COLLOQUIUM MATHEMATICUM

VOL. LXIII 1992 FASC. 2

SOME BOREL MEASURES ASSOCIATED WITH THE GENERALIZED COLLATZ MAPPING

BY

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1. Abstract. This paper is a continuation of a recent paper [2], in which the authors studied some Markov matrices arising from a mapping $T : \mathbb{Z} \to \mathbb{Z}$, which generalizes the famous $3x + 1$ mapping of Collatz. We extended T to a mapping of the polyadic numbers $\widehat{\mathbb{Z}}$ and construct finitely many ergodic Borel measures on $\widehat{\mathbb{Z}}$ which heuristically explain the limiting frequencies in congruence classes, observed for integer trajectories.

2. Introduction. Let $d \geq 2$ be a positive integer and let $m_0, \ldots, m_{d-1}$ be non-zero integers, each relatively prime to d. Also let R be a complete set of integers $\bmod d$ and for i=$0, \ldots, d-1$, the residue $r_i \in R$ is defined by $r_i \equiv im_i \pmod{d}$. Then the *generalized Collatz mapping* $T : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if } x \equiv i \pmod{d}. \tag{1}$$

A central property of the mapping T is that the inverse image of a congruence class $\bmod m$ is a union of congruence classes $\bmod md$. (See [5, Lemma 2.1, page 31].) It is this property that enables T to be extended uniquely to a continuous mapping of the set of d-adic integers into itself (see [4, pages 172–174]) and to a continuous mapping of $\widehat{\mathbb{Z}}$ into itself. This ring can be obtained as the projective limit of the projective system of natural homomorphisms $\phi_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$, where $m \mid n$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.

$\widehat{\mathbb{Z}}$ has a topology for which the congruence classes $\pmod{m}$ form a base for the open sets (see [8, Chapter 3.5]). Denoting the congruence class $\{x \in \widehat{\mathbb{Z}} : x \equiv j \pmod{m}\}$ by $B(j, m)$, there is a unique Haar probability measure σ on $\widehat{\mathbb{Z}}$ with the property that $\sigma(B(j, m)) = 1/m$.

A natural object of study are the ergodic sets $\bmod m$. These are the minimal T-invariant sets composed of congruence classes $\bmod m$. In [2] we went some way in determining how the ergodic sets $\bmod m$ vary with m. We stated two related conjectures which, together with other results of that paper, enable one to complete that program. These conjectures (the second in slightly modified form) are proved in the present paper.

We are also interested in the set $\mathcal{M}_T(\widehat{\mathbb{Z}})$ of T-invariant probability measures on the Borel σ-algebra of $\widehat{\mathbb{Z}}$. In particular, we are interested in the set $\mathcal{M}'_T(\widehat{\mathbb{Z}})$ of those $\mu \in \mathcal{M}_T(\widehat{\mathbb{Z}})$ which satisfy $\mu(B(j, md)) = (1/d)\mu(B(j, m))$.

In studying $\mathcal{M}'_T(\widehat{\mathbb{Z}})$, we are led naturally to a Markov matrix $Q(m)$, as follows: We have

$$\begin{aligned}\mu(B(i,m)) &= \mu(T^{-1}(B(i,m))) = \mu(T^{-1}(B(i,m)) \cap \widehat{\mathbb{Z}}) \\ &= \mu(T^{-1}\Big(B(i,m)) \cap \bigcup_{j=0}^{d-1} B(j,m)\Big) \\ &= \sum_{j=0}^{d-1} \mu(T^{-1}(B(i,m)) \cap B(j,m))\,.\end{aligned}$$

Now $T^{-1}(B(i,m)) \cap B(j,m)$ is a disjoint union of $p_{ij}(m)$ congruence classes $\bmod\, md$, all of which have μ-measure equal to $(1/d)\mu(B(j,m))$. So

$$\mu(B(i,m)) = \sum_{j=0}^{d-1} p_{ij}(m)\frac{1}{d}\,\mu(B(j,m)) = \sum_{j=0}^{d-1} q_{ij}(m)\mu(B(j,m))\,,$$

where $Q(m) = [q_{ij}(m)] = [p_{ij}(m)/d]$ is the Markov matrix introduced in [5]. Hence the column vector $X = (\mu(B(0,m)), \ldots, \mu(B(d-1,m)))^t$ is an eigenvector of $Q(m)$ corresponding to the eigenvalue 1.

