1. Introduction. Suppose that an algebra $B$ is a reduct of an algebra $A$. (In this case we also say that $A$ is an expansion of $B$.) Clearly, the endomorphism monoid $\text{End}(A)$ of $A$ is a submonoid of $\text{End}(B)$. We are interested in the question of which pairs of monoids $(M,N)$ can be represented as $(\text{End}(A), \text{End}(B))$, with $A$ an expansion of $B$, and in the various ways these representable pairs can be represented. Which pairs can be represented was settled in [4]. There it was shown that any representable pair of monoids can be represented by a pair of algebras $A$ and $B$ where $A$ has three unary operations and $B$ has two unary operations. In this paper we are interested in the existence of other kinds of such representations.

Results of this nature are most fully expressed in the language of category theory. So we interrupt our introduction to give some categorical preliminaries. We urge the knowledgeable reader to skip these preliminaries and refer back to them only as necessary.

2. Preliminaries. A concrete category is a pair $(K, \Psi)$ where $\Psi$ is a faithful functor from the category $K$ into Set, the category of sets. Suppose that $(K_0, \Psi)$ and $(H_0, \Lambda)$ are concrete categories. Suppose that the functors $\Phi_0$ and $\Phi_1$ in the commutative diagram (1) are faithful and $\Phi_0$ is full (and thus all four functors in the diagram are faithful):

\[
\begin{array}{ccc}
K_0 & \xrightarrow{\psi} & \text{Set} \\
\Phi_0 & \downarrow & \downarrow \Phi_1 \\
H_0 & \xrightarrow{\Lambda} & \text{Set}
\end{array}
\]

Then $\Phi_0$ is a strong embedding of the concrete category $(K_0, \Psi)$ into the concrete category $(H_0, \Lambda)$ carried by the set functor $\Phi_1$. 

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We remind the reader that a commutative square of functors
\[
\begin{array}{ccc}
P & \xrightarrow{\psi} & K_1 \\
\phi_0 \downarrow & & \downarrow \phi_1 \\
H_0 & \xrightarrow{\Lambda} & H_1
\end{array}
\]
is a pullback iff for any other commutative square of functors
\[
\begin{array}{ccc}
\hat{P} & \xrightarrow{\hat{\psi}} & K_1 \\
\hat{\phi}_0 \downarrow & & \downarrow \phi_1 \\
H_0 & \xrightarrow{\Lambda} & H_1
\end{array}
\]
there exists a unique functor \( \Upsilon \) so that the following diagram commutes:
\[
\begin{array}{ccc}
\hat{P} & = & \hat{P} \\
\hat{\phi}_0 \downarrow & \Upsilon & \downarrow \hat{\psi} \\
H_0 & \xrightarrow{\phi_0} & P & \xrightarrow{\psi} & K_1
\end{array}
\]
We note that the pullback is necessarily unique up to isomorphism.

The pullback \( P = H_0 \star K_1 \) of functors \( \Lambda : H_0 \rightarrow H_1 \) and \( \Phi_1 : K_1 \rightarrow H_1 \) is constructed as follows. The class \( \text{obj}(H_0 \star K_1) \) of all objects of \( H_0 \star K_1 \) consists of all pairs \((a_0, b_1) \in \text{obj}(H_0 \times K_1)\) satisfying \( \Lambda(a_0) = \Phi_1(b_1) \), and the class \( \text{mor}(H_0 \star K_1) \) consists of all morphism pairs \((h_0, k_1) \in \text{mor}(H_0 \times K_1)\) for which \( \Lambda(h_0) = \Phi_1(k_1) \). We use \( \Pi_0 \) and \( \Pi_1 \) to denote the restrictions of the two projections to \( H_0 \star K_1 \subseteq H_0 \times K_1 \), and then the diagram below is the pullback of the functors \( \Lambda \) and \( \Phi_1 \):
\[
\begin{array}{ccc}
H_0 \star K_1 & \xrightarrow{\n_1} & K_1 \\
\n_0 \downarrow & & \downarrow \phi_1 \\
H_0 & \xrightarrow{\Lambda} & H_1
\end{array}
\]
Suppose that \( H_0 \) and \( K_1 \) are varieties of algebras regarded as categories whose morphisms are precisely the algebra homomorphisms. Suppose the functors \( \Lambda : H_0 \rightarrow \text{Set} \) and \( \Phi_1 : K_1 \rightarrow \text{Set} \) are the usual forgetful functors assigning underlying sets and maps to algebras from \( H_0 \) and \( K_1 \). In this case there is an alternative description of the pullback. By an abuse of notation we let \( H_0 \star K_1 \) be the category of all homomorphisms between algebras \((X, F \cup G)\) with \((X, G) \in \text{obj}(H_0)\), \((X, F) \in \text{obj}(K_1)\) and \( F \cap G = \emptyset \). Both \( H_0 \) and \( K_1 \) thus contain reducts of members of \( H_0 \star K_1 \). We let \( \Phi_0 \) and \( \Psi \) be the obvious “reduction” functors. (For example, \( \Phi_0(X, F \cup G) = (X, F) \) and \( \Phi_0(f) = f \).) Then diagram (6) is the pullback of \( \Lambda \) and \( \Phi_1 \):
\[
\begin{array}{ccc}
H_0 \star K_1 & \xrightarrow{\psi} & K_1 \\
\phi_0 \downarrow & & \downarrow \phi_1 \\
H_0 & \xrightarrow{\Lambda} & H_1
\end{array}
\]
Thus, for varieties $U$ and $V$ of algebras, we have two ways of viewing $V \star U$. Since these are isomorphic, we will choose whichever view is most convenient at the moment.

