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OMNIPRESENT HOLOMORPHIC OPERATORS AND MAXIMAL CLUSTER SETS

BY

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1. Introduction and notations. Throughout this paper Ω will stand for a nonempty open set in the complex plane $\mathbb{C}$. $\widehat{\mathbb{C}}$ is the extended complex plane, $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $B(a,r)$ is the ball $\{z \in \mathbb{C} : |z-a| < r\}$ ($a \in \mathbb{C}$, $r > 0$), $\partial\Omega$ is the boundary of Ω in $\widehat{\mathbb{C}}$. $O(\partial\Omega)$ denotes the set of open subsets $V \subset \mathbb{C}$ such that $V \cap \partial\Omega$ is not empty. $H(\Omega)$ denotes, as usual, the linear space of holomorphic functions on Ω, endowed with the topology τ of uniform convergence on each compact subset in Ω. Let $\mathcal{K}$ be the family of compact subsets of Ω. It is known that the family

$$\{D(f,K,\varepsilon) : f \in H(\Omega),\ K \in \mathcal{K},\ \varepsilon > 0\},$$

where

$$D(f,K,\varepsilon) = \{g \in H(\Omega) : |g(z) - f(z)| < \varepsilon \text{ for all } z \in K\},$$

is a basis for τ.

If $f \in H(\Omega)$ and $j \in \mathbb{N}_0$ we denote, as usual, by $f^{(j)}$ the derivative of f of order j. The linear operator $T_j : H(\Omega) \to H(\Omega)$ defined by $T_j f = f^{(j)}$ is continuous. If $j \in \mathbb{N}$ and Ω has simply connected components Ω_i ($i \in I$), then we use the abbreviation $f^{(-j)}$ for an (arbitrary but fixed) antiderivative of f of order j, i.e., we have $(f^{(-j)})^{(j)} = f$. Any other antiderivative of f of order j differs from $f^{(-j)}$ on each Ω_i by a certain polynomial of degree less than j. Fix one point z_i in each Ω_i. We define the linear operator S_j on $H(\Omega)$ as follows. $S_j f$ is the unique j-antiderivative of f on Ω such that

$$(S_j f)^{(k)}(z_i) = 0 \quad (k \in \{0,1,\ldots,j-1\},\ i \in I).$$

Each S_j is continuous. In fact, we have

$$S_j f(z) = \int_{z_i}^{z} f(t)((z-t)^{j-1}/(j-1)!)\,dt \quad (z \in \Omega_i,\ i \in I)$$

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where the integral is taken along any rectifiable curve $\gamma \subset \Omega_i$ joining z_i to z.

A topological space X is a *Baire space* if and only if the intersection of a countable family of open dense subsets is also dense. Baire's Theorem asserts that each completely metrizable topological space is a Baire space. Consequently, $H(\Omega)$ is a Baire space. In a Baire space X, a subset is *residual* when it contains a dense G_δ subset of X; such a subset is "very large" in X. If P is a property on X, we say that $P(x)$ is true for *Baire-almost all* $x \in X$ whenever the subset $\{x \in X : P(x) \text{ is true}\}$ is residual. These notions and results can be found, for instance, in [2, pp. 213–214] and [6, pp. 40–41].

Let $\xi \in \partial\Omega$ and $f \in H(\Omega)$. The *cluster set* $S(f,\xi)$ is defined as the set of all $w \in \widehat{\mathbb{C}}$ for which there exists a sequence of points $\{z_n : n \in \mathbb{N}\}$ such that $z_n \in \Omega$ for all $n \in \mathbb{N}$ and $z_n \to \xi$, $f(z_n) \to w$ as $n \to \infty$. Equivalently,

$$S(f,\xi) = \bigcap\{\mathrm{clos}(f(\Omega \cap V)) : V \text{ is a neighborhood of } \xi\}.$$

Since $\widehat{\mathbb{C}}$ is compact, $S(f,\xi)$ is a nonempty compact set. $S(f,\xi)$ is said to be *maximal* when $S(f,\xi) = \widehat{\mathbb{C}}$ or, equivalently, $S(f,\xi) \supset \mathbb{C}$. For instance, if ξ is an essential singularity of f then the corresponding cluster set is maximal.

