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SOME POSSIBLE COVERS OF MEASURE ZERO SETS

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1. Introduction. Borel was interested in what he called the rarefaction of a measure zero set $\mathcal{X}$ (see [5]), that is, the rate of convergence of the series $\sum_n |I_n|$, for various sequences of open intervals $\langle I_n : n \in N\rangle$ such that $\mathcal{X} \subseteq \bigcap_n \bigcup_{k\geq n} I_k$. A partial ordering was introduced in [8] to identify the measure zero sets with similar covering properties, and it was shown that at least four essential differences exist. In [10], it was shown that this number cannot be improved using the usual axioms of set theory (ZFC); more precisely, we showed that there are exactly four classes of measure zero sets assuming $\mathfrak{u} < \mathfrak{g}$, a forcing axiom known to be relatively consistent with ZFC (see [2]); we describe here explicitly (what should have been done in [10]) the covering properties of those four classes.

In [4], Borel defined regular measure zero sets in order to extended Weierstrass' theory of analytic functions. They are equipped with special covers and Borel needed a regular measure zero set with a fast enough rate of convergence for the series of lengths of one of its covers to pursue his theory. In an attempt to understand which measure zero sets could satisfy Borel's condition, we investigate the rate of convergence of these covers compared with the previous classification; under $\mathfrak{u} < \mathfrak{g}$, both types of covers have the same properties.

2. Notation and preliminaries. We use N for the set of natural numbers, and $N \nearrow N$ for the set of nondecreasing unbounded functions on N. For $f, g \in N \nearrow N$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n \in N$. We use $\mathcal{M}, \mathcal{N}$ to denote families of functions and we recall two orderings:

1. $\mathcal{M} \preceq^1 \mathcal{N}$ iff $(\exists r \in N \nearrow N)(\forall f \in \mathcal{M})(\exists g \in \mathcal{N})(\forall^\infty_n)[r(g(n)) \geq f(n)]$.
2. $\mathcal{M} \preceq^2 \mathcal{N}$ iff $(\exists r \in N \nearrow N)(\forall f \in \mathcal{M})(\exists g \in \mathcal{N})(\forall^\infty_n)[g(r(n)) \geq f(n)]$.

Both orderings are reflexive and transitive and we shall use $\preceq$ and $\sim$ to

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denote either ordering or induced equivalence relation. The $\preceq^1$ has been used in [8] to classify measure zero sets, the $\preceq^2$ ordering was introduced in [6] to study slenderness classes of abelian groups and investigated in [6], [7], [1], [2].

Since both ordering are concerned with the rapidly growing functions in each family, we shall always work with the downward closure under $\leq^*$ of each family; that is, we shall assume that $g \in \mathcal{M}$ whenever $g \leq^* f$ for some $f \in \mathcal{M}$; this will prevent an awkward wording of our results.

A family $\mathcal{M}$ is called *dominating* if

$$(\forall g)(\exists f \in \mathcal{M})(g \leq^* f)\,,$$

and *bounded* if

$$(\exists g)(\forall f \in \mathcal{M})(f \leq^* g)\,,$$

Ultrafilters are maximal proper collections of infinite subsets of N closed under finite intersections and supersets; they will be denoted by $\mathcal{U}$, $\mathcal{V}$.

For any collection $\mathcal{A}$ of infinite subsets of N, a special family of functions is obtained by (the *downward closure* of):

$$\mathcal{M}(\mathcal{A}) = \{f \in N \nearrow N : \{n : f(n) \leq n\} \in \mathcal{A}\}\,.$$

Note that $\mathcal{M}(\mathcal{A})$ is the downward closure of the functions $f(n) = \text{next}(A, n)$ for $A \in \mathcal{A}$, where $\text{next}(A, n)$ denotes the least element of A greater than or equal to n. They will be used mainly with $\mathcal{A}$ being an ultrafilter.