If the functors $\Phi_0$ and $\Phi_1$ of the commutative diagram (7) are full and faithful, the pair $(\Phi_0, \Phi_1)$ is called a simultaneous representation of $\Psi$ in $\Lambda$:

\[
\begin{array}{c}
K_0 \xrightarrow{\Psi} K_1 \\
\Phi_0 \downarrow \quad \downarrow \Phi_1 \\
H_0 \xrightarrow{\Lambda} H_1
\end{array}
\]

(7)

Suppose now that we have a monoid $k_1 \cong \text{End}(B)$, $A$ is an expansion of $B$ and $k_0 \cong \text{End}(A)$. Suppose also that $A$ is the “reduction functor” between the varieties generated by $A$ and $B$, respectively, and that $\Psi$ is the identity inclusion of $k_0$ into $k_1$. Then indeed we have a simultaneous representation of $\Psi$ in $\Lambda$; of course, $\Phi_1$ is an isomorphism of $k_1$ with $\text{End}(B)$ and $\Phi_0$ is the restriction of $\Phi_1$ to $k_0$. Also, we will be interested in more general situations.

Suppose $k_0$ and $k_1$ are small categories and $\Psi : k_0 \to k_1$ is a covariant functor. In this case, we say that $\Psi$ is a small functor. Suppose also that $V$ and $U$ are varieties of algebras and $\Lambda$ is the “reduction functor” from $V \star U$ to $U$ determined by $\Lambda(X, F \cup G) = (X, F)$. We will be interested in simultaneous representations of such small $\Psi$ in such $\Lambda$. The existence of such a representation requires that there be full and faithful functors $\Phi_0$ and $\Phi_1$ which make the diagram (S) below commute:

\[
\begin{array}{c}
k_0 \xrightarrow{\Psi} k_1 \\
\Phi_0 \downarrow \quad \downarrow \Phi_1 \\
V \star U \xrightarrow{\Lambda} U
\end{array}
\]

(S)

According to [4], a functor $\Psi$ is regularly faithful iff $\Psi$ is a composite $\Psi = \Psi_1 \circ \Psi_0$ of a full embedding (that is, a full and faithful functor) $\Psi_1$ and an equalizer $\Psi_0$ in the category of all small categories.

We say that a variety is universal if it contains an isomorphic copy of any small category as a full subcategory.

1. Introduction (continued). Every monoid $M$ is isomorphic to the endomorphism monoid of a unary algebra with at least two unary operations; more generally, the variety $\text{Alg}(\Delta)$ of all algebras of any type $\Delta$ whose sum is $\geq 2$ is universal (see Hedrlín and Pultr [2] and Vopěnka, Hedrlín and Pultr [8], or [6]).

According to [4], any functor $\Psi$ simultaneously representable in a reduction functor between two categories of algebras must be regularly faithful. Conversely, it was shown in [4] that any small regularly faithful functor $\Psi$
has a simultaneous representation in the reduction (or second projection) functor $A : V \star U \to U$ for any universal category of the form $U = \text{Alg}(\Delta)$ and $V = \text{Alg}(1)$.

The present paper extends the latter result to include any universal variety $U$ of unary algebras. Furthermore, we describe all small subcategories selected by expansions of algebras from $U$ by operations of an arbitrary unary variety or a regular non-unary variety $V$. More precisely, we show that any small regularly faithful functor $\Psi$ has a simultaneous representation in the reduction functor $\Lambda$:

$V \star U \to U$

for any universal category of the form $U = \text{Alg}(\Delta)$ and $V = \text{Alg}(1)$.

3. Subpullbacks. The following notion turns out to be of great use to us for our simultaneous representations. A commutative square of functors

\[
\begin{array}{ccc}
K_0 & \xrightarrow{\psi} & K_1 \\
\phi_0 \downarrow & & \downarrow \phi_1 \\
H_0 & \xrightarrow{\Lambda} & H_1
\end{array}
\]

(with $\Omega = \Phi_1 \circ \Psi = \Lambda \circ \Phi_0$) is called a subpullback whenever for any two objects $a, b \in K_0$, the diagram (8) of hom-sets

\[
\begin{array}{ccc}
K_0(a, b) & \xrightarrow{\psi} & K_1(\Psi(a), \Psi(b)) \\
\phi_0 \downarrow & & \downarrow \phi_1 \\
H_0(\Phi_0(a), \Phi_0(b)) & \xrightarrow{\Lambda} & H_1(\Omega(a), \Omega(b))
\end{array}
\]

is a pullback in the category Set of all sets and mappings. This is equivalent to the requirement that, for any $H_1$-morphism $h_1 : \Omega(a) \to \Omega(b)$ such that $h_1 = \Lambda(h_0) = \Phi_1(k_1)$ for some $h_0 \in H_0(\Phi_0(a), \Phi_0(b))$ and $k_1 \in K_1(\Psi(a), \Psi(b))$ there be a unique morphism $k_0 \in K_0(a, b)$ for which $\Phi_0(k_0) = h_0$ and $\Psi(k_0) = k_1$.

Whenever diagram (7) is a subpullback, it is easy to see that $\Phi_0$ is faithful or full, respectively, whenever $\Phi_1$ has one of these properties; similarly for $\Psi$ and $\Lambda$. Throughout the paper, all subpullbacks will consist of faithful functors.

We see that whenever diagram (7) is a simultaneous representation, it is also a subpullback whenever $\Lambda$ is a faithful functor.

Another important instance of a subpullback is the pullback $H_0 \star K_1$ of faithful functors $A : H_0 \to H_1$ and $\Phi_1 : K_1 \to H_1$ in diagram (5). It is clear that the restrictions $H_0 : H_0 \star K_1 \to H_0$ and $\Pi_1 : H_0 \star K_1 \to K_1$ of the
two projections to $H_0 \times K_1 \subseteq H_0 \times K_1$ are faithful and that the diagram (5) above is a subpullback.

Another instance of a subpullback occurs when $\Phi_0$ is a strong embedding of the concrete category $(K_0, \Psi)$ into the concrete category $(H_0, A)$ carried by the set functor $\Phi_1$ as in diagram (1).

All proofs of simultaneous representability will be based on the following simple claim.