Define

$$M(\Omega) = \{f \in H(\Omega) : S(f,\xi) \text{ is maximal for all } \xi \in \partial\Omega\}.$$

It is evident that if Γ is a dense subset of $\partial\Omega$, then $f \in M(\Omega)$ if and only if $S(f,\xi)$ is maximal for all $\xi \in \Gamma$.

In this paper, we introduce the "omnipresent operators" on $H(\Omega)$, in connection with the boundary behavior of holomorphic functions. The derivative and antiderivative operators appear as outstanding examples. It happens that Baire-almost all functions have a chaotic behavior near each boundary point, together with all their derivatives and antiderivatives.

Several authors have published results of similar type. A summary of classical results on cluster sets can be found, for instance, in [1] and [5]. Nevertheless, we point out one of the most interesting, due to Kierst and Szpilrajn (see [3], [4] and also [8, p. 179]), namely: all functions holomorphic in the unit ball $B(0,1)$ except a set of functions of the first category assume all finite complex values infinitely many times in every sector of this ball.

The following elementary lemma is needed only to prove Theorem 4.

LEMMA. *Assume $f \in H(\Omega)$ and g is a polynomial. Then*:

(a) *If $\Omega = \bigcup\{\Omega_i : i \in I\}$, where $I \subset \mathbb{N}$, each Ω_i is a nonempty open set, the family $\{\Omega_i : i \in I\}$ is pairwise disjoint and $f \in M(\Omega_i)$ $(i \in I)$, then $f \in M(\Omega)$.*

(b) *If $f \in M(\Omega)$, then $f + g \in M(\Omega)$.*

Proof. (a) Fix $w \in \mathbb{C}$ and let $\xi \in \partial\Omega$. We first suppose that there exists an $i_0 \in I$ such that $\xi \in \partial\Omega_{i_0}$. Then there is a sequence $\{z_n : n \in \mathbb{N}\} \subset \Omega_{i_0}$

with $z_n \to \xi$ and $f(z_n) \to w$ $(n \to \infty)$. Hence $w \in S(f,\xi)$, because $\Omega_{i_0} \subset \Omega$. Otherwise, if $\xi \notin \partial\Omega_i$ for all $i \in I$ then there must be a sequence $\{i_k : k \in \mathbb{N}\}$ and boundary points $\xi_k \in \partial\Omega_{i_k}$ with $\xi_k \to \xi$ $(k \to \infty)$. For each $k \in \mathbb{N}$ choose a sequence $\{z_n^{(k)} : n \in \mathbb{N}\} \subset \Omega_{i_k}$ with $z_n^{(k)} \to \xi_k$ and $f(z_n^{(k)}) \to w$ $(n \to \infty)$. For fixed $k \in \mathbb{N}$ choose an integer $n_k > k$ such that $\chi(t_k, \xi_k) < 1/k$, where $t_k = z_{n_k}^{(k)}$ and χ denotes the chordal distance on $\widehat{\mathbb{C}}$. Then $\{t_k : k \in \mathbb{N}\} \subset \Omega$, $t_k \to \xi$ and $f(t_k) \to w$ $(k \to \infty)$, so $w \in S(f,\xi)$. Thus $S(f,\xi) \supset \mathbb{C}$ in both cases, so $f \in M(\Omega)$.

