian).

[H] P. Hajłasz, *Co-area formula, Sobolev mappings and related topics*, in preparation.

[He] L. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. 36 (1972), 505–510.

[IM] T. Iwaniec and G. Martin, *Quasiregular mappings in even dimensions*, Mittag-Leffler Report #19, 1989/90 (to appear in Acta Math.).

[K] M. Kirszbraun, *Über die zusammenziehende und Lipschitzsche Transformationen*, Fund. Math. 22 (1934), 77–108.

[M] O. Martio, *Lusin's condition* (N) *and mappings with non-negative Jacobians*, preprint.

[P1] S. P. Ponomarev, *An example of an $ACTL^p$ homeomorphism which is not absolutely continuous in Banach sense*, Dokl. Akad. Nauk SSSR 201 (1971), 1053–1054 (in Russian).

[P2] —, *On the property N for homeomorphisms of class W^1_p*, Sibirsk. Mat. Zh. 28 (2) (1987), 140–148 (in Russian).

[RR] T. Rado and P. V. Reichelderfer, *Continuous Transformations in Analysis*, Springer, 1955.

[R1] Yu. G. Reshetnyak, *Some geometrical properties of functions and mappings with generalized derivatives*, Sibirsk. Mat. Zh. 7 (4) (1966), 886–919 (in Russian).

[R2] —, *On the condition N for mappings of class $W^1_{n,\mathrm{loc}}$*, ibid. 28 (5) (1987), 149–153 (in Russian).

[S] L. Simon, *Lectures on Geometric Measure Theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. 3, 1983.

[V] J. Väisälä, *Quasiconformal maps and positive boundary measure*, Analysis 9 (1989), 205–216.

[W] H. Whitney, *On totally differentiable and smooth functions*, Pacific J. Math. 1 (1951), 143–159.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
BANACHA 2
02-097 WARSZAWA, POLAND
E-mail: HAJLASZ@PLEARN.BITNET

Reçu par la Rédaction le 11.12.1991