**Lemma 3.1.** Assume that $(H_i, A_i)$ and $(K_i, B_i)$ are concrete categories for $i \in \{0, 1\}$. Let the left square of the diagram below be a subpullback and the right square be a strong embedding:

\[
\begin{array}{ccc}
K_0 & \xrightarrow{B_0} & \text{Set} & \xrightarrow{B_1} & K_1 \\
\downarrow F & & \downarrow \Phi_1 & & \\
H_0 & \xrightarrow{A_0} & \text{Set} & \xrightarrow{A_1} & H_1 \\
\end{array}
\]

Then the second natural projection functor $K_0 \times K_1 \to K_1$ has a simultaneous representation in $H_0 \times H_1 \to H_1$.

**Proof.** Define $\Phi_0 : K_0 \times K_1 \to H_0 \times H_1$ by $\Phi_0(k_0, k_1) = (\Gamma(k_0), \Phi_1(k_1))$. For any $(k_0, k_1) \in \text{mor}(K_0 \times K_1)$ we have $B_0(k_0) = B_1(k_1)$ and hence also $A_0 \Gamma(k_0) = F B_0(k_0) = F B_1(k_1) = A_1 \Phi_1(k_1)$. Therefore $\Phi_0(k_0, k_1) \in \text{mor}(H_0 \times H_1)$, and the two second projections form a commutative square with $\Phi_0$ and $\Phi_1$.

We note that the functors $A_1, B_1$ are all faithful by the definition of concrete category. $F$ is faithful and $\Phi_1$ is both full and faithful by the definition of strong embedding. Since $F$ is faithful, it follows from the definition of subpullback that $\Gamma$ is also faithful. And thus $\Phi_0$ is faithful because both $\Gamma$ and $\Phi_1$ are.

It suffices now to show that $\Phi_0$ is full. Let $(h_0, h_1) \in \text{mor}(H_0 \times H_1)$. Since $A_0(h_0) = A_1(h_1)$ and $\Phi_1$ is full, there exists some $k_1 \in K_1$ with $h_1 = \Phi_1(k_1)$. Consequently, $F B_1(k_1) = A_0(h_0)$. Using the subpullback half of the diagram, we obtain a $k_0 \in K_0$ such that $\Gamma(k_0) = h_0$. But then $\Phi_0(k_0, k_1) = (h_0, h_1)$ as required. \qed

4. **Unary expansions.** Throughout the remainder of the paper, the letter $V$ will denote an arbitrarily chosen universal unary variety.

Let $V$ be any unary variety, and let $K(V)$ denote its monoid of (unary) polynomials. We let $Z(V) \subseteq K(V)$ be the set of all constants, that is, the set of all polynomials for which the identity $z(x) = z(y)$ holds in all algebras of $V$. $Z(V)$ is void if and only if $V$ is a regular variety. We let $z$ denote both a member of $Z(V)$ and its constant value. We set $L(V) = K(V) \setminus Z(V)$.

Let $\Psi$ be a functor from the small category $k_0$ to the small category $k_1$. We wish to consider simultaneous representations $(\xi)$ of $\Psi$ in the “reduction"
functor $A : V * U \rightarrow U$. There are three cases to consider. They are (a) the case $K(V) = \{1\}$, (b) the case $L(V) \neq \{1\}$ which we call the nondegenerate case, and (c) the case $L(V) = \{1\}$ and $Z(V) \neq \emptyset$ which we call the essentially nullary case.

(a) In this case $K(V) = \{1\}$, and $V$ is just the category of all sets and mappings. Thus $\Psi : k_0 \rightarrow k_1$ has a simultaneous representation $(S)$ in $A : V * U \rightarrow U$ if and only if $\Psi$ is a full embedding.

(b) In the nondegenerate case, by [4], any functor $\Psi$ with a simultaneous representation $(S)$ in $A : V * U \rightarrow U$ must be regularly faithful. Corollary 4.4 below will establish the converse implication.

(c) In the remaining, essentially nullary case, the morphisms of $V$ are the maps preserving the value of each constant $z \in Z(V)$. The existence of a simultaneous representation $(S)$ of $\Psi$ in $A$ imposes the following condition on the functor $\Psi$:

$$(p) \text{ If } a_0 \in k_0(a, b) \text{ and } \beta_1 \in k_1(\Psi(b), \Psi(c)), \text{ then } \beta_1 = \Psi(\beta_0) \text{ for some } \beta_0 \in k_0(b, c) \text{ exactly when } \beta_0 = \Psi(a_0) = \Psi(\gamma_0) \text{ for some } \gamma_0 \in k_0(a, c).$$

Indeed, suppose $\beta_1 = \Psi(a_0) = \Psi(\gamma_0)$. From the commutativity of $(S)$ it follows that $\Phi_1(\beta_1) = A\Phi_0(a_0) = A\Phi_0(\gamma_0)$. Now using $z_a$ to denote the value of the constant $z \in Z(V)$ in the algebra $\Phi_0(a)$, etc., we have $\Phi_0(a_0)(z_a) = z_b$ and $\Phi_0(\gamma_0)(z_a) = z_c$. So we conclude that $\Phi_1(\beta_1)(z_b) = z_c$. Since $z$ is arbitrary, we conclude that $\Phi_1(\beta_1)$ is also a $V$-homomorphism, and thus $\Phi_1(\beta_1) \in \text{mor}(V * U)$. Therefore by the fullness of $\Phi_0$ we have some $\beta_0 \in \text{mor}(k_0)$ with $\Phi_0(\beta_0) = \Phi_1(\beta_1)$ which we noted is in $\text{mor}(V * U)$. Since $A$ is the “reduction” functor from $V * U$ to $U$, for any $\delta \in \text{mor}(V * U)$ we have $A(\delta) = \delta$. In particular, $A(\Phi_1(\beta_1)) = \Phi_1(\beta_1)$. Now we have $\Phi_1(\beta_1) = A\Phi_0(\beta_0) = \Phi_1(\beta_1)$, and, finally, $\beta_1 = \Psi(\beta_0)$ because $\Phi_1$ is faithful.