Let us relabel the rows and columns of $Q(m)$ so that the transient classes are first, followed by classes of the respective ergodic sets $S_1^{(m)}, \ldots, S_{r(m)}^{(m)}$. Then $Q(m)$ takes on a simpler form as in [2, (1.9)]. For the $S_j^{(m)}$ are in 1-1 correspondence with the irreducible closed sets of $Q(m)$. (See [5, Lemma 3.1].) Also $X = \sum_{k=1}^{r(m)} \lambda_k X_k$, where $\lambda_k \geq 0$ for all k, , $\sum_{k=1}^{r(m)} \lambda_k = 1$ and

$$X_1 = \begin{bmatrix} 0 \\ Y_1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 0 \\ Y_2 \\ 0 \\ \vdots \end{bmatrix}, \ldots$$

Here Y_k is the stationary vector corresponding to $M_m(S_k^{(m)})$, the Markov submatrix of $Q(m)$ corresponding to $S_k^{(m)}$. (See [2, (1.9)] and [7, Theorem 3.3.30].) Hence $\mu(B(j,m)) = 0$ if $B(j,m)$ is a transient class $\bmod\, m$. Now each $S_k^{(m)}$ satisfies $T^{-1}(S_k^{(m)}) \supseteq S_k^{(m)}$. Hence if we also assume that μ is an ergodic measure and use the ergodicity criterion [9, Theorem 1.4, page 17]

$$\lim_{N\to\infty} \frac{1}{N} \sum_{K=0}^{N} \mu(A \cap T^{-K}(B)) = \mu(A)\mu(B) \tag{2}$$

with $A = B = S_k^{(m)}$, we deduce that $\mu(S_k^{(m)}) = 0$ or 1. So precisely one $S_k^{(m)}$ has μ-measure equal to 1. Hence $\lambda_k = 1$ and $\lambda_j = 0$ if $j \neq k$. Hence $\mu(B(j,m)) = 0$ if $B(j,m) \cap S_k^{(m)} = \emptyset$, while if $B(j,m) \subseteq S_k^{(m)}$, then $\mu(B(j,m))$ is the $B(j,m)$th component of Y_k and hence $\mu(B(j,m)) > 0$.

From the theory of Markov matrices, we know that the components of Y_k are given by the following limit, where $B(l,m)$ is any congruence class contained in $S_k^{(m)}$:

$$\begin{aligned}(3)\qquad \mu(B(j,m)) &= \left(\lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{1}{N}[q_{ij}(m)]^K\right)_{jl} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\mathrm{card}_{md^K}(T^{-K}(B(j,m)) \cap B(l,m))}{d^K}.\end{aligned}$$

(We recall that $\mathrm{card}_n(S)$ denotes the number of congruence classes $\bmod\, n$ contained in S.) This can be written more symmetrically by summing over all $B(l,m)) \subseteq S_k^{(m)}$:

$$\begin{aligned}(4)\qquad \mu(B(j,m)) &= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\mathrm{card}_{md^K}(T^{-K}(B(j,m)) \cap S_k^{(m)})}{\mathrm{card}_{md^K}(S_k^{(m)})} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sigma(T^{-K}(B(j,m)) \cap S_k^{(m)})}{\sigma(S_k^{(m)})}.\end{aligned}$$

The assumption that μ is an ergodic measure yields a relation between ergodic sets $S_k^{(m)} \bmod m$ and $S_{k'}^{(n)} \bmod n$, when $m \mid n$ and $\mu(S_k^{(m)}) = \mu(S_{k'}^{(n)}) = 1$, namely

$$S_{k'}^{(n)} = B(j_1,n) \cup \ldots \cup B(j_s,n) \ \Rightarrow\ S_k^{(m)} = B(j_1,m) \cup \ldots \cup B(j_s,m).$$

For $S_k^{(m)}$ and $S_{k'}^{(n)}$ are characterized as consisting of those congruence classes $\bmod\, m$, n respectively, whose μ-measures are positive.

In Section 4 we reverse this analysis and show that the ergodic sets can be linked together to form finitely many projective systems, each system giving rise to an ergodic measure on $\widehat{\mathbb{Z}}$ satisfying (4).

