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HOMOMORPHISMS OF UNARY ALGEBRAS AND OF THEIR EXPANSIONS

BY

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1. Introduction. Suppose that an algebra B is a reduct of an algebra A. (In this case we also say that A is an *expansion* of B.) Clearly, the endomorphism monoid $\mathrm{End}(A)$ of A is a submonoid of $\mathrm{End}(B)$. We are interested in the question of which pairs of monoids (M,N) can be represented as $(\mathrm{End}(A),\mathrm{End}(B))$, with A an expansion of B, and in the various ways these representable pairs can be represented. Which pairs can be represented was settled in [4]. There it was shown that any representable pair of monoids can be represented by a pair of algebras A and B where A has three unary operations and B has two unary operations. In this paper we are interested in the existence of other kinds of such representations.

Results of this nature are most fully expressed in the language of category theory. So we interrupt our introduction to give some categorical preliminaries. We urge the knowledgeable reader to skip these preliminaries and refer back to them only as necessary.

2. Preliminaries. A *concrete category* is a pair (K,Ψ) where Ψ is a faithful functor from the category K into Set, the category of sets. Suppose that (K_0,Ψ) and (H_0,Λ) are concrete categories. Suppose that the functors Φ_0 and Φ_1 in the commutative diagram (1) are faithful and Φ_0 is full (and thus all four functors in the diagram are faithful):

$$\begin{array}{ccc} K_0 & \xrightarrow{\Psi} & \mathrm{Set} \\ {\scriptstyle\Phi_0}\downarrow & & \downarrow{\scriptstyle\Phi_1} \\ H_0 & \xrightarrow{\Lambda} & \mathrm{Set} \end{array} \tag{1}$$

Then Φ_0 is a *strong embedding* of the concrete category (K_0,Ψ) into the concrete category (H_0,Λ) *carried by the set functor* Φ_1.

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We remind the reader that a commutative square of functors

$$\begin{tikzcd} P \arrow[r, "\Psi"] \arrow[d, "\Phi_0"'] & K_1 \arrow[d, "\Phi_1"] \\ H_0 \arrow[r, "\Lambda"] & H_1 \end{tikzcd} \tag{2}$$

is a *pullback* iff for any other commutative square of functors

$$\begin{tikzcd} \widehat{P} \arrow[r, "\widehat{\Psi}"] \arrow[d, "\widehat{\Phi}_0"'] & K_1 \arrow[d, "\Phi_1"] \\ H_0 \arrow[r, "\Lambda"] & H_1 \end{tikzcd} \tag{3}$$

there exists a unique functor Υ so that the following diagram commutes:

$$\begin{tikzcd} \widehat{P} \arrow[r, equal] \arrow[d, "\widehat{\Phi}_0"'] & \widehat{P} \arrow[r, equal] \arrow[d, "\Upsilon"] & \widehat{P} \arrow[d, "\widehat{\Psi}"] \\ H_0 & P \arrow[l, "\Phi_0"'] \arrow[r, "\Psi"] & K_1 \end{tikzcd} \tag{4}$$

We note that the pullback is necessarily unique up to isomorphism.

The pullback $P = H_0 \star K_1$ of functors $\Lambda : H_0 \to H_1$ and $\Phi_1 : K_1 \to H_1$ is constructed as follows. The class $\mathrm{obj}(H_0 \star K_1)$ of all objects of $H_0 \star K_1$ consists of all pairs $(a_0, b_1) \in \mathrm{obj}(H_0 \times K_1)$ satisfying $\Lambda(a_0) = \Phi_1(b_1)$, and the class $\mathrm{mor}(H_0 \star K_1)$ consists of all morphism pairs $(h_0, k_1) \in \mathrm{mor}(H_0 \times K_1)$ for which $\Lambda(h_0) = \Phi_1(k_1)$. We use Π_0 and Π_1 to denote the restrictions of the two projections to $H_0 \star K_1 \subseteq H_0 \times K_1$, and then the diagram below is the pullback of the functors Λ and Φ_1:

$$\begin{tikzcd} H_0 \star K_1 \arrow[r, "\Pi_1"] \arrow[d, "\Pi_0"'] & K_1 \arrow[d, "\Phi_1"] \\ H_0 \arrow[r, "\Lambda"] & H_1 \end{tikzcd} \tag{5}$$

Suppose that H_0 and K_1 are varieties of algebras regarded as categories whose morphisms are precisely the algebra homomorphisms. Suppose the functors $\Lambda : H_0 \to \mathrm{Set}$ and $\Phi_1 : K_1 \to \mathrm{Set}$ are the usual forgetful functors assigning underlying sets and maps to algebras from H_0 and K_1. In this case there is an alternative description of the pullback. By an abuse of notation we let $H_0 \star K_1$ be the category of all homomorphisms between algebras $(X, F \cup G)$ with $(X, G) \in \mathrm{obj}(H_0)$, $(X, F) \in \mathrm{obj}(K_1)$ and $F \cap G = \emptyset$. Both H_0 and K_1 thus contain reducts of members of $H_0 \star K_1$. We let Φ_0 and Ψ be the obvious "reduction" functors. (For example, $\Phi_0(X, F \cup G) = (X, F)$ and $\Phi_0(f) = f$.) Then diagram (6) is the pullback of Λ and Φ_1:

$$\begin{tikzcd} H_0 \star K_1 \arrow[r, "\Psi"] \arrow[d, "\Phi_0"'] & K_1 \arrow[d, "\Phi_1"] \\ H_0 \arrow[r, "\Lambda"] & H_1 \end{tikzcd} \tag{6}$$

Thus, for varieties U and V of algebras, we have two ways of viewing $V \star U$. Since these are isomorphic, we will choose whichever view is most convenient at the moment.

