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ON PETTIS INTEGRALS WITH SEPARABLE RANGE

BY

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1. Introduction. Several techniques have been developed to study Pettis integrability of weakly measurable functions with values in Banach spaces. As shown by M. Talagrand [Ta], it is fruitful to regard a weakly measurable mapping as a pointwise compact set of measurable functions—its Pettis integrability is then a purely measure-theoretic question of an appropriate continuity of a measure. On the other hand, properties of weakly measurable functions can be translated into the language of topological measure theory by means of weak Baire measures on Banach spaces. This approach, originated by G. A. Edgar [E1, E2], was remarkably developed by M. Talagrand.

Following this idea, we show that the Pettis Integral Property of a Banach space E, together with the requirement of separability of E-valued Pettis integrals, is equivalent to the fact that every weak Baire measure on E is, in a certain weak sense, concentrated on a separable subspace. We base on a lemma which is a version of Talagrand's Lemma 5-1-2 from [Ta]. Our lemma easily yields a sequential completeness of the spaces of Grothendieck measures, a related result proved by Pallarés–Vera [PV]. We also present two results on Pettis integrability in the spaces of continuous functions.

2. Preliminaries. As regards terminology concerning measurability in Banach spaces, we follow Edgar [E1, E2] and Talagrand [Ta]. These papers are also excellent sources for the facts we mention below without proof. We now briefly recall the basic concepts.

Let (Ω, Σ, μ) be a finite measure space and let $f : \Omega \to E$ be a function with values in a Banach space E. Then f is *weakly measurable* provided x^*f is measurable for every $x^* \in E^*$. If moreover $|x^*f| \le K$ almost everywhere for some constant K and every $x^* \in E_1^*$ then f is said to be *scalarly bounded*. (We denote by F_r the ball with center at the origin and radius r in a Banach space F.)

A weakly measurable function $f : \Omega \to E$ is *Pettis integrable* if for every $A \in \Sigma$ there exists $\nu(A) \in E$ such that $x^*\nu(A) = \int_A x^*f\,d\mu$, for all $x^* \in E^*$. In that case the vector-valued measure $\nu : \Sigma \to E$ is called an *indefinite*

Pettis integral of f. As is explained in 3-3 of [Ta], it is a little loss of generality to consider only scalarly bounded functions.

A Banach space E is said to have the *Pettis Integral Property* (abbreviated to PIP) if every E-valued scalarly bounded function f (defined on an arbitrary finite measure space) is Pettis integrable.

For a Banach space E, $\mathcal{B}a(E)$ denotes the Baire σ-algebra for the weak topology on E. A *weak Baire measure* on E is a finite measure defined on $\mathcal{B}a(E)$. A Banach space E is *measurable-compact* if every weak Baire measure on E is τ-additive (see [E1] for the definition of τ-additivity). This is equivalent to saying that for every such measure μ there is a separable subspace F of E with $\mu^*(F) = \mu(E)$ (via Tortrat's theorem, cf. [Ta], 2-3-2).

If $f : \Omega \to E$ is a scalarly bounded function then the image measure $f(\mu)$ given by $f(\mu)(B) = \mu(f^{-1}(B))$ is a weak Baire measure on E. Now f is Pettis integrable if and only if the identity map $i_E : E \to E$ is Pettis integrable with respect to $f(\mu)$; this may be easily deduced from Musia/l [M1], Proposition 1, or 4-1-7 of Talagrand [Ta].

Talagrand obtained the following characterization of Pettis integrability in the language of weak Baire measure (see [Ta], 5-2-4).

Given a finite space (Ω, Σ, μ) and a scalarly bounded function $f : \Omega \to E$, f is Pettis integrable if and only if the weak Baire measure $f(\mu)$ is convexly τ-additive, i.e. for every family $\mathcal{H} \subseteq \mathcal{B}a(E)$ of closed convex sets with $\bigcap \mathcal{H} = \emptyset$ there is a countable subfamily $\mathcal{H}_0 \subseteq \mathcal{H}$ such that $f(\mu)(\bigcap \mathcal{H}_0) = 0$.

Thus a Banach space E has PIP if and only if it is convexly measure-compact, that is, if every weak Baire measure on E is convexly τ-additive.

Of some interest is the question for which spaces E every E-valued Pettis integral has a relatively compact or separable range (see Musia/l [M1, M2], Talagrand [T], and [Ta], 4-1-6 and 5-3-2). The latter property means that Pettis integrable functions can be weakly approximated by simple functions (Musia/l [M2], Theorem 3).