We also show that apart from a set of zero σ-measure, all trajectories starting from a transient class $\bmod\, m$ eventually enter the same ergodic set $\bmod\, m$. Also the ergodic theorem tells us that if S is an ergodic set $\bmod\, m$ with corresponding measure μ, then almost all (in the μ-measure sense) trajectories in $\widehat{\mathbb{Z}}$ will enter a given congruence class $B(j,m) \subseteq S$ with limiting frequency given by $\mu(B(j,m))$.

Our interest in ergodic sets and measures arose from computer investigations of divergent integral trajectories, where it appears that such trajectories always have the ergodic properties mentioned above.

3. Determination of the ergodic sets. Let $\mathcal{N}_1$ be the set of positive integers composed of primes which divide at least one m_i, and let $\mathcal{N}_2$ be the set of positive integers which are relatively prime to each m_i.

Also, in the notation of [2, Theorem], for $0 \le i < j \le d-1$ let

$$\Delta_{i,j} = r_j(d - m_i) - r_i(d - m_j)$$

and $\Delta = \gcd_{0\le i<j\le d-1} \Delta_{i,j}$. Moreover, let $S_1^{(m)}, \ldots, S_{r(m)}^{(m)}$ be the ergodic sets mod m. Then we know from the main theorem of [2] that

(i) If $m \in \mathcal{N}_2$ and $\gcd(m, \Delta) = 1$, then $r(m) = 1$ and $S_1(m) = \widehat{\mathbb{Z}}$; while if $\gcd(m, \Delta) = \delta > 1$, then $r(m) = r(\delta)$ and the ergodic sets mod m are the ergodic sets mod δ.

(ii) If $m \in \mathcal{N}_1$, then $r(m) = 1$.

We remark that an ergodic set mod m can split into several ergodic sets mod n, if $m \mid n \mid \Delta$. For example, the mapping $T(x) = x/2 + 12$ if x is even, $T(x) = (3x+1)/2$ if x is odd, has the property that $\Delta = 25$; and using least non-negative representatives mod m to denote congruence classes mod m, we find 4 is an ergodic set mod 5 and splits into two ergodic sets mod 25, namely 24 and 4, 9, 14, 19.

Theorem 3.1. *The following are all the ergodic sets*:

(a) $\widehat{\mathbb{Z}}$;

(b) $S_1^{(m)}, \ldots, S_{r(m)}^{(m)}$, *where* $m \mid \Delta$, $m \in \mathcal{N}_2$;

(c) $S_1^{(m)}$, *where* $m \in \mathcal{N}_1$;

(d) *any intersection of a set of type* (b) *and one of type* (c).

Problem 3.1. There may be infinitely many ergodic sets of type (c) and it would be of interest to classify such mappings T. Consider for example, the mapping $T(x) = 3x/2$ if x is even, $T(x) = (3x+1)/2$ if x is odd. Here $\mathcal{N}_1$ consists of the powers of 3. There are infinitely many ergodic sets, namely the sets $T^n(\widehat{\mathbb{Z}})$, each being composed of 2^n congruence classes mod 3^n and $\sigma(T^n(\widehat{\mathbb{Z}})) = (2/3)^n$. (See [2, Example 1.3].)

Similarly, for the mapping $T(x) = 4x/3$ if $3 \mid x$, $T(x) = (4x - 1)/3$ if $3 \mid (x - 1)$ and $T(x) = (2x - 1)/3$ if $3 \mid (x - 2)$. Here $\mathcal{N}_1$ consists of the powers of 2. Again the sets $T^n(\widehat{\mathbb{Z}})$ are the ergodic sets, but here

$$T^n(\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}} \setminus \bigcup_{i=1}^{n} B(2^{2i-1}, 4^i)$$

and $\sigma(T^n(\widehat{\mathbb{Z}})) = (2 + 2^{-2n})/3$.

These and other examples suggest that there are infinitely many ergodic sets if and only if $T(\mathbb{Z}) \ne \mathbb{Z}$.

Theorem 3.1 is a consequence of the following corrected version of Conjecture 2 of [2], which we can now prove:

LEMMA 3.1. *If* S *and* S' *are ergodic sets* $\bmod m$ *and* $\bmod m'$, *respectively, where* $m \in \mathcal{N}_1$ *and* $m' \in \mathcal{N}_2$, *then* $S \cap S'$ *is an ergodic set* $\bmod mm'$. *More explicitly*:

(i) *If* $M_{m'}(S')$ *is primitive, so is* $M_{mm'}(S \cap S')$.