(b) Fix $w \in \mathbb{C}$ and let $\xi \in \partial\Omega$. If ξ is finite, there exists a sequence $\{z_n : n \in \mathbb{N}\} \subset \Omega$ with $z_n \to \xi$ and $f(z_n) \to w - g(\xi)$ $(n \to \infty)$, whence $f(z_n) + g(z_n) \to w$ $(n \to \infty)$ and $\mathbb{C} \subset S(f+g,\xi)$. Now, assume that $\infty \in \partial\Omega$. If ∞ is isolated in $\partial\Omega$, then it is an essential singularity of f, and so of $f+g$. Then $S(f+g,\infty)$ is maximal by the Casorati–Weierstrass Theorem. If ∞ is not isolated in $\partial\Omega$, then $\Gamma = \partial\Omega \setminus \{\infty\}$ is dense in $\partial\Omega$. But $S(f+g,\xi)$ is maximal for all $\xi \in \Gamma$. Hence $f+g \in M(\Omega)$. ■

2. Omnipresent holomorphic operators. Let $T : H(\Omega) \to H(\Omega)$ be a continuous operator. T need not be linear. If $V \in O(\partial\Omega)$ and $W \subset \mathbb{C}$ is a nonempty set, we define

$$R(T,V,W) = \{f \in H(\Omega) : \text{ there exists } z \in \Omega \cap V \text{ such that } Tf(z) \in W\}.$$

We say that T is *omnipresent* if each $R(T,V,W)$ is dense in $H(\Omega)$. Note that each $R(T,V,W)$ is open in $H(\Omega)$ and, trivially, $R(T,V,W) \subset R(T,V',W')$ whenever $V \subset V'$ and $W \subset W'$. The following theorem gives an alternative definition of omnipresent operators and two immediate properties of them.

THEOREM 1. *Let T and T_j $(j \in \mathbb{N})$ be continuous operators on $H(\Omega)$. Assume that g is a nonconstant entire function. Then*:

(a) *T is omnipresent if and only if Baire-almost all functions $f \in H(\Omega)$ have the property that Tf has maximal cluster set at any boundary point of Ω.*

(b) *If all T_j $(j \in \mathbb{N})$ are omnipresent, then for Baire-almost all $f \in H(\Omega)$ each $T_j f$ $(j \in \mathbb{N})$ has maximal cluster set at any boundary point of Ω.*

(c) *If T is omnipresent, then the operator S on $H(\Omega)$ defined by*

$$Sf(z) = g(Tf(z))$$

for $z \in \Omega$ is also omnipresent.

Proof. (a) Set

$$Q(T) = \{f \in H(\Omega) : Tf \in M(\Omega)\}.$$

Choose a countable dense subset $\{\xi_k : k \in M\}$ in $\partial\Omega$ ($M \subset \mathbb{N}$), a countable basis $\{W_m : m \in \mathbb{N}\}$ for the natural topology of $\mathbb{C}$ and a countable fundamental system of open neighborhoods $\{N(k,l) : l \in \mathbb{N}\}$ for each ξ_k. Evidently,

$$Q(T) = \bigcap\{R(T, N(k,l), W_m) : k \in M,\ l \in \mathbb{N},\ m \in \mathbb{N}\}\,.$$

We have used the fact that cluster sets are closed. If T is omnipresent, then every subset $R(T, N(k,l), W_m)$ is open and dense. Thus $Q(T)$ is dense and G_δ in $H(\Omega)$, so residual. Conversely, suppose $Q(T)$ is residual and fix $V \in O(\partial\Omega)$ and a nonempty open subset $W \subset \mathbb{C}$. There exist $k \in M$, $l \in \mathbb{N}$, $m \in \mathbb{N}$ such that $N(k,l) \subset V$ and $W_m \subset W$. Then $R(T, N(k,l), W_m)$ is residual and contained in $R(T,V,W)$. Hence $R(T,V,W)$ is also residual, and therefore dense.

(b) We have to prove that Q is residual, where

$$Q = \{f \in H(\Omega) : T_j f \in M(\Omega) \text{ for all } j \in \mathbb{N}\}\,.$$

But this is trivial, because every $Q(T_j)$ is residual and, evidently, $Q = \bigcap\{Q(T_j) : j \in \mathbb{N}\}$.