Finally, we define an discuss the cardinals $\mathfrak{u}$ and $\mathfrak{g}$. $\mathfrak{u}$ is the least cardinality of a base for an ultrafilter. To define $\mathfrak{g}$, we call a family $\mathcal{S}$ of infinite sets *groupwise dense* if:

1. $Y' \in \mathcal{S}$ whenever $Y' \subseteq^* Y$ and $Y \in \mathcal{S}$.
2. For every infinite family π of disjoint finite subsets of N, there exists a (necessarily infinite) subfamily of π whose union is a member of $\mathcal{S}$.

Then $\mathfrak{g}$ is defined as the least cardinality of a collection of groupwise dense families with empty intersection.

The inequality $\mathfrak{u} < \mathfrak{g}$ was verified in [2] to hold in known models of ZFC, hence is relatively consistent with ZFC. It was shown there to imply the principle of near-coherence of filters (NCF), in particular the Rudin–Keisler ordering must be downward directed; hence $\mathfrak{u} < \mathfrak{g}$ is incompatible with Martin's axioms for example. In the next section, we shall see the effect of this inequality on families of functions on the natural numbers.

3. Families of functions. If one is interested in the rapidly growing functions of a family $\mathcal{M}$, then the following classes certainly exhibit distinct behaviour.

1. The $\mathcal{D}$-*class* (the dominating class): $\mathcal{M} \in \mathcal{D}$ iff $\mathcal{M}$ is dominating.

2. The $\mathcal{H}$-*class* (the high class): $\mathcal{M} \in \mathcal{H}$ iff
   (i) $(\exists h)(\forall f \in \mathcal{M})(\exists_n^\infty)[f(n) \le h(n)]$.
   (ii) $(\exists g)(\forall f)[\exists_n^\infty f(n) \le g(n) \to f \in \mathcal{M}]$.
3. The $\mathcal{U}$-*class* (for an ultrafilter $\mathcal{U}$): $\mathcal{M} \in \mathcal{U}$ iff
   (i) $(\exists h)(\forall f \in \mathcal{M})[\{n : f(n) \le h(n)\} \in \mathcal{U}]$.
   (ii) $(\exists g)(\forall f)[\{n : f(n) \le g(n)\} \in \mathcal{U} \to f \in \mathcal{M}]$.
4. The $\mathcal{L}$-*class* (the low or bounded class): $\mathcal{M} \in \mathcal{L}$ iff $\mathcal{M}$ is bounded.

Observe that the functions g and h in 2 and 3 must satisfy $g \le^* h$ and $\{n : g(n) \le h(n)\} \in \mathcal{U}$ respectively.

It is somewhat natural to expect families of functions which do not fall in these classes; this is the case assuming the Continuum Hypothesis (CH) for example.

PROPOSITION 1. *Assume CH. There is a family $\mathcal{M}$ which does not belong to any of the classes above.*

Proof. It suffices to build $\mathcal{M}$, closed downward (under $\le^*$) and satisfying the following two conditions:

(i) $(\forall g)(\exists f)[\exists_n^\infty f(n) \le g(n)$ and $(\forall h \in \mathcal{M})(\exists_n^\infty) h(n) < f(n)]$.
(ii) $(\forall g)(\exists h_1, h_2 \in \mathcal{M})(\forall_n^\infty)[\max\{h_1(n), h_2(n)\} \ge g(n)]$.

Indeed, condition (i) will guarantee that $\mathcal{M}$ does not belong to either the $\mathcal{D}$ or the $\mathcal{H}$-class; and condition (ii) guarantees that $\mathcal{M}$ does not belong to the $\mathcal{L}$ or any $\mathcal{U}$-class.

The construction is a straightforward induction. List $N \nearrow N$ as $\langle g_\alpha : \alpha \in \aleph_1\rangle$, and assume as our induction hypothesis that we already have $\langle h_\alpha^1, h_\alpha^2 : \alpha \in \beta\rangle$ and $\langle f_\alpha : \alpha \in \beta\rangle$ such that:

1. $(\forall \alpha \in \beta)(\exists_n^\infty)[f_\alpha(n) \le g_\alpha(n)]$.
2. $(\forall \alpha \in \beta)(\forall \gamma \in \beta)(\forall i = 1, 2)(\exists_n^\infty)[h_\alpha^i(n) < f_\gamma(n)]$.
3. $(\forall \alpha \in \beta)(\forall_n^\infty)[\max\{h_\alpha^1(n), h_\alpha^2(n)\} \ge g_\alpha(n)]$.