(ii) *If* $M_{m'}(S')$ *is periodic with period* t, *so is* $M_{mm'}(S \cap S')$. *Moreover, in the cyclic normal form of* $M_{mm'}(S \cap S')$ (see [5, Lemma 3.5]), *all blocks are square and of the same size.*

Remark 3.1. By virtue of the second part of (ii) above, as observed in [5, Corollary 3.6], we can replace the Cesàro limit in (4) by the usual limit.

Remark 3.2. The structure of the ergodic sets $S_j^{(m)}$, $m \mid \Delta$, can be quite complicated. For example, let $T(x) = x/2+17$ if x is even, $T(x) = (3x+1)/2$ if x is odd. Then T^2 has the property that $\Delta = 35$. Also $r(5) = 2 = r(7)$ and $r(35) = 5$. Using least non-negative representatives, the following are the ergodic sets mod 5, 7 and 35:

$$\begin{aligned}
S_1^{(5)} &: 0,1,2,3; \qquad S_2^{(5)} : 4;\\
S_1^{(7)} &: 0,1,2,3,4,5; \qquad S_2^{(7)} : 6;\\
S_1^{(35)} &: 0,2,3,8,10,11,12,15,16,26,28,32;\\
S_2^{(35)} &: 1,5,7,17,18,21,22,23,25,30,31,32;\\
S_3^{(35)} &: 4,9,14,19,24,29; \qquad S_4^{(35)} : 6,13,20,27; \qquad S_5^{(35)} : 34\,.
\end{aligned}$$

Moreover, $S_1^{(5)} \cap S_1^{(7)} = S_1^{(35)} \cup S_2^{(35)}$, a union of two ergodic sets mod 35.

The proofs of Lemma 3.1 and part (i) follow along the lines of the argument of [2, Example 4.1, page 55] from the following result:

LEMMA 3.2. *Under the conditions of Lemma* 3.1, *there exists a* $K = K(S)$ *such that if* $B(j, mm') \subseteq S \cap S'$, *then there exists a* $B(j', m') \subseteq S'$ *for which*

$$T^{-K}(B(j, mm')) \supseteq B(j', d^K m')\,. \tag{5}$$

Proof. To find $T^{-1}(B(j,n))$, we have to solve the congruence

$$\frac{m_i x - r_i}{d} \equiv j \pmod{n} \tag{6}$$

for $i = 0, \ldots, d-1$. If $d_i = \gcd(m_i, n) > 1$, then $T^{-1}(B(j,n))$ contains a congruence class of the form $B(j', nd/d_i)$.

Now let $B(j,m) \subseteq S$. Then we assert that there exists a $K \geq 1$ such that $T^{-K}(B(j,m))$ contains a congruence class of the form $B(j', d^K)$. For otherwise $\exists K_0$ such that for $K \geq K_0$, $T^{-K}(B(j,m))$ consists wholly of congruence classes $B(j', nd^K)$, where n is divisible by a prime dividing an m_i. Then attempts to solve (6), with n replaced by nd^K, will either give $d_i = \gcd(m_i, nd^K) = \gcd(m_i, n) = 1$, in which case there is one solution $\bmod\, md^{K+1}$, or $d_i \nmid dj + r_i$, in which case there is no solution. Hence $T^{-1}(B(j', nd^K))$ consists of at most $d-1$ congruence classes $\bmod\, nd^{K+1}$, as m certainly contains at least one prime dividing an m_i and for which $p \mid \gcd(m_i, n)$.

Hence

$$\frac{\mathrm{card}_{md^K}\{T^{-K}(B(j,m))\}}{d^K} \leq \frac{\mathrm{card}_{md^{K_0}}\{T^{-K_0}(B(j,m))\}}{d^{K_0}} \left(\frac{d-1}{d}\right)^{K-K_0}$$

if $K \geq K_0$. However, this implies that $\mu_S(B(j,m)) = 0$, contradicting the assumption that $B(j,m) \subseteq S$.