A celebrated example of a weakly measurable function with values in $\ell^\infty(\kappa)$ given by Fremlin–Talagrand [FT] shows that ℓ^∞ does not have PIP and that there is an $\ell^\infty(\kappa)$-valued Pettis integrable function with a nonseparable Pettis integral (where κ is an uncountable cardinal).

3. Pettis integrals with separable range. Let $f : (\Omega, \Sigma, \mu) \to E$ be a scalarly bounded function. Then $X(f) = \{x^* f : x^* \in E_1^*\}$ is a pointwise compact set of measurable functions which is bounded in L^∞-norm. Recall that f is Pettis integrable if and only if the canonical injection $X(f) \to L^1(\mu)$ is pointwise-to-weak continuous (Edgar [E2], Proposition 4.2). We shall denote the topology of pointwise convergence by τ_{p}.

The proof of the following lemma reproduces an argument from Talagrand [Ta], 5-1-2.

LEMMA. *Let (Ω, Σ, μ) be a measurable space and let C be an absolutely convex set of measurable functions which is τ_{p}-compact and bounded in L^∞. Then the following are equivalent*:

(i) *the map $\mu : C \to \mathbb{R}$, $\mu(g) = \int_\Omega g\,d\mu$, is τ_{p}-continuous*;

(ii) *there exists a countable $\Omega_0 \subseteq \Omega$ such that $\int_\Omega g\,d\mu = 0$ whenever $g \in C$ and $g_{|\Omega_0} = 0$.*

Proof. (i)⇒(ii). By τ_{p}-continuity, for every natural number n there exist a finite set $\Omega_0 \subseteq \Omega$ and $\delta_n > 0$ such that $|\int_\Omega g\,d\mu| \le 1/n$ whenever $g \in C$ and $|g_{|\Omega_n}| \le \delta_n$. Clearly $\bigcup_n \Omega_n$ is as desired.

(ii)⇒(i). Assume that (ii) holds but μ is not τ_{p}-continuous; it is then not continuous at 0. There is a $\delta > 0$ such that 0 is in the τ_{p}-closure of the set $Y = \{g \in C : \int_\Omega g\,d\mu \ge \delta\}$. For every pair (F, ε) where $\varepsilon > 0$ and F is a finite subset of Ω we put

$$U(F,\varepsilon) = \{g \in C : |g_{|F}| \le \varepsilon\} \quad \text{and} \quad C(F,\varepsilon) = U(F,\varepsilon) \cap Y\,.$$

Let $H(F,\varepsilon)$ be the closure of $C(F,\varepsilon)$ in the weak topology of $L^1(\mu)$. The sets H form a nested family of convex and weakly compact subsets in $L^1(\mu)$. Hence there is an $h \in \bigcap_{(F,\varepsilon)} H(F,\varepsilon)$; we have $\int_\Omega h\,d\mu \ge \delta$.

For a fixed pair (F,ε) there exists a sequence $(f_n) \subseteq C(F,\varepsilon)$ converging to h almost everywhere (since the latter set is convex, its weak closure coincides with the norm closure). Denote by $f(F,\varepsilon)$ any cluster point of (f_n) in (C, τ_{p}); clearly $f(F,\varepsilon) = h$ almost everywhere.

Now take Ω_0 as in (ii) and let f be a cluster point of the net $(f(F, 1/n) : F \subseteq \Omega_0,\ n = 1, 2, \ldots)$. We have $f_{|\Omega_0} = 0$, so $\int_\Omega f\,d\mu = 0$; on the other hand, $f = h$ almost everywhere, so $\int_\Omega f\,d\mu \ge \delta > 0$, a contradiction.

DEFINITION. Given a weak Baire measure λ on a Banach space E, we say that λ is *scalarly concentrated* on a subspace G of E if $x^*_{|G} = 0$ implies $x^* = 0$ λ-almost everywhere.

THEOREM 1. *Let f be a scalarly bounded function on a finite measure space (Ω, Σ, μ) with values in a Banach space E. Then the following are equivalent*:

(a) *f is Pettis integrable and its Pettis integral has a separable range*;

(b) *the measure $f(\mu)$ is scalarly concentrated on a separable subspace of E.*

Consequently, a Banach space E has the Pettis Integral Property and every E-valued Pettis integral has a separable range if and only if every weak Baire measure on E is scalarly concentrated on a separable subspace.