Since β is countable, it is not difficult to build h_β^1, h_β^2, f_β satisfying the hypothesis. In conclusion, the required family $\mathcal{M}$ is obtained as the downward closure of $\{h_\alpha^1, h_\alpha^2 : \alpha \in \aleph_1\}$. ∎

However, results of [10] imply that every family falls into one of our four classes under $\mathfrak{u} < \mathfrak{g}$; we complete the details here.

THEOREM 2. *Under $\mathfrak{u} < \mathfrak{g}$, every family of functions belongs to one of the four classes*; *further*, *the ultrafilter classes* (*for various $\mathcal{U}$'s*) *are equal.*

Proof. Fix $\mathcal{U}$ any ultrafilter and $\mathcal{M}$ a family of functions. It is proved in [10] that under $\mathfrak{u} < \mathfrak{g}$, $\mathcal{M}$ belongs to either the $\mathcal{L}$, $\mathcal{H}$ or $\mathcal{D}$-class, or $\mathcal{M}$ is equivalent to $\mathcal{M}(\mathcal{U})$ for either ordering; in particular, we have $\mathcal{M} \preceq^2 \mathcal{M}(\mathcal{U})$

and $\mathcal{M}(\mathcal{U}) \preceq^2 \mathcal{M}$ as well. It then suffices to prove that this implies that $\mathcal{M}$ belongs to the $\mathcal{U}$-class. In fact, we prove slightly more.

LEMMA 1. *$\mathcal{M} \preceq^2 \mathcal{M}(\mathcal{U})$ iff* $(\exists h)(\forall f \in \mathcal{M})[\{n : f(n) \le h(n)\} \in \mathcal{U}]$.

Proof. Assume first that $\mathcal{M} \preceq^2 \mathcal{M}(\mathcal{U})$, which means that

$$(\exists h)(\forall f \in \mathcal{M})(\exists X \in \mathcal{U})(\forall n)[f(n) \le h(\mathrm{next}(X, n))]\,.$$

Hence, for all $f \in \mathcal{M}$, for all $x \in X$ as above,

$$f(x) \le h(\mathrm{next}(X, x)) = h(x)\,,$$

so h is the required function.

Conversely, given a function h such that $X_f = \{n : f(n) \le h(n)\} \in \mathcal{U}$ for each $f \in \mathcal{M}$, we have, again for each $f \in \mathcal{M}$ but also for each n,

$$f(n) \le f(\mathrm{next}(X_f, n)) \le h(\mathrm{next}(X_f, n))$$

and this shows that $\mathcal{M} \preceq^2 \mathcal{M}(\mathcal{U})$ as desired. ∎

LEMMA 2. *$\mathcal{M}(\mathcal{U}) \preceq^2 \mathcal{M}$ iff* $(\exists g)(\forall f)[\{n : f(n) \le g(n)\} \in \mathcal{U} \to f \in \mathcal{M}]$.

Proof. Assume first that $\mathcal{M}(\mathcal{U}) \preceq^2 \mathcal{M}$, which means that

$$(\exists r)(\forall X \in \mathcal{U})(\exists h \in \mathcal{M})(\forall n)[\mathrm{next}(X, n) \le r(h(n))]\,.$$

Define $g(n) = m$ iff $r(m) \le n < r(m+1)$; then g is unbounded as $h \in N \nearrow N$. We claim that g is the required function. Indeed, given any f such that $f(n) \le g(n)$ for all $n \in X \in \mathcal{U}$, let $h \in \mathcal{M}$ be the function corresponding to X. Then

$$(\forall n)[f(n) \le f(\mathrm{next}(X, n)) \le g(\mathrm{next}(X, n)) \le g(r(h(n))) = h(n)]\,.$$

Since $h \in \mathcal{M}$, this shows that $f \in \mathcal{M}$ as well.