The more general case of $T^{-K}(B(j,mm'))$ then follows. For if we have $T^{-K}(B(j,m)) \supseteq B(j', d^K)$, there will be a sequence of congruences of type (6) with $n = mm'd^K$, $0 \leq k \leq K-1$. Now as $\gcd(m_i, nm'd^K) = \gcd(m_i, nd^K)$ and (6) has a solution of the form

$$x \equiv \left(\frac{m_i}{\gcd(m_i,n)}\right)^{-1}(dj + r_i) \left(\bmod\ \frac{n}{\gcd(m_i,n)}\right), \tag{7}$$

we can choose the inverse in (7), not just $\bmod\, n$, but $\bmod\, nm'$, thereby deriving a corresponding sequence of congruences, which have the effect of removing any primes dividing some m_i from the starting modulus mm'.

Part (ii) is any easy exercise in set theory, in conjunction with a reduction of the problem to the primitive case, as in the proof of [2, Lemma 3.3].

If $m \mid n$, each ergodic set $S_i^{(n)}$ mod n is contained in exactly one ergodic set $S_j^{(m)}$ mod m. The next corollary describes a precise relation between these sets:

Corollary 3.1. *If $m \mid n$ and $S = \bigcup_{k=1}^t B(i_k, n)$ is an ergodic set* $\bmod\, n$ *and $\Phi_{n,m}(S) = \bigcup_{k=1}^t B(i_k, m)$, then $\Phi_{n,m}(S)$ is an ergodic set* $\bmod\, m$.

Proof. This divides naturally into several cases. We write $m = MM'$, $n = NN'$, where $M, N \in \mathcal{N}_1$ and $M', N' \in \mathcal{N}_2$ with $M \mid N$, $M' \mid N'$.

(i) $m = M'$, $n = N'$. This is straightforward and uses Lemmas 2.7 and 3.5 of [2]. For by an examination of the orbit nature of equivalence classes of $Q(M')$ and $Q(N')$, it is easy to prove that if an ergodic set S' mod M' splits into a union $S_1 \cup \ldots \cup S_t$ of ergodic sets $\bmod\, N'$, then each S_i is intersected by every congruence class in S' in the same number of congruence classes $\bmod N'$.

(ii) $m = M$, $n = N$. This was Remark 2.1 in [2].

(iii) The remaining cases use Lemma 3.1 to reduce the problem to cases (i) and (ii).

Remark 3.3. From Corollary 3.1 and Theorem 3.1, it follows that the ergodic sets $\bmod m$ may be linked together as m varies, to form $r(\Delta)$ disjoint projective systems $\mathcal{D} = \{S_{j_m}^{(m)}\}$, where $m \mid n$ implies $\Phi_{n,m}(S_{j_n}^{(n)}) = S_{j_m}^{(m)}$.

Example 3.1. The mapping $T(x) = x/2$ if x is even, $T(x) = (5x-3)/2$ if x is odd (Example 1.2 of [2]). Here $\Delta = 3$ and there are finitely many ergodic sets:

(i) $S_1^{(m)} = \widehat{\mathbb{Z}}$ if $\gcd(m, 15) = 1$;
(ii) $S_1^{(m)} = 3\widehat{\mathbb{Z}}$ and $S_2^{(m)} = \widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}$ if $3 \mid m$ and $5 \nmid m$;
(iii) $S_1^{(m)} = \widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}$ if $3 \nmid m$ and $5 \mid m$;
(iv) $S_1^{(m)} = 3\widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}$ and $S_2^{(m)} = (\widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}) \setminus 5\widehat{\mathbb{Z}}$ if $15 \mid m$.

There are two projective systems of ergodic sets. We have, for example,

(a) $\Phi_{15,5}(3\widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}$; (b) $\Phi_{15,5}((\widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}) \setminus 5\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}$;

(c) $\Phi_{15,3}(3\widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}) = 3\widehat{\mathbb{Z}}$; (d) $\Phi_{15,3}((\widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}) \setminus 5\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}$.

Example 3.2. The mapping $T(x) = 7x/2$ if x is even, $T(x) = (7x+3)/2$ if x is odd. Here $\Delta = 3$ and there are infinitely many ergodic sets:

(i) $S_1^{(m)} = \widehat{\mathbb{Z}}$ if $\gcd(m, 21) = 1$;
(ii) $S_1^{(m)} = 3\widehat{\mathbb{Z}}$ and $S_2^{(m)} = \widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}$ if $3 \mid m$ and $7 \nmid m$;
(iii) $S_1^{(m)} = T^t(\widehat{\mathbb{Z}})$ if $m = 7^t n$ and $\gcd(21, n) = 1$;
(iv) $S_1^{(m)} = S_1^{(7^t)} \cap 3\widehat{\mathbb{Z}}$ and $S_2^{(m)} = S_1^{(7^t)} \cap (\widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}})$ if $m = 7^t n$, $3 \mid n$, $7 \nmid n$.