(c) Let V and W be as above. Since $g(\mathbb{C})$ is dense, the set $g^{-1}(W)$ is open and nonempty. In addition, $R(S,V,W) \supset R(T,V,g^{-1}(W))$. Thus $R(S,V,W)$ is dense. ■

The next results furnish two outstanding examples of this kind of operators.

THEOREM 2. (a) *Every j-derivative operator on $H(\Omega)$ ($j \in \mathbb{N}_0$) is omnipresent.*

(b) *For Baire-almost all $f \in H(\Omega)$ each derivative $f^{(j)}$ ($j \in \mathbb{N}_0$) has maximal cluster set at any boundary point of Ω.*

Proof. Part (b) is immediate from (a) and Theorem 1(b). For (a), fix $j \in \mathbb{N}_0$ and consider the operator $Tf = f^{(j)}$. Let V, W be as above, $\varepsilon > 0$, $f \in H(\Omega)$ and $K \in \mathcal{K}$. Fix a ball $B(w,\delta) \subset W$. Choose $L \in \mathcal{K}$ such that $L \supset K$ and such that each connected component of $\widehat{\mathbb{C}} \setminus K$ contains some connected component of $\widehat{\mathbb{C}} \setminus \Omega$, i.e., such that each "hole" of K contains at least one "hole" of Ω. Choose a closed ball $\overline{B}(a,r) \subset V \cap \Omega \setminus L$. There are open sets G_1, G_2 in Ω such that $L \subset G_1$, $\overline{B}(a,r) \subset G_2$ and $G_1 \cap G_2 = \emptyset$. If $S = L \cup \overline{B}(a,r)$ and $G = G_1 \cup G_2$, then S is compact, G is open, $S \subset G \subset \Omega$ and each "hole" of S contains at least one "hole" of Ω. Define g on G by

$$g(z) = \begin{cases} f(z) & \text{for } z \in G_1, \\ wz^j/j! & \text{for } z \in G_2. \end{cases}$$

Trivially $g \in H(G)$. By Runge's Theorem [7, p. 288], there exists a rational function h, with poles outside Ω, such that

$$|g(z) - h(z)| < \min(\varepsilon, \delta r^j/j!)$$

for every $z \in S$. Then $h \in H(\Omega)$ and $|f(z) - h(z)| < \varepsilon$ on K. In addition, if $\gamma = \{t : |t - a| = r\}$, Cauchy's formula gives

$$|Th(a) - w| = \left| \frac{j!}{2\pi i} \oint_\gamma \frac{h(t) - wz^j/j!}{(t-a)^{j+1}} dt \right|$$
$$< \frac{j! 2\pi r}{2\pi r^{j+1}} \sup\{|h(t) - g(t)| : t \in \overline{B}(a,r)\} < \delta\,,$$

so $Th(a) \in W$. Consequently, $h \in D(f, K, \varepsilon) \cap R(T, V, W)$, i.e., $R(T, V, W)$ is dense. ■

THEOREM 3. *Let $G \subset \mathbb{C}$ be a simply connected domain, $a \in G$, and let $\varphi : G \times G \to \mathbb{C}$ be a function not identically zero and analytic with respect to both variables. Consider the linear operator T on $H(G)$ defined by*

$$Tf(z) = \int_a^z \varphi(z,t) f(t)\, dt \qquad (z \in G)$$

where the integral is taken along any rectifiable curve in G joining a to z. Then T is omnipresent.

Proof. Firstly, the integral in the statement is evidently well-defined for each $z \in G$. It is a standard exercise to prove that T is continuous.