Proof. (a)⇒(b). Let G be a separable subspace of E containing the values of the Pettis integral of f. Now $x^*_{|G} = 0$ implies $\int_A x^* f \, d\mu = 0$ for every $A \in \Sigma$, and it follows that $x^* f = 0$ μ-almost everywhere, so $x^* = 0$ $f(\mu)$-almost everywhere.

(b)⇒(a). Put $\lambda = f(\mu)$ and let G, the closure of a countable set $D \subseteq E$, be a subspace of E such that λ is scalarly concentrated on G. Now $x^*_{|D} = 0$ implies $x^* = 0$ λ-almost everywhere so by the Lemma the map $x^* \to \int_B x^* \, d\lambda$ is τ_{p} (=weak*) continuous for every $B \in \mathcal{B}a(E)$. Thus the canonical embedding $E^* \to L^1(\lambda)$ is τ_{p}-to-weak continuous and $i_E : E \to E$ is Pettis integrable with respect to λ. If $x_0 = \int_B i_E \, d\lambda$ then $x^*_{|G} = 0$ implies $x^* x_0 = 0$; hence $x_0 \in G$. It follows that f is Pettis integrable and the range of its Pettis integral is contained in G.

The last statement of the theorem follows from the equivalence (a)⇔(b) (in the proof of necessity one can reduce the problem to the case of a weak Baire measure λ with $\lambda^*(E_1) = \lambda(E)$).

Although the material above is not very far from Talagrand's ideas from [Ta], Theorem 1 seems to be worth spelling out as the condition involved in this characterization of Pettis integrability is more transparent than that of convex τ-additivity.

If λ is a τ-additive weak Baire measure on a Banach space E then $\lambda^*(G) = \lambda(E)$ for some separable G; clearly λ is then scalarly concentrated on G. As will be explained in the last section, this cannot be reversed. On the other hand, there are Pettis integrals with non-separable range, and so there are convexly τ-additive weak Baire measures that are not scalarly concentrated on separable subspaces. However, the following seems to be open.

PROBLEM. *Suppose that a Banach space E enjoys PIP. Does it follow that every Pettis integral in E has a separable range? In other words, is it true that every weak Baire measure on E is scalarly concentrated on a separable subspace provided each is convexly τ-additive?*

4. A note on Grothendieck measures. In this section we shall show that the sequential completeness of the space of Grothendieck measures on a topological space is a trivial consequence of our Lemma.

Let X be a completely regular topological space. We adhere to the standard notation and denote by $\mathcal{M}_\sigma(X)$ the space of Baire measures on X. Wheeler [Wh] introduced the subspace $\mathcal{M}_{\mathrm{g}}(X)$ of $\mathcal{M}_\sigma(X)$ and called its elements *Grothendieck measures*. A Baire measure is a Grothendieck measure if it is τ_{p}-continuous on absolutely convex and τ_{p}-compact sets in $C_{\mathrm{b}}(X)$, the space of all continuous bounded functions on X.

Pallarés–Vera [PV] showed that Grothendieck measures are tightly connected with Pettis integration of weakly continuous functions. They also proved the following result.

COROLLARY ([PV], Corollaries 8 and 9). *If a measure* $\mu \in \mathcal{M}_\sigma^+(X)$ *is a weak* cluster point of a sequence* $(\mu_n) \subseteq \mathcal{M}_{\mathrm{g}}^+(X)$ *then* $\mu \in \mathcal{M}_{\mathrm{g}}^+(X)$. *Consequently*, $\mathcal{M}_{\mathrm{g}}(X)$ *is weak* sequentially complete.*

Proof. Let $C \subseteq C_{\mathrm{b}}(X)$ be absolutely convex and τ_{p}-compact. Then C is uniformly bounded (Wheeler [Wh], p. 119). For every n there is a countable $X_n \subseteq X$ such that $g \in C$, $g_{|X_n} = 0$ implies $\int_X g\,d\mu_n = 0$. It follows that the set $\bigcup_n X_n$ satisfies condition (ii) of the Lemma so μ is τ_{p}-continuous on C. This proves that $\mu \in \mathcal{M}_{\mathrm{g}}^+(X)$. The second statement follows by a standard argument (see [PV], proof of Corollary 9).