Theorem 4.6 below shows that any $\Psi$ satisfying $(p)$ has a simultaneous representation $(S)$ in any $A : V * U \rightarrow U$ for which $V$ is an essentially nullary variety.

A functor $F : \text{Set} \rightarrow \text{Set}$ is called linear if there exist sets $A \neq \emptyset$ and $B$ such that, for any set $X$, $F(X) = (X \times A) \sqcup B$ is the sum (i.e. the disjoint union) of $X \times A$ with $B$ and the image $F(f)$ of any mapping $f : X \rightarrow X'$ is the sum $F(f) = (f \times \text{id}_A) \sqcup \text{id}_B$.

**Lemma 4.1.** For any universal unary variety $U$ and any infinite cardinal $\gamma$ there is a strong embedding of $\text{Alg}(1, 1)$ into $U$ carried by a linear functor $F(X) = (X \times A) \sqcup B$ with $\text{card}(A), \text{card}(B) \geq \gamma$.

**Proof.** For every universal unary variety $U$ there exists a strong embedding of $\text{Alg}(1, 1)$ into $U$ carried by a linear functor (see [7]). Since any composite of linear functors is linear, it suffices then to find a strong embedding $\Sigma : \text{Alg}(1, 1) \rightarrow \text{Alg}(1, 1)$ carried by a linear functor $F$ satisfying
the cardinality requirements. The functor $\Sigma$ will be a composite of three strong embeddings given below.

First, the linear functor $F_0(X) = (X \times 3) \sqcup 2$ carries a functor $\Sigma_0 : \text{Alg}(1, 1) \to \text{Alg}(1, 1)$ defined by $\Sigma_0(X, \alpha, \beta) = (F_0(X), \varphi, \psi)$, where $\varphi(x, i) = (x, i + 1)$ for $(x, i) \in X \times 3$ with the addition modulo three, $\varphi(0) = 1$, $\varphi(1) = 0$, and $\psi(x, 1) = (\alpha(x), 0)$, $\psi(x, 2) = (\beta(x), 0)$, $\psi(x, 0) = (x, 0)$, $\psi(0) = \psi(1) = 0$.

It is easy to check that $\Sigma_0$ is a faithful functor. So we only need to prove its fullness. Let $g$ be a homomorphism from $\Sigma_0(X, \alpha, \beta)$ to $\Sigma_0(X', \alpha, \beta')$. Orbits of the permutation $\varphi$ on $X \times 3$ and 2 have different prime orders, and hence these subsets are preserved by any homomorphism. Furthermore, $X \times \{0\}$ is the set of all fixed points of $\psi$ in $X \times 3$; so there is a mapping $f$ such that $g(x, 0) = (f(x), 0)$. Using $\varphi$, we obtain $g(x, i) = (f(x), i)$ for all $i \in 3$, and $\psi$ on $X \times \{1, 2\}$ guarantees that $f$ is a homomorphism. Similarly we find that $g$ is the identity on $\{0, 1\}$ and so $g = \Sigma_0(f)$.

Secondly, from [8] and [6, p. 180] we obtain the existence of a connected, endomorphism-free algebra $C \in \text{Alg}(1, 1)$ defined on the cardinal $\gamma$. (The graph of a unary algebra has $(a, b)$ as an edge iff $b = f(a)$ for some operation $f$. Recall that a unary algebra is connected iff this graph is connected.) For any $A = (Y, \varphi, \psi) \in \text{Alg}(1, 1)$ define $\Sigma_1(A) \in \text{Alg}(1, 1, 1)$ on the set $Y \times \gamma$ as follows.

We define $\varphi'$ and $\psi'$ on $A$ by $\varphi'(y, c) = (y, \varphi(c))$ and $\psi'(y, c) = (y, \psi(c))$. We define a third operation $\chi$ by $\chi(y, 0) = (\varphi(y), 0)$, $\chi(y, 1) = (\psi(y), 1)$ and as the identity elsewhere. With $\Sigma_1(f) = f \times \text{id}_\gamma$, it is easy to check that $\Sigma_1$ is a faithful functor.

To prove that $\Sigma_1$ is full, we first note that any homomorphism $g$ between $\Sigma_1$-images of algebras from $\text{Alg}(1, 1)$ maps any set of the form $\{y\} \times \gamma$ into a set of the form $\{z\} \times \gamma$. Since $C$ has only the trivial endomorphism, there exists a mapping $f$ such that $g = f \times \text{id}_\gamma$. But then $f$ must be compatible with the operations $\varphi$ and $\psi$ of the original algebras via the operation $\chi$. Altogether, $\Sigma_1$ is a strong embedding carried by the linear functor $F_1(Y) = Y \times \gamma$.

For the third strong embedding $\Sigma_2$ of $\text{Alg}(1, 1, 1)$ into $\text{Alg}(1, 1)$ carried by the linear functor $F_2(Z) = Z \times 5$, we refer the reader to [7].

The composite strong embedding $\Sigma = \Sigma_2 \circ \Sigma_1 \circ \Sigma_0$ is, therefore, carried by the set functor $F = F_2 \circ F_1 \circ F_0$ with $F(X) = X \times (\gamma \times 15) \sqcup (\gamma \times 10)$ as required.

Now we turn to the case (b) of a nondegenerate unary variety $V$.

Let $F_\nu(Y) \in V$ denote the algebra freely generated by a nonvoid set $Y$, and let $\Theta$ be its least congruence collapsing the set $\{z(y) \mid y \in Y, z \in Z(V)\}$. Since the latter set is a subalgebra of $F_\nu(Y)$, all other classes of $\Theta$ are trivial.
So we may assume that \((L(V) \times Y) \cup \{z\}\) is the underlying set of the quotient algebra \(Q_Y = F_Y(Y)/\Theta\) when \(V\) is not regular. For any regular variety \(V\), clearly \(Q_Y = F_Y(Y) = K(V) \times Y\).

Let Inv denote the variety of all mono- unary algebras \((X, \delta)\) defined by the identity \(\delta^2(x) = x\).