Secondly, we claim that for each nonempty open subset $U \subset G$ there is a closed ball $\overline{B}(c,r) \subset U$ such that

$$\int_z^c \varphi(c,t)\, dt \neq 0 \qquad \text{for all } z \in \overline{B}(c,r)\,.$$

Otherwise, for each $c \in U$ the set $\{z \in G : \int_z^c \varphi(c,t)\, dt = 0\}$ would have a point of accumulation in G, whence $\int_z^c \varphi(c,t)\, dt$ would be zero for $z \in G$, by the analytic continuation principle. But this yields, by derivation, that $\varphi(c,z) = 0$ for $(c,z) \in U \times G$. By the analytic continuation principle, φ would be identically zero on $G \times G$, which is a contradiction.

Now, we prove that T is omnipresent. For this, fix a compact set $K \subset G$, $V \in O(\partial G)$, a nonempty open set $W \subset \mathbb{C}$, $\varepsilon > 0$, $f \in H(G)$ and a ball $B(w,\delta) \subset W$. It is clear that we can find b, c, r, γ and L satisfying:

(a) L is compact, $\{a\} \cup K \subset L \subset G$ and the complement of L is connected,

(b) $r > 0$ and $\overline{B}(c,r) \subset V \cap G \setminus L$,

(c) γ is a rectifiable Jordan arc in G joining a to c and $\gamma = \gamma_1 \cdot \gamma_2 \cdot \gamma_3$, where γ_k $(k = 1, 2, 3)$ are rectifiable Jordan arcs too, $\gamma_1 = \gamma \cap L$ and $\gamma_3 = \gamma \cap \overline{B}(c,r)$,

(d) b is the initial point of γ_3 and $\lambda \equiv \int_b^c \varphi(c,z)\, dz \neq 0$.

Define the constants α, β, η by

$$\alpha = \sup\{|\varphi(c,z)| : z \in \gamma\}, \qquad \beta = ||f||_\gamma = \sup_\gamma |f|,$$
$$\eta = 2\alpha(1 + \beta + |w/\lambda| + \alpha\beta \operatorname{length}(\gamma)/|\lambda|).$$

Let us partition γ_2 as $\gamma_2 = \gamma_4 \cdot \gamma_5$, where γ_k $(k = 4, 5)$ are rectifiable Jordan arcs and

$$\operatorname{length}(\gamma_5) < \delta/\eta. \tag{1}$$

We can suppose that γ_5 has a parametrization $u \in [0,1] \mapsto \omega(u)$ where ω is injective. Define the constant g_0 by

$$g_0 = \Big(w - \int_{\gamma_1 \cdot \gamma_4} \varphi(c,z) f(z)\, dz\Big)\Big/\lambda. \tag{2}$$

Let $K_0 = L \cup \gamma \cup \overline{B}(c,r)$. Define the function g on K_0 as

$$g(z) = \begin{cases} f(z) & \text{if } z \in L \cup \gamma_4, \\ f(\omega(0))(1-u) + g_0 u & \text{if } z = \omega(u) \in \gamma_5, \\ g_0 & \text{if } z \in \overline{B}(c,r). \end{cases}$$

Then K_0 is a compact set whose complement is connected, g is continuous on K_0 and holomorphic in its interior $K_0^0 = L^0 \cup B(c,r)$. By Mergelyan's Theorem, there is a polynomial P such that

$$|P(z) - g(z)| < \min(\varepsilon, \delta, 1)/(1 + 2\alpha \operatorname{length}(\gamma)) \tag{3}$$

for all $z \in K_0$. We have

$$\begin{aligned} TP(c) - w &= \int_\gamma \varphi(c,z) P(z)\, dz - w \\ &= \int_{\gamma_1 \cdot \gamma_4} \varphi(c,z)(P(z) - g(z))\, dz + \int_{\gamma_1 \cdot \gamma_4} \varphi(c,z) f(z)\, dz \\ &\quad + \int_{\gamma_5} \varphi(c,z) P(z)\, dz + \int_{\gamma_3} \varphi(c,z)(P(z) - g(z))\, dz \\ &\quad + \int_{\gamma_3} g_0 \cdot \varphi(c,z)\, dz - w = I + J, \end{aligned}$$

because of (2), where

$$I = \int_{\gamma_1 \cdot \gamma_4 + \gamma_3} \varphi(c,z)(P(z) - g(z))\, dz,$$
$$J = \int_{\gamma_5} \varphi(c,z) P(z)\, dz.$$