5. Pettis integration in $C(K)$. In this section K always stands for a compact Hausdorff space. The dual of $C(K)$, the Banach space of continuous functions with the supremum norm, is identified with the space of signed Radon measures on K of bounded variation, and will be denoted by $\mathcal{M}(K)$ ($\mathcal{M}^+(K)$ stands for its positive cone). In particular, δ_t denotes the Dirac measure at $t \in K$.

We shall say that K has *property* $(*)$ if for every function $f : K \to \mathbb{R}$, f is continuous on K provided it is sequentially continuous (i.e. $\lim f(t_n) = f(t)$ for every sequence (t_n) converging to t in K).

As mentioned in Section 2, a Pettis integral with values in $C(K)$ need not have a separable range (recall that the Banach space $\ell^\infty(\kappa)$ is isometric to $C(K)$, where K is the Čech–Stone compactification of κ with the discrete topology). On the other hand, if K is the support of a Radon measure then every weakly compact set in $C(K)$ is separable (this is a theorem due to Rosenthal, see Talagrand [Ta], 12-1-5), so in that case every $C(K)$-valued Pettis integral has a separable range, as the range of a vector measure in a Banach space is relatively weakly compact.

THEOREM 2. *If K has property* $(*)$ *then every Pettis integral in* $C(K)$ *has a separable range.*

Proof. Let μ be a weak Baire measure on $C(K)$ such that the identity $i : C(K) \to C(K)$ is scalarly bounded and Pettis integrable with respect to μ. For every $t, s \in K$ put $\varrho(s,t) = \int_{C(K)} |\delta_t - \delta_s|\,d\mu$. Clearly this defines a pseudometric on K. For a fixed $t \in K$, the function $\varrho(\cdot, t)$ is sequentially continuous on K from the Lebesgue theorem. Property $(*)$ implies that ϱ is a continuous pseudometric on K. Let K' be a quotient space of K and $\pi : K \to K'$ be the canonical map. Since K' is compact and metrizable (see Engelking [En], 3.2.11 and 4.2.I), the space

$E = \{h \circ \pi : h \in C(K')\}$ is separable. We shall check that $\int i\,d\mu$ takes its values in E.

Let $B \in \mathcal{B}a(C(K))$ and let $f = \int_B i\,d\mu \in C(K)$. If $t, s \in K$ and $\pi(t) = \pi(s)$ then $\delta_t = \delta_s$ μ-almost everywhere. Therefore

$$f(t) = \int_B \delta_t\,d\mu = \int_B \delta_s\,d\mu = f(s)\,,$$

and it follows that $f \in E$.

Let μ be a weak Baire measure on $C(K)$ and let $B \in \mathcal{B}a(C(K))$. As remarked by Edgar [E2] (p. 568), there are two reasons for which $\int_B i\,d\mu$ may not exist in $C(K)$. Simply the only candidate for $\int_B i\,d\mu$ is the function φ defined by $\varphi(t) = \int_B \delta_t\,d\mu$. This φ is sequentially continuous on K but need not be continuous. Even if φ is continuous on K, the condition $\lambda(\varphi) = \int_B \lambda = d\mu$ may fail for some non-atomic $\lambda \in \mathcal{M}_1^+(K)$ (cf. Edgar's remark on the Fremlin–Talagrand example, [E2], p. 569).

The condition $(*)$ for K we have defined above might seem to be a reasonable way to overcome the first obstacle in seeking Pettis integrals in $C(K)$. Unfortunately, $(*)$ is not necessary for $C(K)$ having PIP, at least when the continuum hypothesis (CH) holds. This may be seen by analyzing Talagrand's example 16-4-1 from [Ta]. The space K he has constructed under CH is such that $C(K)$ is measure-compact (hence has PIP), and there exists a non-isolated point $t_0 \in K$ which is not a limit of a sequence from $K \setminus \{t_0\}$. The latter property means that K fails $(*)$ since χ_{t_0} is sequentially continuous but not continuous.

THEOREM 3. *If K is a first-countable compact space then $C(K)$ has the Pettis Integral Property and every $C(K)$-valued Pettis integral has a separable range.*

Proof. We shall check that if $z \in C(K)^{**}$ is weak* sequentially continuous on $C(K)^*$ then z is weak* continuous (so $z \in C(K)$). This property, sometimes called the Mazur Property, is known to imply PIP (cf. [E2]). The rest will follow from Theorem 2.