Here $M_m(S_1^{(m)})$ is primitive, whereas $M_m(S_2^{(m)})$ is periodic of order 2. Again there are two projective systems of ergodic sets.

4. Construction of ergodic measures on $\widehat{\mathbb{Z}}$. Let $\mathcal{B}(m)$ denote the σ-algebra generated by all congruence classes $B(j, l)$, where $l \mid m$. If $\mathcal{D}$ is a projective system of ergodic sets and $S \in \mathcal{D}$ is an ergodic set $\bmod m$, then (4) defines a measure μ_S on $\mathcal{B}(m)$:

$$\mu_S(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{K \le N} \frac{\sigma(T^{-K}(A) \cap S)}{\sigma(S)}\,. \tag{8}$$

Remark 4.1. We will have occasion to use the fact that in (8), S can be replaced by any subset consisting of one or more congruence classes $\bmod m$ contained in S.

The next result shows that each of the $r(\Delta)$ families of probability measures μ_S defined by (8) is consistent:

LEMMA 4.1. *If* $m \mid n$, S *is an ergodic set* $\bmod n$ *and* $A \in \mathcal{B}(m)$, *then*

$$\mu_{\Phi_{n,m}(S)}(A) = \mu_S(A)\,. \tag{9}$$

Proof. This divides naturally into several cases. We write $m = MM'$, $n = NN'$, where $M, N \in \mathcal{N}_1$, $M', N' \in \mathcal{N}_2$ and $M \mid N$, $M' \mid N'$. Let $S' = \Phi_{n,m}(S)$ and assume $A = B(j,m)$.

(i) $m = M'$, $n = N'$. Here $Q(m)$ and $Q(n)$ are doubly stochastic and $\mu_S(B(j,n)) = 1/\operatorname{card}_n(S)$ and $\mu_{S'}(B(j,m)) = 1/\operatorname{card}_m(S')$. Then case (i) of the proof of Corollary 3.1 gives the desired result. For each member of S' intersects S in the same number r of congruence classes $\bmod n$ and $B(j,m)$ is the union of such classes. Hence

$$\mu_{S'}(B(j,m)) = \frac{r}{\operatorname{card}_n(S)} = \frac{1}{\operatorname{card}_m(S')}\,,$$

as S is the union of rt congruence classes $\bmod n$, where $t = \operatorname{card}_m(S')$ and hence $\operatorname{card}_m(S) = rt$.

(ii) $m = M$, $n = N$. Here $S' = S_0 \cup S$, where S_0 consists of the transient classes $\bmod n$. Then if $B(j,m) \subseteq S'$, we have $B(j,m) = B_0 \cup B$, where $B_0 = B(j,m) \cap S_0$ is composed of transient classes $\bmod n$ and $B = B(j,m) \cap S$.

Now by [5, Lemma 3.3], $\{Q(n)\}^K$ tends to a matrix whose columns are identical and where the rows corresponding to transient classes are zero. Then from (3), we have

$$\begin{aligned}
\mu_{S'}(B(j,m)) &= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\operatorname{card}_m(T^{-K}(B(j,m)) \cap B(j,m))}{d^K} \\
&= \frac{1}{\frac{n}{m}} \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\operatorname{card}_n(T^{-K}(B_0 \cup B) \cap (B_0 \cup B))}{d^K} \\
&= \frac{1}{\frac{n}{m}} \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\operatorname{card}_n(T^{-K}(B) \cap (B_0 \cup B))}{d^K} \\
&= \frac{1}{\frac{n}{m}} \frac{n}{m} \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\operatorname{card}_n(T^{-K}(B) \cap B(k,n))}{d^K}\,, \\
&\qquad\qquad B(k,n) \subseteq S\,, \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\operatorname{card}_n(T^{-K}(B_0 \cup B) \cap B(k,n))}{d^K} \\
&= \mu_S(B(j,m))\,.
\end{aligned}$$

(iii) The remaining cases use Lemma 3.1 to reduce the problem to cases (i) and (ii).