Conversely, we assume that

$$(\exists g)(\forall f)[\{n : f(n) \le g(n)\} \in \mathcal{U} \to f \in \mathcal{M}]\,.$$

Define $r(n) = \max\{k : g(k) \le n\}$; r is well defined since g is unbounded monotone. Also, $r(g(n)) \ge n$ for each n. Now fix $X \in \mathcal{U}$, and put $f(n) = g(\mathrm{next}(X, n))$. Since for each $x \in X$, $f(x) = g(\mathrm{next}(X, x)) = g(x)$, we must have $f \in \mathcal{M}$ by hypothesis; but also

$$(\forall n)[r(f(n)) = r(g(\mathrm{next}(X, n))) \ge \mathrm{next}(X, n)]$$

and this shows that $\mathcal{M}(\mathcal{U}) \preceq^2 \mathcal{M}$ as desired. ∎

It is significant that not only each family belongs to one the four classes under $\mathfrak{u} < \mathfrak{g}$, but that all $\mathcal{U}$-classes are equal; it is shown in [1] that under Martin's axiom, there are $2^{2^{\aleph_0}}$ distinct ultrafilter classes.

4. Covers of measure zero sets. In this section, we shall use (following Borel) the word *cover* for a measure zero set $\mathcal{X} \subseteq [0, 1]$ to mean a sequence of open intervals $\mathcal{I} = \langle I_n : n \in N \rangle$ such that

$$\mathcal{X} \subseteq \bigcap_n \bigcup_{k \geq n} I_k \quad \text{and} \quad \sum |I_n| < \infty .$$

It is easily seen that $\mathcal{X}$ is of measure zero if and only if it has such a cover.

Now define (see [8]) $f_{\mathcal{I}} \in N \nearrow N$ by $f_{\mathcal{I}}(n) = \lceil 1/\sum_{k\geq n} |I_k| \rceil$, and put $\mathcal{M}(\mathcal{X}) = \{f_{\mathcal{I}} : \mathcal{I}$ is a cover of $\mathcal{X}\}$. The family $\mathcal{M}(\mathcal{X})$ estimates the rate of convergence of the covers of $\mathcal{X}$. Hence, under $\mathfrak{u} < \mathfrak{g}$, $\mathcal{M}(\mathcal{X})$ must fall into one of the four classes of Section 3, and these transform into the only four possibilities for the covers of $\mathcal{X}$:

1. $\mathcal{X}$ is of strong measure zero, i.e. for all sequences of positive reals $\langle \varepsilon_n : n \in N \rangle$, there is a cover $\mathcal{I} = \langle I_n : n \in N \rangle$ of $\mathcal{X}$ such that $\sum_{k \geq n} |I_k| < \varepsilon_n$ for each n.
2. $\mathcal{M}(\mathcal{X})$ is in the $\mathcal{H}$-class:
   - (i) There is a sequence of positive reals $\langle \varepsilon_n : n \in N \rangle$ such that for all covers $\mathcal{I} = \langle I_n : n \in N \rangle$ of $\mathcal{X}$, $\sum_{k \geq n} |I_k| \geq \varepsilon_n$ for infinitely many n.
   - (ii) There is a sequence of positive reals $\langle \delta_n : n \in N \rangle$ converging to zero such that for all (non-increasing) sequences $\langle \gamma_n : n \in N \rangle$, if $\gamma_n \geq \delta_n$ infinitely often then $\mathcal{X}$ has a cover $\mathcal{I}$ satisfying $\sum_{k \geq n} |I_k| \leq \gamma_n$ for each n.
3. $\mathcal{M}(\mathcal{X})$ belongs to the $\mathcal{U}$-class:
   - (i) There is a sequence of positive reals $\langle \varepsilon_n : n \in N \rangle$ such that for all covers $\mathcal{I} = \langle I_n : n \in N \rangle$ of $\mathcal{X}$, $\sum_{k \geq n} |I_k| \geq \varepsilon_n$ for a set of n in the ultrafilter $\mathcal{U}$.
   - (ii) There is a sequence of positive reals $\langle \delta_n : n \in N \rangle$ converging to zero such that for all (non-increasing) sequences $\langle \gamma_n : n \in N \rangle$, if $\gamma_n \geq \delta_n$ on a set of n in $\mathcal{U}$, then $\mathcal{X}$ has a cover $\mathcal{I}$ satisfying $\sum_{k \geq n} |I_k| \leq \gamma_n$ for each n.
4. $\mathcal{M}(\mathcal{X})$ is bounded: There is a sequence of positive reals $\langle \varepsilon_n : n \in N \rangle$ such that all covers $\mathcal{I}$ of $\mathcal{X}$ satisfy $\sum_{k \geq n} |I_k| \geq \varepsilon_n$ for all but finitely many n.