If $z \in C(K)^{**}$ is weak* sequentially continuous on $C(K)^*$ then the function φ given by $\varphi(t) = z(\delta_t)$ is continuous. Put $w(\lambda) = z(\lambda) - \lambda(\varphi)$. Now we have $w(\delta_t) = 0$; we are to prove that $w = 0$.

Fix $\varepsilon > 0$. We shall prove that for every $t \in K$ there is a neighbourhood V of t such that $|w(\nu)| \le \varepsilon\nu(K)$ for every $\nu \in \mathcal{M}^+(K)$ concentrated on V. Suppose otherwise: let (V_n) be a countable base at t and let $\nu_n \in \mathcal{M}_1^+(K)$ be such that $|w(\nu_n)| \ge \varepsilon$ and $\nu_n(V_n) = 1$ for every n. Note that the sequence ν_n converges weak* to δ_t; hence $|w(\delta_t)| \ge \varepsilon$, a contradiction.

It follows that there is a finite cover $\{V_1, \ldots, V_k\}$ of K such that $|w(\nu)| \le \varepsilon\nu(K)$ for every $\nu \in \mathcal{M}^+(K)$ with $\nu(V_i) = \nu(K)$ for some $i \le k$. Let

$\{A_1, \ldots, A_m\}$ be the collection of all atoms of this partition. For $\lambda \in \mathcal{M}_1^+(K)$ we have

$$|w(\lambda)| \le \sum_{i\le m} |w(\lambda_{A_i})| \le \sum_{i\le m} \varepsilon\lambda(A_i) = \varepsilon\,,$$

where λ_A denotes the restriction of a measure λ to a set A. Thus $w = 0$ and the proof is complete.

We do not know if property $(*)$ implies that $C(K)$ has PIP. It is worth recalling that $(*)$ is much weaker than the assumption of first-countability. For instance, it is relatively consistent with the usual axioms of set theory to assume that every Cantor cube 2^κ has property $(*)$ (cf. [Ma] and [AC]; see also [P1], where the measure-compactness of $C(2^\kappa)$ is derived from that fact).

Theorem 3 is applicable to K being the two arrows space (cf. [En], 3.10.C) which is separable and first-countable. Here $C(K)$ has PIP but is not measure-compact (cf. [E2], Example 5.7, see also [SW]). This means that there exists a weak Baire measure on $C(K)$ that vanishes on all separable subspaces but is scalarly concentrated on a certain separable subspace.

REFERENCES

[AC] M. Ya. Antonovskiĭ and D. Chudnovsky, *Some questions of general topology and Tikhonov semifields. II*, Russian Math. Surveys 31 (3) (1976), 69–128.

[E1] G. A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. 26 (1977), 663–677.

[E2] —, *Measurability in a Banach space, II*, ibid. 28 (1979), 559–579.

[En] R. Engelking, *General Topology*, PWN, Warszawa 1977.

[FT] D. H. Fremlin and M. Talagrand, *A decomposition theorem for additive set functions, with application to Pettis integrals and ergodic means*, Math. Z. 168 (1979), 117–142.

[Ma] S. Mazur, *On continuous mappings on Cartesian products*, Fund. Math. 39 (1952), 229–238.

[M1] K. Musiał, *Martingales of Pettis integrable functions*, in: Measure Theory, Oberwolfach 1979, Lecture Notes in Math. 794, Springer, 1980, 324–339.

[M2] —, *Pettis integration*, in: Proc. 13th Winter School on Abstract Analysis, Suppl. Rend. Circ. Mat. Palermo 10 (1985), 133–142.

[PV] A. J. Pallarés and G. Vera, *Pettis integrability of weakly continuous functions and Baire measures*, J. London Math. Soc. 32 (1985), 479–487.

[P1] G. Plebanek, *On the space of continuous functions on a dyadic set*, Mathematika 38 (1991), 42–49.

[SW] F. D. Sentilles and R. F. Wheeler, *Pettis integration via the Stonian transform*, Pacific J. Math. 107 (1983), 473–496.

[T] M. Talagrand, *Sur les mesures vectorielles définies par une application Pettis-intégrable*, Bull. Soc. Math. France 108 (1980), 475–483.

[Ta] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. 307 (1984).

[Wh] R. F. Wheeler, *A survey of Baire measures and strict topologies*, Exposition. Math. 1 (1983), 97–190.

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