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A CHARACTERIZATION OF COMPLETELY BOUNDED MULTIPLIERS OF FOURIER ALGEBRAS

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1. Introduction. Given a locally compact group G, we denote by λ the left regular representation of G, by $C_{\mathrm{r}}^*(G)$ the reduced C^*-algebra of G and by $W^*(G)$ its weak closure in $B(L^2(G))$. The Fourier algebra of G is the space of coefficients of λ and it is the predual of $W^*(G)$. A function φ on G is a *multiplier* of $A(G)$ if $\varphi\psi$ belongs to $A(G)$ for every ψ in $A(G)$. We denote by $MA(G)$ the space of multipliers of $A(G)$. Every φ in $MA(G)$ defines an operator m_φ on $A(G)$ whose transpose gives rise to a σ-weakly continuous operator M_φ on $W^*(G)$ such that $M_\varphi\lambda(s) = \varphi(s)\lambda(s)$ for $s \in W^*(G)$ (cf. [dCH], Prop. 1.2). One says that $\varphi \in MA(G)$ is a *completely bounded multiplier* of $A(G)$ if M_φ is completely bounded on $W^*(G)$ (or equivalently on $C_{\mathrm{r}}^*(G)$), which means that $||M_\varphi||_{\mathrm{cb}} = \sup_{n\geq 1} ||M_\varphi \otimes i_n||$ is finite, where i_n denotes the identity map on $M_n(C)$. The corresponding subspace of $MA(G)$ is denoted by $M_0A(G)$, and it is a Banach algebra with the norm

$$||\varphi||_{M_0A} = ||M_\varphi||_{\mathrm{cb}}\,.$$

It constitutes a remarkable class for the study of harmonic analysis on G: see for instance [dCH] and [CH]. Moreover, the authors of [BF] proved that $M_0A(G)$ coincides with the space $B_2(G)$ of Herz–Schur multipliers of G. To do that, they used a characterization of these multipliers due to J. E. Gilbert [Gi], but the latter was never published. The aim of this note is to present a short proof of the following theorem, where condition (2) is a well-known and useful variant of Gilbert's theorem (cf. [CH], p. 508):

THEOREM. *Let G be as above and let φ be a function on G. Then the following conditions are equivalent*:

(1) *φ belongs to $M_0A(G)$*;

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(2) *there exist a Hilbert space K and bounded continuous functions ξ, η from G to K such that $\varphi(t^{-1}s) = \langle \xi(s), \eta(t)\rangle$ for all s, t in G.*

Moreover, if these conditions are satisfied, then $||\varphi||_{M_0A} = \inf ||\xi||_\infty ||\eta||_\infty$ where the infimum is taken over all pairs as in condition (2).

2. Proof of the theorem. The proof of (1) ⇒ (2) rests on a representation theorem for completely bounded maps on unital C^*-algebras ([Pau], Theorem 7.4): If φ is an element of $M_0A(G)$, then there exist a Hilbert space K, a nondegenerate *-representation π: $C^*_{\mathrm r}(G) \to B(K)$ and two bounded operators v_1, v_2 from $L^2(G)$ to K such that $M_\varphi(a) = v_2^*\pi(a)v_1$ for a in $C^*_{\mathrm r}(G)$ and

$$||\varphi||_{M_0A} = ||v_1||\,||v_2||\,.$$

By [DC*], 13.3, the nondegenerate *-representation $\sigma = \pi\circ\lambda$ of $L^1(G)$ is associated with a unique continuous unitary representation of G, still denoted by σ. Then we claim that we have, for every $s \in G$,

$$(*)\qquad M_\varphi(\lambda(s)) = \varphi(s)\lambda(s) = v_2^*\sigma(s)v_1\,.$$

In fact, if $s \in G$ is fixed, let W_s be the set of compact neighbourhoods of s, ordered by reverse inclusion. For $V \in W_s$, choose a positive continuous function f_V supported in V such that $\int f_V = 1$. Then again by [DC*], 13.3, $\lambda(f_V)$ converges σ-strongly to $\lambda(s)$ in $W^*(G)$ and $\sigma(f_V)$ converges σ-strongly to $\sigma(s)$ in $B(K)$. As $M_\varphi(\lambda(f_V)) = v_2^*\sigma(f_V)v_1$ for every V, we get $(*)$ by σ-weak continuity of M_φ.

Finally, take some unit vector $\xi_0 \in L^2(G)$ and set

$$\xi(s) = \sigma(s)v_1\lambda(s^{-1})\xi_0 \quad\text{and}\quad \eta(s) = \sigma(s)v_2\lambda(s^{-1})\xi_0 \quad \text{for } s\in G\,.$$

Then ξ and η are bounded and continuous, and we have, for s, t in G,

$$\begin{aligned}\langle\xi(s),\eta(t)\rangle &= \langle v_2^*\sigma(t^{-1}s)v_1\lambda(s^{-1})\xi_0, \lambda(t^{-1})\xi_0\rangle\\ &= \varphi(t^{-1}s)\langle\lambda(t^{-1}s)\lambda(s^{-1})\xi_0, \lambda(t^{-1})\xi_0\rangle = \varphi(t^{-1}s)\,.\end{aligned}$$

Moreover, $||\xi||_\infty||\eta||_\infty \le ||v_1||\,||v_2|| = ||\varphi||_{M_0A}$.

The proof of (2)⇒(1) is straightforward, so we only sketch it: If φ satisfies condition (2), by Theorem 1.6 of [dCH], it is enough to check that φ belongs to $MA(G)$. If $\psi = \langle\lambda(\cdot)f, g\rangle$ is in $A(G)$ (with f, g in $L^2(G)$), and if $||\psi||_A = ||f||\,||g||$, then choose an orthonormal basis (ε_i) of K and set $\xi_i(s) = \langle\xi(s^{-1}),\varepsilon_i\rangle f(s)$ and $\eta_i(s) = \langle\eta(s^{-1}),\varepsilon_i\rangle g(s)$. Then $\varphi\psi(s) = \sum_i\langle\lambda(s)\xi_i,\eta_i\rangle$ and it is easy to see that

$$||\varphi\psi||_A \le ||\xi||_\infty||\eta||_\infty||\psi||_A\,.$$

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