**Lemma 4.2.** If \(V\) is a nondegenerate unary variety and if \(\alpha, \beta \geq \text{card}(L(V))\) are infinite cardinals, then there exists a subpullback

\[
\begin{array}{ccc}
\text{Inv} & \longrightarrow & \text{Set} \\
\downarrow & & \downarrow F \\
V & \longrightarrow & \text{Set}
\end{array}
\]

in which \(F\) is a linear functor given by \(F(X) = (X \times A) \cup B\) with \(\text{card}(A) \geq \alpha\) and \(\text{card}(B) \geq \beta\).

**Proof.** To define the functor \(F\), set \(A = \alpha \times L(V)\) and \(B = \beta \times L(V) \cup \{0\}\).

For any \((X, \delta) \in \text{Inv}\), first define an auxiliary algebra \(B(X)\) on \(F(X)\) by \(k(x, a, l) = (x, a, kl)\) and \(k(b, l) = (b, kl)\) whenever \(kl \in L(V)\), and by \(k(z) = 0\) in all other cases. The algebra \(B(X)\) is isomorphic to \(Q_Y \in V\) with \(Y = X \times \alpha \cup \beta\), and \(F(f)\) is a homomorphism of \(B(X)\) into \(B(X')\) for any mapping \(f : X \rightarrow X'\).

Next we define \(\sigma : F(X) \rightarrow F(X)\) as \(\sigma(x, a, 1) = (\delta(x), a, 1)\) on \(X \times \alpha \times \{1\}\) and as an identity elsewhere; clearly \(\sigma^2\) is the identity on \(F(X)\).

Finally, for each \(k \in K(V)\) set \(k^* = \sigma \circ k \circ \sigma\) and define

\[\Gamma(X, \delta) = (F(X), \{k^* \mid k \in K\}).\]

Since \(\sigma\) is an involution, we obtain

\[\sigma \circ k^* = k \circ \sigma\quad \text{and}\quad \sigma \circ k = k^* \circ \sigma.\]

So \(\sigma\) is an isomorphism between the algebras \(B(X)\) and \(\Gamma(X, \delta)\). This shows that \(\Gamma(X, \delta) \in V\).

For \(k \in Z(V) \cup \{1\}\) we have \(k = k^*\). For \(l \in L(V) \setminus \{1\}\) we find that \(l\) and \(l^*\) coincide on \(F(X) \setminus (X \times \alpha \times \{1\})\). For any such \(l\) and for all \((x, a, 1) \in X \times \alpha \times \{1\}\),

\[
l^*F(f)(x, a, 1) = \sigma l \sigma f(x, a, 1) = \sigma l (\delta f(x, a, 1)) = \sigma l (\delta f(x, a, l)),
\]

\[
F(f)l^*(x, a, 1) = F(f) \sigma l \sigma(x, a, 1) = F(f) \sigma l (\delta(x, a, 1)) = F(f) \sigma l (\delta(x, a, l)).
\]

Since \(V\) is nondegenerate, there is at least one such \(l \in L(V) \setminus \{1\}\). Thus \(F(f)\) is a morphism of \(\Gamma(X, \delta)\) exactly when \(f\) is a morphism of \(X\). Consequently, the square is a subpullback as claimed. \(\blacksquare\)
Theorem 4.3. The reduction functor $\text{Inv} \star \text{Alg}(1,1) \to \text{Alg}(1,1)$ has a simultaneous representation in $V \star U \to U$ for any nondegenerate unary variety $V$ and any universal unary variety $U$.

Proof. Apply 4.1, 4.2 and 3.1. ■

In [4] it is shown that any regularly faithful small functor has a simultaneous representation in $\text{Inv} \star \text{Alg}(1,1) \to \text{Alg}(1,1)$; in view of this, the result below follows immediately from Theorem 4.3.

Corollary 4.4. Let $U$ and $V$ be unary varieties, $U$ universal and $V$ nondegenerate. A small functor $\Psi : k_0 \to k_1$ has a simultaneous representation in $V \star U \to U$ if and only if $\Psi$ is regularly faithful.

The remainder of the section deals with the case of an essentially nullary variety $V$.

Lemma 4.5. If a small functor $\Psi : k_0 \to k_1$ satisfies the condition (p), then it has a simultaneous representation $(\Phi_0, \Phi_1)$ in $\text{Alg}(0) \star \text{Alg}(\Delta) \to \text{Alg}(\Delta)$ for some type $\Delta$.

Proof. First we construct a full embedding $\Phi_1 : k_1 \to \text{Alg}(\Delta)$. This embedding will be a composite of two contravariant functors.

According to pp. 50–51 of [6], for a suitable unary type $\Delta'$ there exists a full embedding $I_0 : k_1^{\text{opp}} \to \text{Alg}(\Delta')$ carried by $M \sqcup 2$, where $M$ is the Cayley–MacLane representation of $k_1^{\text{opp}}$. Equivalently, the underlying set functor of $I_0$ is the sum $D \sqcup \overline{2}$ of the dual Cayley–MacLane representation $D$ of $k_1$ and the contravariant constant functor $\overline{2}$ with a two-element value.

By Lemma 4.1 and from [7] it now follows that there are sets $H$ and $E$ with $\text{obj}(k_0) \subseteq H$ and a full faithful functor $\Gamma : k_1 \to \text{Alg}(1,1)$ such that the algebra $\Gamma(k) = (F(k), \varphi_k, \psi_k)$ is carried by the linear contravariant functor given by $F(k) = (D(k) \times H) \sqcup E$. More precisely, since $D(k) = \bigsqcup \{k_1(k, l) \mid l \in \text{obj}(k_1)\}$ and $D(\kappa)(\lambda) = \lambda \circ \kappa$, for any $\kappa \in k_1(k, k')$, the mapping $F(\kappa) : F(k') \to F(k)$ is given by $F(\kappa)(\lambda, h) = (\lambda \circ \kappa, h)$ for $(\lambda, h) \in D(k') \times H$ and $F(\kappa)(e) = e$ for all $e \in E$.