Remark 4.2. Because $\bigcup \mathcal{B}(m)$ generates the Borel σ-algebra on $\widehat{\mathbb{Z}}$, corresponding to each projective system $\mathcal{D}_i$, $i = 1, \ldots, r(\Delta)$ of ergodic sets, we can define a probability measure μ_i on $\widehat{\mathbb{Z}}$, using a version of the Kolmogorov extension theorem in [6, page 143]. We now give some properties of these measures.

LEMMA 4.2. *$\mu_i \in \mathcal{M}'_T(\widehat{\mathbb{Z}})$ for $i = 1, \ldots, r(\Delta)$.*

Proof. We have to prove

$$\mu_S(B(j, md)) = \frac{1}{d}\mu_{\Phi_{md,m}(S)}(B(j, m))$$

if $S \in \mathcal{D}_i$ is an ergodic set mod md. By [2, Lemma 2.7] we have $\Phi_{md,m}(S) = S$. Hence

$$\mu_S(B(j, md)) = \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\mathrm{card}_{md^{K+1}}\{T^{-K}(B(j, md)) \cap S\}}{\mathrm{card}_{md^{K+1}}(S)} \tag{10}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sum_{k=1}^t p_{Kjj_k}(md, m)}{\mathrm{card}_{md^{K+1}}(S)} \,.$$

Here $S = \bigcup_{i=1}^t B(j_i, m)$ and $p_{Kjl}(n, m) = \mathrm{card}_{nd^k}(T^{-K}(B(j, n)) \cap B(l, m))$, where $m \mid n$.

Now the proof of [5, Lemma 2.8] shows that

$$p_{Kjl}(mm', m) = p_{Kjl}(m, m) \tag{11}$$

if $\gcd(m', m_i) = 1$ for $i = 0, \ldots, d-1$. Hence (10) becomes

$$\mu_S(B(j, md)) = \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sum_{k=1}^t p_{Kjj_k}(m, m)}{\mathrm{card}_{md^{K+1}}(S)}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sum_{k=1}^t p_{Kjj_k}(m, m)}{d\,\mathrm{card}_{md^K}(S)}$$

$$= \frac{1}{d} \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\mathrm{card}_{md^K}(T^{-K}(B(j, m) \cap S)}{\mathrm{card}_{md^K}(S)}$$

$$= \frac{1}{d}\mu_S(B(j, m))\,.$$

Finally, $\mu_i(T^{-1}(A)) = \mu_i(A)$ holds if $A \in \mathcal{B}(m)$ and hence by [1, Theorem 1.1, page 4], it also holds if $A \in \mathcal{B}(\widehat{\mathbb{Z}})$.

Lemma 4.2 is a special case of a more general result, which reduces the calculation of $\mu_S(B(j, m))$ to the case where $m \in \mathcal{N}_1$:

LEMMA 4.3. *If S is an ergodic set* $\bmod mm'$, *where* $m' \in \mathcal{N}_2$ *and* $B(j, mm') \subseteq S$, *then*

$$\mu_S(B(j, mm')) = \mu_{\Phi_{mm',m}(S)}(B(j, m))/r\,, \tag{12}$$

where $r = \mathrm{card}_{mm'}(S)/\,\mathrm{card}_m(\Phi_{mm',m}(S))$.

Proof. We have

$$\mu_S(B(j, mm')) = \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\mathrm{card}_{mm'd^K}\{T^{-K}(B(j, mm')) \cap S\}}{\mathrm{card}_{mm'd^K}(S)}\,. \tag{13}$$

Let $\Phi_{mm',m}(S) = S'$. Then, in part using Lemma 3.1, we deduce that S' is a union $S_1 \cup \ldots \cup S_t$ of ergodic sets $\bmod mm'$, where $S_1 = S$. Also as $B(j, mm') \subseteq S_1$, we have $T^{-K}(B(j, mm')) \cap S_i = \emptyset$ for $i = 2, \ldots, t$. Hence

$$T^{-K}(B(j, mm') \cap S_1) = T^{-K}(B(j, mm') \cap S'\,.$$

Then in view of (11), (13) gives

$$\begin{aligned}
\mu_S(B(j, mm')) &= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sum_{B(l,m)\subseteq S'} p_{Kjl}(mm', m)}{\mathrm{card}_{mm'd^K}(S_1)} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sum_{B(l,m)\subseteq S'} p_{Kjl}(m, m)}{\dfrac{\mathrm{card}_{mm'}(S_1)}{\mathrm{card}_m(S')}\,\mathrm{card}_{md^K}(S')} \\
&= \frac{1}{r}\mu_{S'}(B(j, m))\,.
\end{aligned}$$