All countable sets are of strong measure zero. It was proved in [8] that $\mathcal{X}$ belongs to the $\mathcal{H}$-class if it is not of strong measure zero but included in a F_σ-set of measure zero; in particular, all uncountable closed sets of measure zero belong to that class. It was also proved there that if $\mathcal{X}$ is comeager relative to a self-supporting closed set of positive measure, then $\mathcal{X}$ belongs to the fourth class, that is, $\mathcal{M}(\mathcal{X})$ is bounded. An example of a set $\mathcal{X}$ falling in the $\mathcal{U}$-class was also built in [8], and hence all those classes are nonempty (in ZFC) and those are the only ones assuming $\mathfrak{u} < \mathfrak{g}$.

I do not know if assuming CH (or any other relatively consistent axiom), a family of the form $\mathcal{M}(\mathcal{X})$, $\mathcal{X}$ of measure zero, can be built (as in Proposition 1) which does not belong to any of these four classes.

5. Regular measure zero sets. In [4], Borel defined regular measure zero sets of the plane in order to extend Weierstrass' theory of analytic functions; he needed a measure zero set equipped with a cover of fast enough rate of convergence. For convenience, we shall study subsets of the real line without essential modifications.

Definition 3. Let $\langle a_n : n \in N\rangle$ be a sequence of points (not necessarily distinct) that we shall call *fundamental*, and for each $k, n \in N$, let I_n^k be an open interval containing a_n. We assume further that $I_n^{k+1} \subseteq I_n^k$, $\lim_{k\to\infty} |I_n^k| = 0$ and that $\sum_n |I_n^0| < \infty$.

We conclude that $\mathcal{X} = \bigcap_k \bigcup_n I_n^k$ is a measure zero set containing each a_n; such an $\mathcal{X}$ is called *regular* by Borel. We call the sequence $\langle I_n^k : k, n \in N\rangle$ a *fundamental cover* of $\mathcal{X}$, or of any subset $\mathcal{Y}$ of $\mathcal{X}$.

Borel showed in [4] that any measure zero set is contained in a regular measure zero set.

Lemma 3 (Borel). *Let $\mathcal{X}$ be a measure zero set. Then it has a fundamental cover with fundamental points belonging to $\mathcal{X}$.*

We can now classify measure zero sets according to the rate of convergence of their fundamental covers in the same way as we did for covers.

Definition 4. For a fundamental cover $\mathcal{I} = \langle I_n^k : k, n \in N\rangle$ of a measure zero set $\mathcal{X}$, define

$$f_{\mathcal{I}}^k(n) = \Big\lceil 1/\sum_{m\ge n} |I_m^k| \Big\rceil \in N \nearrow N$$

and put

$$\mathcal{R}(\mathcal{X}) = (\text{the downward closure of})\ \{f_{\mathcal{I}}^k : \mathcal{I} \text{ is a fundamental cover of } \mathcal{X},\ k \in N\}.$$

We see that $\mathcal{M}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{X})$; and as a partial converse, we have:

Lemma 4. *For each $f \in \mathcal{R}(\mathcal{X})$, there is a $g \in \mathcal{M}(\mathcal{X})$ such that*

$$(\forall n)[f(n) \le (n+1)g(n^2)].$$

Corollary 5. *$\mathcal{R}(\mathcal{X})$ is bounded if and only if $\mathcal{M}(\mathcal{X})$ is bounded.*

In particular, by Theorem 4.2 of [8], $\mathcal{R}(\mathcal{X})$ is bounded as soon as $\mathcal{X}$ is comeager relative to a self-supporting closed set of positive measure. An even stronger result follows from the proof of that Theorem 4.2; namely, under the same hypothesis (in particular, if $\mathcal{X}$ is comeager of measure zero) there are $\delta > 0$ and a positive sequence $\langle \varepsilon_n : n \in N \rangle$ such that if $\mathcal{X} \subseteq \bigcup I_n$ and $\sum_n |I_n| < \delta$, then $\sum_{k \ge n} |I_k| \ge \varepsilon_n$ for all but finitely many n.

We also have:

COROLLARY 6. *$\mathcal{R}(\mathcal{X})$ is dominating iff $\mathcal{M}(\mathcal{X})$ is dominating.*

COROLLARY 7. *$\mathcal{M}(\mathcal{X}) \preceq^2 \mathcal{M}(\mathcal{U})$ for some ultrafilter $\mathcal{U}$ if and only if $\mathcal{R}(\mathcal{X}) \preceq^2 \mathcal{M}(\mathcal{V})$ for some $\mathcal{V}$.*

Proof. One direction follows immediately from $\mathcal{M}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{X})$.

Now suppose that $\mathcal{M}(\mathcal{X}) \preceq^2 \mathcal{M}(\mathcal{U})$. Then by Lemma 1 we can fix some g such that $\{n : f(n) \le g(n)\} \in \mathcal{U}$ for each $f \in \mathcal{M}(\mathcal{X})$. Define $s(n) = \lfloor \sqrt{n} \rfloor$, put $\mathcal{V} = s(\mathcal{U})$ and $g'(n) = (n+1)g((n+1)^2)$. If $f' \in \mathcal{R}(\mathcal{X})$, then there exists $f \in \mathcal{M}(\mathcal{X})$ such that $f'(n) \le (n+1)f(n^2)$ for each n by Lemma 4. Now choose $X \in \mathcal{U}$ such that $f(x) \le g(x)$ for each $x \in X$. We conclude that for each $y = s(x) \in s(X)$,

$$\begin{aligned} f'(y) &\le (y+1)f(y^2) \le (y+1)f(x) \quad \text{[because } y^2 \le x < (y+1)^2\text{]} \\ &\le (y+1)g(x) \le (y+1)g((y+1)^2) = g'(y)\,. \end{aligned}$$

Hence, for each $f' \in \mathcal{R}(\mathcal{X})$, $\{n : f'(n) \le g'(n)\} \in s(\mathcal{U}) = \mathcal{V}$. This shows that $\mathcal{R}(\mathcal{X}) \preceq^2 \mathcal{M}(\mathcal{V})$ by Lemma 1 again and completes the proof. ■

We conclude that under $\mathfrak{u} < \mathfrak{g}$, the families $\mathcal{R}(\mathcal{X})$, $\mathcal{M}(\mathcal{X})$ are in the same class for each $\mathcal{X}$. However, does it follow from ZFC alone? In particular, is the fact that $\mathcal{R}(\mathcal{X})$ belongs to the $\mathcal{H}$-class sufficient to guarantee the same for $\mathcal{M}(\mathcal{X})$? This would be the case if a function g witnessing the $\mathcal{H}$-property of $\mathcal{R}(\mathcal{X})$ could be found so that the new function $g'(n) = g(\lfloor \sqrt{n} \rfloor - 1)/\lfloor \sqrt{n} \rfloor$ is unbounded; for g' would witness the $\mathcal{H}$-property of $\mathcal{M}(\mathcal{X})$; however, this is not always possible by results of Zakrzewski [11].

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