If the functors Φ_0 and Φ_1 of the commutative diagram (7) are full and faithful, the pair (Φ_0, Φ_1) is called a *simultaneous representation* of Ψ in Λ:

$$\begin{array}{ccc} K_0 & \xrightarrow{\Psi} & K_1 \\ {\scriptstyle \Phi_0}\downarrow & & \downarrow{\scriptstyle \Phi_1} \\ H_0 & \xrightarrow{\Lambda} & H_1 \end{array} \tag{7}$$

Suppose now that we have a monoid $\mathbf{k}_1 \cong \mathrm{End}(B)$, A is an expansion of B and $\mathbf{k}_0 \cong \mathrm{End}(A)$. Suppose also that Λ is the "reduction functor" between the varieties generated by A and B, respectively, and that Ψ is the identity inclusion of $\mathbf{k}_0$ into $\mathbf{k}_1$. Then indeed we have a simultaneous representation of Ψ in Λ; of course, Φ_1 is an isomorphism of $\mathbf{k}_1$ with $\mathrm{End}(B)$ and Φ_0 is the restriction of Φ_1 to $\mathbf{k}_0$. Also, we will be interested in more general situations.

Suppose $\mathbf{k}_0$ and $\mathbf{k}_1$ are small categories and $\Psi : \mathbf{k}_0 \to \mathbf{k}_1$ is a covariant functor. In this case, we say that Ψ is a *small* functor. Suppose also that V and U are varieties of algebras and Λ is the "reduction functor" from $V \star U$ to U determined by $\Lambda(X, F \cup G) = (X, F)$. We will be interested in simultaneous representations of such small Ψ in such Λ. The existence of such a representation requires that there be full and faithful functors Φ_0 and Φ_1 which make the diagram (S) below commute:

$$\begin{array}{ccc} \mathbf{k}_0 & \xrightarrow{\Psi} & \mathbf{k}_1 \\ {\scriptstyle \Phi_0}\downarrow & & \downarrow{\scriptstyle \Phi_1} \\ V \star U & \xrightarrow{\Lambda} & U \end{array} \tag{S}$$

According to [4], a functor Ψ is *regularly faithful* iff Ψ is a composite $\Psi = \Psi_1 \circ \Psi_0$ of a full embedding (that is, a full and faithful functor) Ψ_1 and an equalizer Ψ_0 in the category of all small categories.

We say that a variety is *universal* if it contains an isomorphic copy of any small category as a full subcategory.

1. Introduction (continued). Every monoid M is isomorphic to the endomorphism monoid of a unary algebra with at least two unary operations; more generally, the variety $\mathrm{Alg}(\Delta)$ of all algebras of any type Δ whose sum is ≥ 2 is universal (see Hedrlín and Pultr [2] and Vopěnka, Hedrlín and Pultr [8], or [6]).

According to [4], any functor Ψ simultaneously representable in a reduction functor between two categories of algebras must be regularly faithful. Conversely, it was shown in [4] that any small regularly faithful functor Ψ

has a simultaneous representation in the reduction (or second projection) functor $\Lambda : V \star U \to U$ for any universal category of the form $U = \mathrm{Alg}(\Delta)$ and $V = \mathrm{Alg}(1)$.

The present paper extends the latter result to include any universal variety U of unary algebras. Furthermore, we describe all small subcategories selected by expansions of algebras from U by operations of an arbitrary unary variety or a regular non-unary variety V. More precisely, we show that any small regularly faithful functor Ψ has a simultaneous representation in the reduction functor $\Lambda : V \star U \to U$, as in the diagram (S), whenever U is a universal unary variety and V is either an arbitrary non-unary regular variety or a unary variety not polynomially equivalent to the variety of sets or to a nullary variety. In addition, we characterize functors simultaneously representable in Λ for an expansion $V \star U$ of a universal unary variety U by an essentially nullary variety V.

3. Subpullbacks. The following notion turns out to be of great use to us for our simultaneous representations. A commutative square of functors

$$\begin{array}{ccc} K_0 & \xrightarrow{\Psi} & K_1 \\ {\scriptstyle \Phi_0}\downarrow & & \downarrow{\scriptstyle \Phi_1} \\ H_0 & \xrightarrow{\Lambda} & H_1 \end{array} \tag{7}$$

(with $\Omega = \Phi_1 \circ \Psi = \Lambda \circ \Phi_0$) is called a *subpullback* whenever for any two objects $a, b \in K_0$, the diagram (8) of hom-sets

$$\begin{array}{ccc} K_0(a,b) & \xrightarrow{\Psi} & K_1(\Psi(a),\Psi(b)) \\ {\scriptstyle \Phi_0}\downarrow & & \downarrow{\scriptstyle \Phi_1} \\ H_0(\Phi_0(a),\Phi_0(b)) & \xrightarrow{\Lambda} & H_1(\Omega(a),\Omega(b)) \end{array} \tag{8}$$

is a pullback in the category Set of all sets and mappings. This is equivalent to the requirement that, for any H_1-morphism $h_1 : \Omega(a) \to \Omega(b)$ such that $h_1 = \Lambda(h_0) = \Phi_1(k_1)$ for some $h_0 \in H_0(\Phi_0(a), \Phi_0(b))$ and $k_1 \in K_1(\Psi(a), \Psi(b))$ there be a unique morphism $k_0 \in K_0(a, b)$ for which $\Phi_0(k_0) = h_0$ and $\Psi(k_0) = k_1$.

Whenever diagram (7) is a subpullback, it is easy to see that Φ_0 is faithful or full, respectively, whenever Φ_1 has one of these properties; similarly for Ψ and Λ. Throughout the paper, all subpullbacks will consist of faithful functors.

We see that whenever diagram (7) is a simultaneous representation, it is also a subpullback whenever Λ is a faithful functor.

Another important instance of a subpullback is the pullback $H_0 \star K_1$ of faithful functors $\Lambda : H_0 \to H_1$ and $\Phi_1 : K_1 \to H_1$ in diagram (5). It is clear that the restrictions $\Pi_0 : H_0 \star K_1 \to H_0$ and $\Pi_1 : H_0 \star K_1 \to K_1$ of the two projections to $H_0 \star K_1 \subseteq H_0 \times K_1$ are faithful and that the diagram (5) above is a subpullback.