Let $P^- = \text{hom}(-, 2)$ denote the contravariant power set functor. Set $G = P^- \circ F$. The composite $G : k_1 \to \text{Set}$ is then a covariant faithful functor. To define $\Phi_1(k)$ for $k \in \text{obj}(k_1)$, we assemble the following set $Q_k$ of operations on the set $G(k)$:

the two unary operations $P^-(\varphi_k)$ and $P^-(\psi_k)$,

a unary constant $\zeta$ with the value $\emptyset$,

a unary operation $\gamma$ of complementation,

a $\lambda$-ary union $\nu$, where $\lambda \geq \text{card}(G(k))$ for all $k \in \text{obj}(k_1)$.

We set $\Phi_1(k) = (G(k), Q_k) \in \text{Alg}(\Delta)$ for all $k \in \text{obj}(k_1)$, where $\Delta$ is the similarity type $(1, 1, 1, 1, \lambda)$. We also set $\Phi_1(\kappa) = G(\kappa)$ for all $\kappa \in \text{mor}(k_1)$. 
Since $\Phi_1(k) = G(k)$ is a homomorphism for every morphism $\kappa$ of $k_1$, $\Phi_1$ is a well-defined covariant faithful functor.

To see that $\Phi_1$ is full, we observe that the three set-theoretical operations ensure that any homomorphism $g : \Phi_1(k) \to \Phi_1(k')$ is a complete Boolean algebra homomorphism of the respective power sets. Hence $g = P^-(f)$ for some $f : \Gamma(k') \to \Gamma(k)$. The faithfulness of $P^-$ implies that $P^- (\varphi_k \circ g) = g \circ P^- (\varphi_k)$ only when $f \circ \varphi_k = \varphi_k \circ f$. From this and from a similar observation about $\psi$ it follows that the mapping $f$ is a homomorphism and then, because the functor $\Gamma$ is full, that $g = G(k)$ for some $\kappa \in k_1(k,k')$. This shows that $\Phi_1$ is full.

To define the other component $\Phi_0 : k_0 \to \text{Alg}(0) \ast \text{Alg}(\Delta)$ of the simultaneous representation in question, we first recall that $\text{obj}(k_0) \subseteq H$. Using this fact, we expand each algebra $\Phi_1(\Psi(a))$ with $a \in \text{obj}(k_0)$ by a nullary operation $z_a \in G(\Psi(k))$ defined by

$$z_a = \{(\Psi(\chi), b) \mid b \in \text{obj}(k_0) \text{ and } \chi \in k_0(a,b)\},$$

and define $\Phi_0(a) = (G(\Psi(a)), Q_{\Psi(a)} \cup \{z_a\})$ for every $a \in \text{obj}(k_0)$. For every $\alpha \in k_0(a,b)$ we set $\Phi_0(\alpha) = G(\Psi(\alpha))$.

Let $\alpha \in k_0(a,b)$ and $(\lambda, h) \in D(b) \times H$. Then $F(\Psi(\alpha))(\lambda, h) = (\lambda \circ \Psi(\alpha), h) \in z_a$ exactly when $\lambda \circ \Psi(\alpha) = \Psi(\gamma)$ for some $\gamma \in k_0(a,b)$, and also $(\lambda, h) \in z_b$ iff $\lambda = \Psi(\beta)$ for some $\beta \in k_0(b,h)$. Using (p), we conclude that $G(\Psi(\alpha))(z_a) = z_b$, so that $\Phi_0$ is a well-defined faithful functor.

To prove that $\Phi_0$ is a full functor, choose $a_1 \in k_1(\Psi(a),\Psi(b))$ arbitrarily and note that $(\text{id}_\Psi(b), b) \in z_b$. If $G(a_1)(z_a) = z_b$ then $(a_1,b) = F(\alpha_1)(\Psi(\text{id}_b), b) \in z_a$, and $a_1 = \Psi(\alpha)$ for some $\alpha \in k_0(a,b)$ follows from the definition of $z_a$.

**Theorem 4.6.** If a variety $V$ is essentially nullary and if $U$ is a universal unary variety, then a small functor $\Psi : k_0 \to k_1$ has a simultaneous representation in $V \ast U \to U$ if and only if it satisfies (p).

**Proof.** The necessity of (p) has been established earlier. So we suppose now that (p) holds.

Let $\Delta$ be an arbitrary similarity type. According to [7], there is a strong embedding $\Phi_1 : \text{Alg}(\Delta) \to U$ carried by a set functor $F(X) = (X^\kappa \times A) \sqcup B$ with a nonzero cardinal $\kappa$ and with some $A \neq \emptyset$ and $B$.

Now we wish to define a faithful functor $\Gamma : \text{Alg}(0) \to V$ which together with $F$ and the forgetful functors will form a subpullback. Suppose $(X,c) \in \text{Alg}(0)$. Then we set $\Gamma(X,c) = (F(X), Z) \in V$ where each $z \in Z$ is to have the value const$_{X \times \{a\}}(z) \in X^\kappa \times A$.

Now by Lemma 3.1 we have a simultaneous representation of the reduction functor $\text{Alg}(0) \ast \text{Alg}(\Delta) \to \text{Alg}(\Delta)$ in the reduction functor $V \ast U \to U$. The theorem follows from Lemma 4.5.
5. Regular non-unary expansions. In this section we prove the following result.

**Theorem 5.1.** For any regular non-unary variety $V$ and a unary universal variety $U$, a small functor $\Psi : k_0 \to k_1$ has a simultaneous representation in $V \star U \to U$ if and only if $\Psi$ is regularly faithful.

Simultaneous representability of a functor $\Psi$ in any functor between categories of algebras requires that $\Psi$ be regularly faithful; on the other hand, any small regularly faithful $\Psi$ is simultaneously representable in $\text{Alg}(1) \star \text{Alg}(2) \to \text{Alg}(2)$ (see [4]). Hence it suffices to show that, for any pair of varieties $U$ and $V$ specified by Theorem 5.1, $\text{Alg}(1) \star \text{Alg}(2) \to \text{Alg}(2)$ has a simultaneous representation in $V \star U \to U$.