Inequality (3) yields

$$|I| < \delta/2\,.$$

From (3) and the definition of g, it follows that

$$\begin{aligned}|P(z)| &< 1 + |g(z)| < 1 + |f(\omega(0))| + |g_0| \\ &< 1 + \beta + |w/\lambda| + \alpha\beta\,\mathrm{length}(\gamma)/|\lambda|\end{aligned}$$

for all $z \in \gamma_5$. Thus

$$|J| < ||P||_{\gamma_5}\alpha\,\mathrm{length}(\gamma_5) < \delta/2\,,$$

because of (1). Consequently, $|TP(c) - w| < \delta$, so

$$TP(c) \in B(w, \delta) \subset W\,. \tag{4}$$

But (3) gives

$$|P(z) - f(z)| < \varepsilon \quad \text{on } K \tag{5}$$

because $K \subset L \subset K_0$. Then (4) and (5) tell us that $P \in D(f, K, \varepsilon) \cap R(T, V, W)$. Hence $R(T, V, W)$ is dense, as required. ∎

THEOREM 4. *Assume that Ω has simply connected components Ω_i ($i \in I$). For each $j \in \mathbb{N}$, consider the j-antiderivative operator S_j defined in the introduction. Then*:

(a) *Every S_j is omnipresent.*

(b) *For Baire-almost all $f \in H(\Omega)$ all derivatives $f^{(k)}$ ($k \in \mathbb{N}_0$) and all antiderivatives $f^{(-j)}$ ($j \in \mathbb{N}$) have maximal cluster set at any boundary point of Ω.*

Proof. (a) $H(\Omega)$ has the initial topology for the countable family $\{(H(\Omega_i), p_i) : i \in I\}$, where p_i is the restriction map $p_i : H(\Omega) \to H(\Omega_i)$. Apply Theorem 3 for $G = \Omega_i$, $a = z_i$ and $\varphi(z,t) = (z-t)^{j-1}/(j-1)!$ to deduce that $S_{j,i}$ is omnipresent on $H(\Omega_i)$ ($i \in I$). Here $S_{j,i}$ denotes the restriction of S_j to $H(\Omega_i)$. We set

$$\begin{aligned}Q &= \{f \in H(\Omega) : S_j f \in M(\Omega)\}\,,\\ Q_i &= \{g \in H(\Omega_i) : S_{j,i} g \in M(\Omega_i)\}\ (i \in I)\,.\end{aligned}$$

By Theorem 1(a), each Q_i is residual. From the lemma we derive that $Q \supset \bigcap_{i\in I} p_i^{-1}(Q_i)$. But $p_i^{-1}(Q_i)$ is residual, because p_i is continuous and open. Thus Q is residual, so S_j is omnipresent by Theorem 1(a) again.

(b) Denote by P the set of functions $f \in H(\Omega)$ satisfying the property given in (b). Let P^* be the set

$$P^* = \{f \in H(\Omega) : T_k f \text{ and } S_j f \text{ are in } M(\Omega) \text{ for all } k \in \mathbb{N}_0 \text{ and all } j \in \mathbb{N}\}\,,$$

where T_k is the k-derivative operator. Trivially $P^* \supset P$. From (a), Theorem 1(b) and Theorem 2(a), we conclude that P^* is residual. It suffices to show that $P = P^*$. But this is readily derived from the lemma, because every $f^{(-j)}$ is equal to $S_j f + P_i$ on Ω_i, where P_i is a certain polynomial of degree less than j. ∎

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