Another instance of a subpullback occurs when Φ_0 is a strong embedding of the concrete category (K_0, Ψ) into the concrete category (H_0, Λ) carried by the set functor Φ_1 as in diagram (1).

All proofs of simultaneous representability will be based on the following simple claim.

LEMMA 3.1. *Assume that (H_i, A_i) and (K_i, B_i) are concrete categories for $i \in \{0, 1\}$. Let the left square of the diagram below be a subpullback and the right square be a strong embedding*:

$$\begin{array}{ccccc} K_0 & \xrightarrow{B_0} & \mathrm{Set} & \xleftarrow{B_1} & K_1 \\ {\scriptstyle\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle \Phi_1} \\ H_0 & \xrightarrow{A_0} & \mathrm{Set} & \xleftarrow{A_1} & H_1 \end{array}$$

Then the second natural projection functor $K_0 \star K_1 \to K_1$ has a simultaneous representation in $H_0 \star H_1 \to H_1$.

Proof. Define $\Phi_0 : K_0 \star K_1 \to H_0 \star H_1$ by $\Phi_0(k_0, k_1) = (\Gamma(k_0), \Phi_1(k_1))$. For any $(k_0, k_1) \in \mathrm{mor}(K_0 \star K_1)$ we have $B_0(k_0) = B_1(k_1)$ and hence also $A_0\Gamma(k_0) = FB_0(k_0) = FB_1(k_1) = A_1\Phi_1(k_1)$. Therefore $\Phi_0(k_0, k_1) \in \mathrm{mor}(H_0 \star H_1)$, and the two second projections form a commutative square with Φ_0 and Φ_1.

We note that the functors A_i, B_i are all faithful by the definition of concrete category. F is faithful and Φ_1 is both full and faithful by the definition of strong embedding. Since F is faithful, it follows from the definition of subpullback that Γ is also faithful. And thus Φ_0 is faithful because both Γ and Φ_1 are.

It suffices now to show that Φ_0 is full. Let $(h_0, h_1) \in \mathrm{mor}(H_0 \star H_1)$. Since $A_0(h_0) = A_1(h_1)$ and Φ_1 is full, there exists some $k_1 \in K_1$ with $h_1 = \Phi_1(k_1)$. Consequently, $FB_1(k_1) = A_0(h_0)$. Using the subpullback half of the diagram, we obtain a $k_0 \in K_0$ such that $\Gamma(k_0) = h_0$. But then $\Phi_0(k_0, k_1) = (h_0, h_1)$ as required. ∎

4. Unary expansions. Throughout the remainder of the paper, the letter U will denote an arbitrarily chosen universal unary variety.

Let V be any unary variety, and let $K(V)$ denote its monoid of (unary) polynomials. We let $Z(V) \subseteq K(V)$ be the set of all constants, that is, the set of all polynomials for which the identity $z(x) = z(y)$ holds in all algebras of V. $Z(V)$ is void if and only if V is a regular variety. We let z denote both a member of $Z(V)$ and its constant value. We set $L(V) = K(V) \setminus Z(V)$.

Let Ψ be a functor from the small category $\mathbf{k}_0$ to the small category $\mathbf{k}_1$. We wish to consider simultaneous representations (S) of Ψ in the "reduction" functor $\Lambda : V \star U \to U$. There are three cases to consider. They are (a) the case $K(V) = \{1\}$, (b) the case $L(V) \neq \{1\}$ which we call the *nondegenerate* case, and (c) the case $L(V) = \{1\}$ and $Z(V) \neq \emptyset$ which we call the *essentially nullary* case.

(a) In this case $K(V) = \{1\}$, and V is just the category of all sets and mappings. Thus $\Psi : \mathbf{k}_0 \to \mathbf{k}_1$ has a simultaneous representation (S) in $\Lambda : V \star U \to U$ if and only if Ψ is a full embedding.

(b) In the nondegenerate case, by [4], any functor Ψ with a simultaneous representation (S) in $\Lambda : V \star U \to U$ must be regularly faithful. Corollary 4.4 below will establish the converse implication.

(c) In the remaining, essentially nullary case, the morphisms of V are the maps preserving the value of each constant $z \in Z(V)$. The existence of a simultaneous representation (S) of Ψ in Λ imposes the following condition on the functor Ψ:

(p) If $\alpha_0 \in \mathbf{k}_0(a, b)$ and $\beta_1 \in \mathbf{k}_1(\Psi(b), \Psi(c))$, then $\beta_1 = \Psi(\beta_0)$ for some $\beta_0 \in \mathbf{k}_0(b, c)$ exactly when $\beta_1 \circ \Psi(\alpha_0) = \Psi(\gamma_0)$ for some $\gamma_0 \in \mathbf{k}_0(a, c)$.

Indeed, suppose $\beta_1 \circ \Psi(\alpha_0) = \Psi(\gamma_0)$. From the commutativity of (S) it follows that $\Phi_1(\beta_1) \circ \Lambda\Phi_0(\alpha_0) = \Lambda\Phi_0(\gamma_0)$. Now using z_a to denote the value of the constant $z \in Z(V)$ in the algebra $\Phi_0(a)$, etc., we have $\Phi_0(\alpha_0)(z_a) = z_b$ and $\Phi_0(\gamma_0)(z_a) = z_c$. So we conclude that $\Phi_1(\beta_1)(z_b) = z_c$. Since z is arbitrary, we conclude that $\Phi_1(\beta_1)$ is also a V-homomorphism, and thus $\Phi_1(\beta_1) \in \mathrm{mor}(V \star U)$. Therefore by the fullness of Φ_0 we have some $\beta_0 \in \mathrm{mor}(\mathbf{k}_0)$ with $\Phi_0(\beta_0) = \Phi_1(\beta_1)$ which we noted is in $\mathrm{mor}(V \star U)$. Since Λ is the "reduction" functor from $V \star U$ to U, for any $\delta \in \mathrm{mor}(V \star U)$ we have $\Lambda(\delta) = \delta$. In particular, $\Lambda(\Phi_1(\beta_1)) = \Phi_1(\beta_1)$. Now we have $\Phi_1(\beta_1) = \Lambda\Phi_0(\beta_0) = \Phi_1\Psi(\beta_0)$, and, finally, $\beta_1 = \Psi(\beta_0)$ because Φ_1 is faithful.