Since any regular non-unary variety $V$ contains the variety $\text{Sl}$ of join-semilattices or the variety $\text{Sl}_0$ of join-semilattices with zero ([1], [3], [5]), we only need to show that, for any universal non-unary variety $U$, the functor $\text{Alg}(1) \star \text{Alg}(2) \to \text{Alg}(2)$ has a simultaneous representation in $\text{Sl} \star U \to U$ and a simultaneous representation in $\text{Sl}_0 \star U \to U$. Since the composite of simultaneous representations is a simultaneous representation, Theorem 5.1 follows directly from the two lemmata below.

**Lemma 5.2.** The reduction functor $\text{Alg}(1) \star \text{Alg}(2) \to \text{Alg}(2)$ has a simultaneous representation in $\text{Sl} \star \text{Alg}(2) \to \text{Alg}(2)$.

**Lemma 5.3.** The reduction functor $\text{Sl} \star \text{Alg}(2) \to \text{Alg}(2)$ has a simultaneous representation in $\text{Sl} \star U \to U$ and in $\text{Sl}_0 \star U \to U$ for any unary universal variety $U$.

**Proof of Lemma 5.2.** Once again, an application of Lemma 3.1 will assemble the simultaneous representation.

Let $P : \text{Set} \to \text{Set}$ be the functor assigning the set $P(X)$ of all finite $s \subseteq X$ to any set $X$ and, to any $f : X \to X'$, the mapping $P(f) : P(X) \to P(X')$ given by $[P(f)](s) = \{f(x) \mid x \in s\} = f^+(s)$ for every $s \in P(X)$.

For the joint carrier $F : \text{Set} \to \text{Set}$, we select the factorfunctor of $P \times 4$ modulo the equivalence $\Theta$ which collapses $(s, 2)$ and $(s, 3)$ for each $s \in P(X)$ with card$(s) \leq 1$, and nothing else. It is easy to see that $F$ is, indeed, a well-defined set functor. To simplify the notation, we replace $(\emptyset, i)$ by $i$, and $(s, i)$ by $s_i$ for all $s \neq \emptyset$ and all $i \in 4$; also, $s_2$ shall denote the class $\{(s, 2), (s, 3)\}$ of $\Theta$. With these conventions in mind, for each $(X, \cdot) \in \text{Alg}(2)$ we define a binary operation $\cdot$ of $\text{Alg}(1) \times (\cdot) = (F(X), \cdot)$ as follows,

- (1) $0 \cdot 0 = 1, 2 \cdot 2 = 0$, and $i \cdot j = 2$ for all other $i, j \in \{0, 1, 2\}$;
- (2) $s_0 \cdot 0 = 0 \cdot s_0 = 0$ and $s_i \cdot 0 = 0 \cdot s_i = 1$ for $i \in \{1, 2, 3\}$ and $s \neq \emptyset$;
- (3) if $s \neq \emptyset$ then $s_i \cdot 2 = 2 \cdot s_i = s_2$ for all $i \in 4$;

Theorem 5.2, which follows directly from the two lemmata below.

- (i) $s_0 \cdot 1 = 1 \cdot s_0 = s_1$ and $s_1 \cdot 1 = 1 \cdot s_1 = s_2$;
- $s_2 \cdot 1 = 1 \cdot s_2 = s_3$ and $s_3 \cdot 1 = 1 \cdot s_3 = s_2$ when card$(s) > 1$;
- $s_2 \cdot 1 = 1 \cdot s_2 = s_2$ when card$(s) = 1$;
(4) $s_0 \ast t_0 = (s \cup t)_0$ when $s \neq \emptyset$ and $t \neq \emptyset$;
(5) $s_1 \ast t_1 = \{x \cdot y \mid x \in s, y \in t\}_2$ when $s \neq \emptyset$ and $t \neq \emptyset$;
(6) $u \ast v = 2$ for all other $u, v \in F(X)$.

It is routine to verify that $F(f)$ is a homomorphism whenever $f$ is; thus $\Phi_2$ is a well-defined faithful functor.

To show that $\Phi_1$ is full, assume that $g : (F(X), *) \to (F(X'), *)$ is a homomorphism in Alg(2).

Observe first that, for the unary term $t(z) = z \ast z$, the equation $t^1(z) = z$ is solvable just when $z \in \{0, 1, 2\}$; the remainder of (1) then implies that $g(i) = i$ for $i \in \{0, 1, 2\}$.

Secondly, from the fact that (2) fully describes solution sets of equations $z \ast 0 = 0$ and $z \ast 1 = 1$ we conclude that $g(s_0) \in (P(X') \setminus \{\emptyset\}) \times \{0\}$ and $g(s_i) \in (P(X') \setminus \{\emptyset\}) \times \{1, 2, 3\}$ for any nonvoid $s \in P(X)$ and for all $i \in \{1, 2, 3\}$. In particular, $g$ determines a unique mapping $h : P(X) \setminus \{\emptyset\} \to P(X') \setminus \{\emptyset\}$ such that $g(s_0) = h(s)_0$.

Next, from (3) it follows that $g(s_i) = h(s_i)$, for all $i \in 4$, and also that card($h(s)$) = 1 whenever card($s$) = 1. This, in turn, yields the existence of a unique mapping $f : X \to X'$ such that $h(\{x\}) = \{f(x)\}$ for all $x \in X$. Using (4), we then find that $h = P(f)$, and hence also that $g = F(f)$.

Finally, (5) ensures that $f : (X, \cdot) \to (X', \cdot)$ is a morphism in Alg(2), so that $\Phi_1$ is a full embedding as claimed.