Other simple properties of our measures μ_i follow from (8):

LEMMA 4.4. (a) $\mu_S(S) = 1$ *if S is an ergodic set* $\bmod m$.
(b) *If $A \in \mathcal{B}(m)$ and $A \cap S = \emptyset$, then $\mu_S(A) = 0$.*

Properties of the irreducible Markov submatrix corresponding to an ergodic set $\bmod m$ imply

LEMMA 4.5. *Each μ_i is ergodic with respect to T.*

Proof. By [9, Theorem 1.4, page 17], it suffices to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{K=0}^{N} \mu_S(A \cap T^{-K}(B)) = \mu_S(A)\mu_S(B) \tag{14}$$

if $A = B(j, m)$ and $B = B(k, m)$.

If A is a transient class, then both sides are zero by [3, Theorem 4(I), page 31]. So we assume $A \subseteq S$. By Definition 8, we have to prove that

$$\lim_{M\to\infty} \frac{1}{M} \sum_{L=0}^{M} \lim_{N\to\infty} \frac{1}{N} \sum_{K\le N} \frac{\sigma(T^{-K}(A \cap T^{-L}(B)))}{\sigma(S)} = \mu_S(A)\mu_S(B)\,.$$

This follows from $T^{-K}(A \cap T^{-L}(B)) = T^{-K}(A) \cap T^{-(K+L)}(B)$ and by replacing S by $T^{-K}(A) \cap S$, using Remark 4.1.

Finally, ergodic sets have an attracting property.

LEMMA 4.6. *Except for a set of zero σ-measure, all trajectories starting in a transient class* $\bmod m$ *will enter an ergodic set* $\bmod m$.

Remark 4.3. We can be more explicit: if there is more than one ergodic set $\bmod m$, $m = m_1 m_1'$, where $m_1 \in \mathcal{N}_1$ and $m_1' \in \mathcal{N}_2$, then by Theorem 3.1, each transient class has the form $B(j,m) = B(j,m_1) \cap B(k,m_1')$, where $B(j,m_1)$ is a transient class and $B(k,m_1')$ is contained in an ergodic set S_k' mod m_1'. Consequently, almost all trajectories starting in $B(j,m)$ will eventually enter the ergodic set $S_k = S \cap S_k'$, where S is the unique ergodic set $\bmod m_1$.

For example in Example 3.1 above, almost all trajectories starting in $B(0,15)$ will enter $3\widehat{\mathbb{Z}} \setminus 5\widehat{\mathbb{Z}}$, while almost all starting in $B(5,15)$ or $B(10,15)$ enter $(\widehat{\mathbb{Z}} \setminus 3\widehat{\mathbb{Z}}) \setminus 5\widehat{\mathbb{Z}}$.

Proof. Let $S_1, \ldots, S_{r(m)}$ be the ergodic sets $\bmod m$ and let S_0 denote the union of the transient classes. Then if $B(j,m)$ is a transient class $\bmod m$, noting that $T^{-(k+1)}(S_0) \subseteq T^{-k}(S_0)$, we have (see [3, Theorem 4(I), page 31])

$$\begin{aligned}
\sigma(x \in B(j,m) : \forall K \geq 0,\ T^K(x) \in S_0) &= \sigma\Big(\bigcap_{K\geq 0} T^{-K}(S_0) \cap B(j,m)\Big) \\
&= \lim_{K\to\infty} \sigma(T^{-K}(S_0) \cap B(j,m)) \\
&= \lim_{K\to\infty} \sum_{B(i,m)\subseteq S_0} \frac{p_{Kij}(m)}{md^K} = 0\,.
\end{aligned}$$

In conclusion, the author would like to express his gratitude to Drs. Phil Pollett and Bob Buttsworth for helpful conversations.

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Reçu par la Rédaction le 7.12.1990;
en version modifiée le 11.1.1991