Theorem 4.6 below shows that any Ψ satisfying (p) has a simultaneous representation (S) in any $\Lambda : V \star U \to U$ for which V is an essentially nullary variety.

A functor $F : \mathrm{Set} \to \mathrm{Set}$ is called *linear* if there exist sets $A \neq \emptyset$ and B such that, for any set X, $F(X) = (X \times A) \sqcup B$ is the sum (i.e. the disjoint union) of $X \times A$ with B and the image $F(f)$ of any mapping $f : X \to X'$ is the sum $F(f) = (f \times \mathrm{id}_A) \sqcup \mathrm{id}_B$.

LEMMA 4.1. *For any universal unary variety U and any infinite cardinal γ there is a strong embedding of* $\mathrm{Alg}(1, 1)$ *into U carried by a linear functor* $F(X) = (X \times A) \sqcup B$ *with* $\mathrm{card}(A), \mathrm{card}(B) \geq \gamma$.

P r o o f. For every universal unary variety U there exists a strong embedding of $\mathrm{Alg}(1, 1)$ into U carried by a linear functor (see [7]). Since any composite of linear functors is linear, it suffices then to find a strong embedding $\Sigma : \mathrm{Alg}(1, 1) \to \mathrm{Alg}(1, 1)$ carried by a linear functor F satisfying the cardinality requirements. The functor Σ will be a composite of three strong embeddings given below.

First, the linear functor $F_0(X) = (X \times 3) \sqcup 2$ carries a functor $\Sigma_0 : \mathrm{Alg}(1,1) \to \mathrm{Alg}(1,1)$ defined by $\Sigma_0(X, \alpha, \beta) = (F_0(X), \varphi, \psi)$, where $\varphi(x,i) = (x, i+1)$ for $(x,i) \in X \times 3$ with the addition modulo three, $\varphi(0) = 1$, $\varphi(1) = 0$, and $\psi(x,1) = (\alpha(x), 0)$, $\psi(x,2) = (\beta(x), 0)$, $\psi(x,0) = (x,0)$, $\psi(0) = \psi(1) = 0$.

It is easy to check that Σ_0 is a faithful functor. So we only need to prove its fullness. Let g be a homomorphism from $\Sigma_0(X, \alpha, \beta)$ to $\Sigma_0(X', \alpha, \beta)$. Orbits of the permutation φ on $X \times 3$ and 2 have different prime orders, and hence these subsets are preserved by any homomorphism. Furthermore, $X \times \{0\}$ is the set of all fixed points of ψ in $X \times 3$; so there is a mapping f such that $g(x,0) = (f(x), 0)$. Using φ, we obtain $g(x,i) = (f(x), i)$ for all $i \in 3$, and ψ on $X \times \{1,2\}$ guarantees that f is a homomorphism. Similarly we find that g is the identity on $\{0,1\}$ and so $g = \Sigma_0(f)$.

Secondly, from [8] and [6, p. 180] we obtain the existence of a connected, endomorphism-free algebra $C \in \mathrm{Alg}(1,1)$ defined on the cardinal γ. (The graph of a unary algebra has (a,b) as an edge iff $b = f(a)$ for some operation f. Recall that a unary algebra is connected iff this graph is connected.) For any $A = (Y, \varphi, \psi) \in \mathrm{Alg}(1,1)$ define $\Sigma_1(A) \in \mathrm{Alg}(1,1,1)$ on the set $Y \times \gamma$ as follows.

We define φ' and ψ' on A by $\varphi'(y,c) = (y, \varphi(c))$ and $\psi'(y,c) = (y, \psi(c))$. We define a third operation χ by $\chi(y,0) = (\varphi(y), 0)$, $\chi(y,1) = (\psi(y), 1)$ and as the identity elsewhere. With $\Sigma_1(f) = f \times \mathrm{id}_\gamma$, it is easy to check that Σ_1 is a faithful functor.

To prove that Σ_1 is full, we first note that any homomorphism g between Σ_1-images of algebras from $\mathrm{Alg}(1,1)$ maps any set of the form $\{y\} \times \gamma$ into a set of the form $\{z\} \times \gamma$. Since C has only the trivial endomorphism, there exists a mapping f such that $g = f \times \mathrm{id}_\gamma$. But then f must be compatible with the operations φ and ψ of the original algebras via the operation χ. Altogether, Σ_1 is a strong embedding carried by the linear functor $F_1(Y) = Y \times \gamma$.

For the third strong embedding Σ_2 of $\mathrm{Alg}(1,1,1)$ into $\mathrm{Alg}(1,1)$ carried by the linear functor $F_2(Z) = Z \times 5$, we refer the reader to [7].

The composite strong embedding $\Sigma = \Sigma_2 \circ \Sigma_1 \circ \Sigma_0$ is, therefore, carried by the set functor $F = F_2 \circ F_1 \circ F_0$ with $F(X) = X \times (\gamma \times 15) \sqcup (\gamma \times 10)$ as required. ∎

Now we turn to the case (b) of a nondegenerate unary variety V.

Let $F_V(Y) \in V$ denote the algebra freely generated by a nonvoid set Y, and let Θ be its least congruence collapsing the set $\{z(y) \mid y \in Y,\ z \in Z(V)\}$. Since the latter set is a subalgebra of $F_V(Y)$, all other classes of Θ are trivial.

So we may assume that $(L(V)\times Y)\cup\{z\}$ is the underlying set of the quotient algebra $Q_Y = F_V(Y)/\Theta$ when V is not regular. For any regular variety V, clearly $Q_Y = F_V(Y) = K(V)\times Y$.

Let Inv denote the variety of all mono-unary algebras (X,δ) defined by the identity $\delta^2(x) = x$.