For any $(X, \alpha) \in \text{Alg}(1)$ we need to define a join semilattice $\Gamma(X, \alpha) = (F(X), \lor)$ in such a way that $\Gamma$, the two underlying set functors $\text{Alg}(2) \to \text{Set}$, and the set functor $P$ constitute a subpullback.

First we define a binary relation $\leq$ on $F(X)$ as follows:

(7) $s_i \leq t_j$ whenever $s \subseteq t_i$ and $0 < i \leq j \leq 4$ or $i = j = 0$;
(8) $s_0 \leq t_j$ whenever $\alpha^+(s) \subseteq t_j$ for all $j \in \{1, 2, 3\}$.

The relation $\leq$ is, obviously, a partial order on $F(X)$. To see that $\leq$ defines a join semilattice, we first note that, for $i, j > 0$, the least upper bound $s_i \lor t_j$ of $\{s_i, t_j\}$ is just $s \lor t_{k}$ with $k = \max\{i, j\}$. Clearly $s_0, t_0 \leq s \lor t_0$; if $s_0, t_0 \leq u_j$ for some $j > 0$, then $\alpha^+(s \lor t) = \alpha^+(s) \lor \alpha^+(t) \leq u_j$, that is, $s \lor t_0 \leq u_j$. Whence $s_0 \lor t_0 = (s \lor t)_0$. Finally, $(\alpha^+(s) \lor t)_0$ is an upper bound of $\{s_0, t_j\}$ whenever $j > 0$; if $u_k$ is any upper bound of the latter pair then $k \geq j$ and $\alpha^+(s) \lor t_j \leq u_k$. Hence $s_0 \lor t_j = (\alpha^+(s) \lor t)_j$. The partial order $\leq$ thus defines a semilattice $(F(X), \lor)$.

Since $P(f) = f^+$ preserves unions, the mapping $F(f)$ is a semilattice homomorphism if and only if $F(f)(s_0 \lor t_j) = (f^+(\alpha^+(s) \lor t)_j = (f^+(\alpha^+(s) \lor f^+(t))_j$ coincides with $F(f)(s_0) \lor F(f)(t_j) = (f^+(\alpha^+(s)) \lor f^+(t))_j = (\alpha^+(f^+(s) \lor f^+(t))_j$ for $j > 0$ and all appropriate $s$ and $t$. Since $P \times \{0, 1\}$ is a faithful
subfunctor of $F$, this occurs exactly when $f \circ \alpha = \alpha \circ f$. Consequently, the functor $\Gamma$ completes the subpullback needed to apply Lemma 3.1. \hfill \blacksquare 

Proof of Lemma 5.3. For any $(X, *) \in \text{Alg}(2)$, let $\Psi_t = (X^2, \pi_0, \pi_1, g)$ \in $\text{Alg}(1, 1, 1)$, where $\pi_j : X^2 \rightarrow X^2$ is defined by $\pi_j(x_0, x_1) = (x_j, x_j)$ for $j \in 2$, and $g(x, y) = (x * y, x * y)$ for all $(x, y) \in X^2$. Since a mapping $g : X^2 \rightarrow (X')^2$ is compatible with both $\pi_j$ if and only if $g = Q_2(f) = f \times f$ for some $f : X \rightarrow X'$, it follows that $\Psi_t : \text{Alg}(2) \rightarrow \text{Alg}(1, 1, 1)$ is a strong embedding carried by the cartesian square functor $Q_2$.

Let $U$ be an arbitrary universal unary variety.

Since there is a strong embedding $\text{Alg}(1, 1, 1) \rightarrow \text{Alg}(1, 1)$ carried by a linear functor [7], an application of Lemma 4.1 yields a strong embedding $\Phi_t : \text{Alg}(2) \rightarrow U$ whose carrier is a set functor given by $F(X) = X^2 \times A \sqcup B$ with card($B$) $\geq 2$.

We now define functors $\Gamma : \text{Sl} \rightarrow \text{Sl}$ and $\Gamma'_0 : \text{Sl} \rightarrow \text{Sl}_0$ which, together with $F$ and the underlying set functors, will form the subpullbacks needed to apply Lemma 3.1.

For any semilattice $(X, \lor) \in \text{Sl}$, let $(X^2, \lor)$ denote its square. Select a well-ordering $\leq$ of $B$; let $b_0 \in B$ denote the least element of $B$ and $b_1 \in B$ its successor. Extend $\leq$ to a partial order on $F(X) = X^2 \times A \sqcup B$ by requiring that:

(a) $b_0 \leq (x_0, x_1, a)$ $\leq$ $b_1$ for all $(x_0, x_1, a) \in X^2 \times A$, and
(b) $(x_0, x_1, a) \leq (x'_0, x'_1, a')$ if and only if $a = a'$ and $(x_0, x_1) \leq (x'_0, x'_1)$ in $(X^2, \lor)$.

Clearly, the partial order $(F(X), \leq)$ determines a semilattice $\Gamma(X, \lor) = (F(X), +)$ with zero $b_0$; for each $a \in A$, it contains a subsemilattice $X^2 \times \{a\}$ isomorphic to $(X^2, \lor)$, while $(x_0, x_1, a) + (x'_0, x'_1, a') = b_1$ whenever $a \neq a'$. Finally, we set $\Gamma_0(X, \lor) = (F(X), +, b_0)$.

It is easy to see that a mapping $f : (X, \lor) \rightarrow (X', \lor)$ is a semilattice homomorphism if and only if $F(f) : (F(X), +) \rightarrow (F(X'), +)$ is an Sl-morphism and only if $F(f) : (F(X), +, b_0) \rightarrow (F(X'), +, b_0)$ is an $\text{Sl}_0$-morphism. Whence the respective subpullbacks required by Lemma 3.1 exist, and the proof is complete. \hfill \blacksquare 

We are unaware of any general result about non-unary non-regular expansions of a universal variety $U$ of any type. In particular, does every small regularly faithful functor have a simultaneous representation in $\text{Grp} \ast U \rightarrow U$ for the variety $\text{Grp}$ of groups and for any universal (unary) variety $U$?

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