LEMMA 4.2. *If V is a nondegenerate unary variety and if $\alpha,\beta \geq \mathrm{card}(L(V))$ are infinite cardinals, then there exists a subpullback*

$$\begin{array}{ccc} \mathrm{Inv} & \longrightarrow & \mathrm{Set} \\ {\scriptstyle \Gamma}\downarrow & & \downarrow{\scriptstyle F} \\ V & \longrightarrow & \mathrm{Set} \end{array}$$

in which F is a linear functor given by $F(X) = (X\times A)\sqcup B$ with $\mathrm{card}(A)\geq\alpha$ *and* $\mathrm{card}(B)\geq\beta$.

Proof. To define the functor F, set $A{=}\alpha\times L(V)$ and $B{=}\beta\times L(V)\sqcup\{0\}$.

For any $(X,\delta)\in\mathrm{Inv}$, first define an auxiliary algebra $B(X)$ on $F(X)$ by $k(x,a,l) = (x,a,kl)$ and $k(b,l) = (b,kl)$ whenever $kl\in L(V)$, and by $k(z) = 0$ in all other cases. The algebra $B(X)$ is isomorphic to $Q_Y\in V$ with $Y = X\times\alpha\sqcup\beta$, and $F(f)$ is a homomorphism of $B(X)$ into $B(X')$ for any mapping $f : X\to X'$.

Next we define $\sigma : F(X)\to F(X)$ as $\sigma(x,a,1) = (\delta(x),a,1)$ on $X\times\alpha\times\{1\}$ and as an identity elsewhere; clearly σ^2 is the identity on $F(X)$.

Finally, for each $k\in K(V)$ set $k^* = \sigma\circ k\circ\sigma$ and define

$$\Gamma(X,\delta) = (F(X),\{k^*\mid k\in K\})\,.$$

Since σ is an involution, we obtain

$$\sigma\circ k^* = k\circ\sigma \quad\text{and}\quad \sigma\circ k = k^*\circ\sigma\,.$$

So σ is an isomorphism between the algebras $B(X)$ and $\Gamma(X,\delta)$. This shows that $\Gamma(X,\delta)\in V$.

For $k\in Z(V)\cup\{1\}$ we have $k = k^*$. For $l\in L(V)\setminus\{1\}$ we find that l and l^* coincide on $F(X)\setminus(X\times\alpha\times\{1\})$. For any such l and for all $(x,a,1)\in X\times\alpha\times\{1\}$,

$$\begin{aligned} l^*F(f)(x,a,1) &= \sigma l\sigma(f(x),a,1) = \sigma l(\delta f(x),a,1) \\ &= \sigma(\delta f(x),a,l) = (\delta f(x),a,l), \\ F(f)l^*(x,a,1) &= F(f)\sigma l\sigma(x,a,1) = F(f)\sigma l(\delta(x),a,1) \\ &= F(f)\sigma(\delta(x),a,l) = (f\delta(x),a,l)\,. \end{aligned}$$

Since V is nondegenerate, there is at least one such $l\in L(V)\setminus\{1\}$. Thus $F(f)$ is a morphism of $\Gamma(X,\delta)$ exactly when f is a morphism of X. Consequently, the square is a subpullback as claimed. ∎

THEOREM 4.3. *The reduction functor* $\mathrm{Inv}\star\mathrm{Alg}(1,1) \to \mathrm{Alg}(1,1)$ *has a simultaneous representation in* $V\star U \to U$ *for any nondegenerate unary variety* V *and any universal unary variety* U.

Proof. Apply 4.1, 4.2 and 3.1. ∎

In [4] it is shown that any regularly faithful small functor has a simultaneous representation in $\mathrm{Inv}\star\mathrm{Alg}(1,1) \to \mathrm{Alg}(1,1)$; in view of this, the result below follows immediately from Theorem 4.3.

COROLLARY 4.4. *Let* U *and* V *be unary varieties*, U *universal and* V *nondegenerate. A small functor* $\Psi : \mathbf{k}_0 \to \mathbf{k}_1$ *has a simultaneous representation in* $V\star U \to U$ *if and only if* Ψ *is regularly faithful.*

The remainder of the section deals with the case of an essentially nullary variety V.

LEMMA 4.5. *If a small functor* $\Psi : \mathbf{k}_0 \to \mathbf{k}_1$ *satisfies the condition* (p), *then it has a simultaneous representation* (Φ_0, Φ_1) *in* $\mathrm{Alg}(0)\star\mathrm{Alg}(\Delta) \to \mathrm{Alg}(\Delta)$ *for some type* Δ.

Proof. First we construct a full embedding $\Phi_1 : \mathbf{k}_1 \to \mathrm{Alg}(\Delta)$. This embedding will be a composite of two contravariant functors.

According to pp. 50–51 of [6], for a suitable unary type Δ' there exists a full embedding $\Gamma_0 : \mathbf{k}_1^{\mathrm{opp}} \to \mathrm{Alg}(\Delta')$ carried by $M \sqcup 2$, where M is the Cayley–MacLane representation of $\mathbf{k}_1^{\mathrm{opp}}$. Equivalently, the underlying set functor of Γ_0 is the sum $D \sqcup \overline{2}$ of the dual Cayley–MacLane representation D of $\mathbf{k}_1$ and the contravariant constant functor $\overline{2}$ with a two-element value.

By Lemma 4.1 and from [7] it now follows that there are sets H and E with $\mathrm{obj}(\mathbf{k}_0) \subseteq H$ and a full faithful functor $\Gamma : \mathbf{k}_1 \to \mathrm{Alg}(1,1)$ such that the algebra $\Gamma(k) = (F(k), \varphi_k, \psi_k)$ is carried by the linear contravariant functor given by $F(k) = (D(k) \times H) \sqcup E$. More precisely, since $D(k) = \coprod\{\mathbf{k}_1(k,l) \mid l \in \mathrm{obj}(\mathbf{k}_1)\}$ and $D(\kappa)(\lambda) = \lambda \circ \kappa$, for any $\kappa \in \mathbf{k}_1(k,k')$, the mapping $F(\kappa) : F(k') \to F(k)$ is given by $F(\kappa)(\lambda, h) = (\lambda\circ\kappa, h)$ for $(\lambda, h) \in D(k') \times H$ and $F(\kappa)(e) = e$ for all $e \in E$.

Let $P^- = \mathrm{hom}(-,2)$ denote the contravariant power set functor. Set $G = P^- \circ F$. The composite $G : \mathbf{k}_1 \to \mathrm{Set}$ is then a covariant faithful functor. To define $\Phi_1(k)$ for $k \in \mathrm{obj}(\mathbf{k}_1)$, we assemble the following set Q_k of operations on the set $G(k)$:

the two unary operations $P^-(\varphi_k)$ and $P^-(\psi_k)$,
a unary constant ζ with the value $\emptyset$,
a unary operation γ of complementation,
a λ-ary union ν, where $\lambda \geq \mathrm{card}(G(k))$ for all $k \in \mathrm{obj}(\mathbf{k}_1)$.

We set $\Phi_1(k) = (G(k), Q_k) \in \mathrm{Alg}(\Delta)$ for all $k \in \mathrm{obj}(\mathbf{k}_1)$, where Δ is the similarity type $(1,1,1,1,\lambda)$. We also set $\Phi_1(\kappa) = G(\kappa)$ for all $\kappa \in \mathrm{mor}(\mathbf{k}_1)$.

Since $\Phi_1(\kappa) = G(\kappa)$ is a homomorphism for every morphism κ of $\mathbf{k}_1$, Φ_1 is a well-defined covariant faithful functor.

To see that Φ_1 is full, we observe that the three set-theoretical operations ensure that any homomorphism $g : \Phi_1(k) \to \Phi_1(k')$ is a complete Boolean algebra homomorphism of the respective power sets. Hence $g = P^-(f)$ for some $f : \Gamma(k') \to \Gamma(k)$. The faithfulness of P^- implies that $P^-(\varphi_{k'}) \circ g = g \circ P^-(\varphi_k)$ only when $f \circ \varphi_{k'} = \varphi_k \circ f$. From this and from a similar observation about ψ it follows that the mapping f is a homomorphism and then, because the functor Γ is full, that $g = G(\kappa)$ for some $\kappa \in \mathbf{k}_1(k, k')$. This shows that Φ_1 is full.

To define the other component $\Phi_0 : \mathbf{k}_0 \to \mathrm{Alg}(0) \star \mathrm{Alg}(\Delta)$ of the simultaneous representation in question, we first recall that $\mathrm{obj}(\mathbf{k}_0) \subseteq H$. Using this fact, we expand each algebra $\Phi_1(\Psi(a))$ with $a \in \mathrm{obj}(\mathbf{k}_0)$ by a nullary operation $z_a \in G(\Psi(k))$ defined by

$$z_a = \{(\Psi(\chi), b) \mid b \in \mathrm{obj}(\mathbf{k}_0) \text{ and } \chi \in \mathbf{k}_0(a, b)\},$$

and define $\Phi_0(a) = (G(\Psi(a)), Q_{\Psi(a)} \cup \{z_a\})$ for every $a \in \mathrm{obj}(\mathbf{k}_0)$. For every $\alpha \in \mathbf{k}_0(a, b)$ we set $\Phi_0(\alpha) = G(\Psi(\alpha))$.

Let $\alpha \in \mathbf{k}_0(a, b)$ and $(\lambda, h) \in D(b) \times H$. Then $F\Psi(\alpha)(\lambda, h) = (\lambda \circ \Psi(\alpha), h) \in z_a$ exactly when $\lambda \circ \Psi(\alpha) = \Psi(\gamma)$ for some $\gamma \in \mathbf{k}_0(a, h)$, and also $(\lambda, h) \in z_b$ iff $\lambda = \Psi(\beta)$ for some $\beta \in \mathbf{k}_0(b, h)$. Using (p), we conclude that $G(\Psi(\alpha))(z_a) = z_b$, so that Φ_0 is a well-defined faithful functor.

To prove that Φ_0 is a full functor, choose $\alpha_1 \in \mathbf{k}_1(\Psi(a), \Psi(b))$ arbitrarily and note that $(\mathrm{id}_{\Psi(b)}, b) \in z_b$. If $G(\alpha_1)(z_a) = z_b$ then $(\alpha_1, b) = F(\alpha_1)(\Psi(\mathrm{id}_b), b) \in z_a$, and $\alpha_1 = \Psi(\alpha)$ for some $\alpha \in \mathbf{k}_0(a, b)$ follows from the definition of z_a. ∎

THEOREM 4.6. *If a variety V is essentially nullary and if U is a universal unary variety, then a small functor $\Psi : \mathbf{k}_0 \to \mathbf{k}_1$ has a simultaneous representation in $V \star U \to U$ if and only if it satisfies* (p).

Proof. The necessity of (p) has been established earlier. So we suppose now that (p) holds.

Let Δ be an arbitrary similarity type. According to [7], there is a strong embedding $\Phi_1 : \mathrm{Alg}(\Delta) \to U$ carried by a set functor $F(X) = (X^\kappa \times A) \sqcup B$ with a nonzero cardinal κ and with some $A \neq \emptyset$ and B.

Now we wish to define a faithful functor $\Gamma : \mathrm{Alg}(0) \to V$ which together with F and the forgetful functors will form a subpullback. Suppose $(X, c) \in \mathrm{Alg}(0)$. Then we set $\Gamma(X, c) = (F(X), Z) \in V$ where each $z \in Z$ is to have the value $\mathrm{const}_c \times \{a\} \in X^\kappa \times A$.

Now by Lemma 3.1 we have a simultaneous representation of the reduction functor $\mathrm{Alg}(0) \star \mathrm{Alg}(\Delta) \to \mathrm{Alg}(\Delta)$ in the reduction functor $V \star U \to U$. The theorem follows from Lemma 4.5. ∎

5. Regular non-unary expansions. In this section we prove the following result.

THEOREM 5.1. *For any regular non-unary variety V and a unary universal variety U, a small functor $\Psi : \mathbf{k}_0 \to \mathbf{k}_1$ has a simultaneous representation in $V \star U \to U$ if and only if Ψ is regularly faithful.*

Simultaneous representability of a functor Ψ in any functor between categories of algebras requires that Ψ be regularly faithful; on the other hand, any small regularly faithful Ψ is simultaneously representable in $\mathrm{Alg}(1) \star \mathrm{Alg}(2) \to \mathrm{Alg}(2)$ (see [4]). Hence it suffices to show that, for any pair of varieties U and V specified by Theorem 5.1, $\mathrm{Alg}(1) \star \mathrm{Alg}(2) \to \mathrm{Alg}(2)$ has a simultaneous representation in $V \star U \to U$.

Since any regular non-unary variety V contains the variety Sl of join-semilattices or the variety Sl_0 of join-semilattices with zero ([1], [3], [5]), we only need to show that, for any universal unary variety U, the functor $\mathrm{Alg}(1) \star \mathrm{Alg}(2) \to \mathrm{Alg}(2)$ has a simultaneous representation in $\mathrm{Sl} \star U \to U$ and a simultaneous representation in $\mathrm{Sl}_0 \star U \to U$. Since the composite of simultaneous representations is a simultaneous representation, Theorem 5.1 follows directly from the two lemmata below.

LEMMA 5.2. *The reduction functor* $\mathrm{Alg}(1) \star \mathrm{Alg}(2) \to \mathrm{Alg}(2)$ *has a simultaneous representation in* $\mathrm{Sl} \star \mathrm{Alg}(2) \to \mathrm{Alg}(2)$.

LEMMA 5.3. *The reduction functor* $\mathrm{Sl} \star \mathrm{Alg}(2) \to \mathrm{Alg}(2)$ *has a simultaneous representation in* $\mathrm{Sl} \star U \to U$ *and in* $\mathrm{Sl}_0 \star U \to U$ *for any unary universal variety U.*

Proof of Lemma 5.2. Once again, an application of Lemma 3.1 will assemble the simultaneous representation.

Let $P : \mathrm{Set} \to \mathrm{Set}$ be the functor assigning the set $P(X)$ of all finite $s \subseteq X$ to any set X and, to any $f : X \to X'$, the mapping $P(f) : P(X) \to P(X')$ given by $[P(f)](s) = \{f(x) \mid x \in s\} = f^+(s)$ for every $s \in P(X)$.

For the joint carrier $F : \mathrm{Set} \to \mathrm{Set}$, we select the factorfunctor of $P \times 4$ modulo the equivalence Θ which collapses $(s, 2)$ and $(s, 3)$ for each $s \in P(X)$ with $\mathrm{card}(s) \leq 1$, and nothing else. It is easy to see that F is, indeed, a well-defined set functor. To simplify the notation, we replace $(\emptyset, i)$ by i, and (s, i) by s_i for all $s \neq \emptyset$ and all $i \in 4$; also, s_2 shall denote the class $\{(s, 2), (s, 3)\}$ of Θ. With these conventions in mind, for each $(X, \cdot) \in \mathrm{Alg}(2)$ we define a binary operation $*$ of $\Phi_1(X, \cdot) = (F(X), *)$ as follows.

(1) $0 * 0 = 1$, $2 * 2 = 0$, and $i * j = 2$ for all other $i, j \in \{0, 1, 2\}$;

(2) $s_0 * 0 = 0 * s_0 = 0$ and $s_i * 0 = 0 * s_i = 1$ for $i \in \{1, 2, 3\}$ and $s \neq \emptyset$;

(3) if $s \neq \emptyset$ then $s_i * 2 = 2 * s_i = s_2$ for all $i \in 4$;
$s_0 * 1 = 1 * s_0 = s_1$ and $s_1 * 1 = 1 * s_1 = s_2$;
$s_2 * 1 = 1 * s_2 = s_3$ and $s_3 * 1 = 1 * s_3 = s_2$ when $\mathrm{card}(s) > 1$;
$s_2 * 1 = 1 * s_2 = s_2$ when $\mathrm{card}(s) = 1$;

(4) $s_0 * t_0 = (s \cup t)_1$ when $s \neq \emptyset$ and $t \neq \emptyset$;
(5) $s_1 * t_1 = \{x \cdot y \mid x \in s, y \in t\}_2$ when $s \neq \emptyset$ and $t \neq \emptyset$;
(6) $u * v = 2$ for all other $u, v \in F(X)$.

It is routine to verify that $F(f)$ is a homomorphism whenever f is; thus Φ_1 is a well-defined faithful functor.

To show that Φ_1 is full, assume that $g : (F(X), *) \to (F(X'), *)$ is a homomorphism in Alg(2).

Observe first that, for the unary term $t(z) = z*z$, the equation $t^3(z) = z$ is solvable just when $z \in \{0, 1, 2\}$; the remainder of (1) then implies that $g(i) = i$ for $i \in \{0, 1, 2\}$.

Secondly, from the fact that (2) fully describes solution sets of equations $z * 0 = 0$ and $z * 0 = 1$ we conclude that $g(s_0) \in (P(X') \setminus \{\emptyset\}) \times \{0\}$ and $g(s_i) \in (P(X') \setminus \{\emptyset\}) \times \{1, 2, 3\}$ for any nonvoid $s \in P(X)$ and for all $i \in \{1, 2, 3\}$. In particular, g determines a unique mapping $h : P(X) \setminus \{\emptyset\} \to P(X') \setminus \{\emptyset\}$ such that $g(s_0) = h(s)_0$.

Next, from (3) it follows that $g(s_i) = h(s)_i$ for all $i \in 4$, and also that $\mathrm{card}(h(s)) = 1$ whenever $\mathrm{card}(s) = 1$. This, in turn, yields the existence of a unique mapping $f : X \to X'$ such that $h(\{x\}) = \{f(x)\}$ for all $x \in X$. Using (4), we then find that $h = P(f)$, and hence also that $g = F(f)$. Finally, (5) ensures that $f : (X, \cdot) \to (X', \cdot)$ is a morphism in Alg(2), so that Φ_1 is a full embedding as claimed.

For any $(X, \alpha) \in$ Alg(1) we need to define a join semilattice $\Gamma(X, \alpha) = (F(X), \vee)$ in such a way that Γ, the two underlying set functors Alg(2) $\to$ Set, and the set functor F constitute a subpullback.

First we define a binary relation $\leq$ on $F(X)$ as follows:

(7) $s_i \leq t_j$ whenever $s \subseteq t$, and $0 < i \leq j \in 4$ or $i = j = 0$;
(8) $s_0 \leq t_j$ whenever $\alpha^+(s) \subseteq t$ and $j \in \{1, 2, 3\}$.

The relation $\leq$ is, obviously, a partial order on $F(X)$. To see that $\leq$ defines a join semilattice, we first note that, for $i, j > 0$, the least upper bound $s_i \vee t_j$ of $\{s_i, t_j\}$ is just $(s \cup t)_k$ with $k = \max\{i, j\}$. Clearly $s_0, t_0 \leq (s \cup t)_0$; if $s_0, t_0 \leq u_j$ for some $j > 0$, then $\alpha^+(s \cup t) = \alpha^+(s) \cup \alpha^+(t) \leq u$, that is, $(s \cup t)_0 \leq u_j$. Whence $s_0 \vee t_0 = (s \cup t)_0$. Finally, $(\alpha^+(s) \cup t)_j$ is an upper bound of $\{s_0, t_j\}$ whenever $j > 0$; if u_k is any upper bound of the latter pair then $k \geq j$ and $\alpha^+(s) \cup t \subseteq u$, so that $(\alpha^+(s) \cup t)_j \leq u_k$. Hence $s_0 \vee t_j = (\alpha^+(s) \cup t)_j$. The partial order $\leq$ thus defines a semilattice $(F(X), \vee)$.

Since $P(f) = f^+$ preserves unions, the mapping $F(f)$ is a semilattice homomorphism if and only if $F(f)(s_0 \vee t_j) = F(f)(\alpha^+(s) \cup t)_j = (f^+\alpha^+(s) \cup f^+(t))_j$ coincides with $F(f)(s_0) \vee F(f)(t_j) = (f^+(s))_0 \vee (f^+(t))_j = (\alpha^+ f^+(s) \cup f^+(t))_j$ for $j > 0$ and all appropriate s and t. Since $P \times \{0, 1\}$ is a faithful subfunctor of F, this occurs exactly when $f \circ \alpha = \alpha \circ f$. Consequently, the functor Γ completes the subpullback needed to apply Lemma 3.1. ■

Proof of Lemma 5.3. For any $(X, *) \in \mathrm{Alg}(2)$, let $\Psi_1 = (X^2, \pi_0, \pi_1, \varrho) \in \mathrm{Alg}(1,1,1)$, where $\pi_j : X^2 \to X^2$ is defined by $\pi_j(x_0, x_1) = (x_j, x_j)$ for $j \in 2$, and $\varrho(x, y) = (x * y, x * y)$ for all $(x, y) \in X^2$. Since a mapping $g : X^2 \to (X')^2$ is compatible with both π_j if and only if $g = Q_2(f) = f \times f$ for some $f : X \to X'$, it follows that $\Psi_1 : \mathrm{Alg}(2) \to \mathrm{Alg}(1,1,1)$ is a strong embedding carried by the cartesian square functor Q_2.

Let U be an arbitrary universal unary variety.

Since there is a strong embedding $\mathrm{Alg}(1,1,1) \to \mathrm{Alg}(1,1)$ carried by a linear functor [7], an application of Lemma 4.1 yields a strong embedding $\Phi_1 : \mathrm{Alg}(2) \to U$ whose carrier is a set functor given by $F(X) = X^2 \times A \sqcup B$ with $\mathrm{card}(B) \geq 2$.

We now define functors $\Gamma : \mathrm{Sl} \to \mathrm{Sl}$ and $\Gamma_0 : \mathrm{Sl} \to \mathrm{Sl}_0$ which, together with F and the underlying set functors, will form the subpullbacks needed to apply Lemma 3.1.

For any semilattice $(X, \vee) \in \mathrm{Sl}$, let $(X^2, \vee)$ denote its square. Select a well-ordering $\leq$ of B; let $b_0 \in B$ denote the least element of B and $b_1 \in B$ its successor. Extend $\leq$ to a partial order on $F(X) = X^2 \times A \sqcup B$ by requiring that:

(a) $b_0 \leq (x_0, x_1, a) \leq b_1$ for all $(x_0, x_1, a) \in X^2 \times A$, and
(b) $(x_0, x_1, a) \leq (x_0', x_1', a')$ if and only if $a = a'$ and $(x_0, x_1) \leq (x_0', x_1')$ in $(X^2, \vee)$.

Clearly, the partial order $(F(X), \leq)$ determines a semilattice $\Gamma(X, \vee) = (F(X), +)$ with zero b_0; for each $a \in A$, it contains a subsemilattice $X^2 \times \{a\}$ isomorphic to $(X^2, \vee)$, while $(x_0, x_1, a) + (x_0', x_1', a') = b_1$ whenever $a \neq a'$. Finally, we set $\Gamma_0(X, \vee) = (F(X), +, b_0)$.

It is easy to see that a mapping $f : (X, \vee) \to (X', \vee)$ is a semilattice homomorphism if and only if $F(f) : (F(X), +) \to (F(X'), +)$ is an Sl-morphism if and only if $F(f) : (F(X), +, b_0) \to (F(X'), +, b_0)$ is an Sl_0-morphism. Whence the respective subpullbacks required by Lemma 3.1 exist, and the proof is complete. ■

We are unaware of any general result about non-unary non-regular expansions of a universal variety U of any type. In particular, does every small regularly faithful functor have a simultaneous representation in $\mathrm{Grp} \star U \to U$ for the variety Grp of groups and for any universal (unary) variety U?

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