COLLOQUIUM MATHEMATICUM

VOL. LXIII 1992 FASC. 2

FINITE-DIMENSIONAL IDEALS IN BANACH ALGEBRAS

BY

BERTRAM YOOD (UNIVERSITY PARK, PENNSYLVANIA)

Let A be a semi-prime Banach algebra. By an ideal in A we shall always mean a two-sided ideal unless otherwise specified. Smyth [9] has shown that, for x in A, xA is finite-dimensional if and only if Ax is finite-dimensional. Let F be the set of all x in A for which xA is finite-dimensional. We extend Smyth's theorem as follows. Let K be any ideal in A. Then, for x in A, xK is finite-dimensional if and only if Kx is finite-dimensional. Note that a distinction between this result and the Smyth case where $K = A$ is that x need not lie in K. Then we describe and study F and its role in Banach algebra theory.

Let Γ be the set of non-zero central idempotents p in the socle of A for which pA is a simple algebra. All these are in F and F is the direct sum of the ideals pA for p in Γ.

In the theory of commutative Banach algebras much attention is devoted to seeing when an ideal must be contained in a modular maximal ideal. We consider the non-commutative case where A has a dense socle and B is the completion of A in some normed algebra norm on A. An ideal W of B is contained in a modular maximal ideal of B if and only if W does not contain F. Easy examples show this can fail if A does not have a dense socle.

First we treat some preliminaries. Throughout A is a semi-prime Banach algebra over the complex field with socle S and center Z. For an ideal W in A let $L(W) = \{x \in A : xW = (0)\}$ and $R(W) = \{x \in A : Wx = (0)\}$. Then $L(W) = R(W)$ by [3, p. 162]. Let W^a denote the common value of $L(W)$ and $R(W)$. The socle of W is $S \cap W = SW = WS$ (see [12, Lemma 3.10]). Each minimal right (left) ideal of A has the form $pA\,(Ap)$ where p is an idempotent. Such an element p we call a *minimal idempotent*. An idempotent $q \neq 0$ is said to be a *simple idempotent* if qAq is a simple algebra. Every minimal idempotent is simple. For a simple idempotent q and an ideal W either $q \in W$ or $q \in W^a$ by [13, Lemma 5.1].

We compare F with another notion of finite-dimensionality in Banach algebra theory which has been studied. In [11] Vala called an element $w \in A$ *finite* if the mapping $x \to xwx$ of A into A has finite-dimensional range. We refer to [7] for further references and work on this notion. Let Φ be the set of all elements in A finite in the above sense. Of course $\Phi \supset F$. In [4, Theorem 7] it was shown that $\Phi = S$. In Corollary 1 below we see that if A is primitive and infinite-dimensional then $F = (0)$. On the other hand, $\Phi = S$ can be non-zero for such A as is the case for $B(X)$, the Banach algebra of all bounded linear operators on an infinite-dimensional Banach space X.

LEMMA 1. *F is the union of all the finite-dimensional ideals of A.*

Proof. Let $x \in F$. By definition xA (Ax) is the linear span of a finite number of elements $xv_1, \ldots, xv_n$ ($w_1x, \ldots, w_rx$). For each a and b in A we have $xb = \sum_{k=1}^{n} \beta_k xv_k$ and $ax = \sum_{j=1}^{r} \alpha_j w_j x$ where the α_j and β_k are scalars. Then

$$axb = \sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_j \beta_k w_j x v_k$$

so that the rn elements $w_j x v_k$ span AxA. Let V be the set of scalar multiples of x. Then x lies in the finite-dimensional ideal $V + xA + Ax + AxA$.

Conversely, if K is a finite-dimensional ideal then clearly zA is finite-dimensional for each $z \in K$ so that $K \subset F$.

LEMMA 2. *A finite-dimensional one-side ideal K of A is contained in a finite-dimensional ideal of A.*

Proof. Let $v_1, \ldots, v_n$ be a basis for K. Clearly $K \subset F$. By Lemma 1 each v_j is contained in a finite-dimensional ideal W_j of A. Then $K \subset W_1 + \ldots + W_n$.

THEOREM 1. *Let I be an ideal of A and $x \in A$. The following statements are equivalent.*

(a) *xI is finite-dimensional.*
(b) *Ix is finite-dimensional.*
(c) *$x \in F + I^a$.*

Proof. Suppose (a). By Lemma 2, xI is contained in a finite-dimensional ideal W. Now W is Artinian and semi-simple by [5, Theorem 1.3.1] so that by [5, Lemma 1.4.2] there is an idempotent $v \in W$ where $xI = vW = vA$. Then as v is a left identity for xI we have $(vx - x)I = (0)$. Hence $vx - x \in I^a$. However, $v \in F$ so that $x \in F + I^a$. Thus (a) implies (c). Conversely, if $x \in F + I^a$ then clearly xI is finite-dimensional so that (a) and (c) are equivalent. Interchanging the roles of left and right we see in the same way that (b) and (c) are equivalent.

LEMMA 3. $S = F \oplus (S \cap F^a)$.

P r o o f. A finite-dimensional ideal K in A is equal to its socle. Therefore $K \subset S$ and so, by Lemma 1, $F \subset S$. This also follows from [4, Theorem 7] where it is shown that, for $x \in A$, we have $x \in S$ if and only if xAx is finite-dimensional. Since each minimal idempotent of A is either in F or in F^a we have the given decomposition of S.

LEMMA 4. *Let q be an idempotent in $S \cap Z$. Then qA is finite-dimensional.*

P r o o f. As qA is closed in A it is a Banach algebra. As $q \in Z$, qA is an ideal in A and is therefore semi-prime. Moreover, qA is its own socle. It follows by [10, Theorem 5] or [4, Theorem 11] that qA is finite-dimensional.

LEMMA 5. *Let p be a minimal idempotent in F. Then $ApA = eA$ where $e \in S \cap Z$ and e is a simple idempotent. Moreover*,

$$(1-e) = \{x \in A : Ax \subset (1-p)A\} = \{x \in A : xA \subset A(1-p)\}\,.$$

P r o o f. As in the proof of Lemma 1, ApA is finite-dimensional. Therefore ApA is, by [5, p.20], semi-simple as well as semi-prime. Hence [5, p. 30] applies so that we can express $ApA = eA$ where e is a central idempotent. By Lemma 3 we have $e \in S$. Next we see that ApA is a simple algebra. For if W is an ideal of ApA then either $p \in W$ or $p \in W^a$. If $p \in W$ then $W = ApA$. If $pW = Wp = (0)$, then $W^2 = (0)$ and $W = (0)$. In summary, e is a simple central idempotent lying in S.

As $A = eA \oplus (1-e)A$ and eA is simple we see that $(1-e)A$ is a modular maximal ideal of A. The set of $x \in A$ for which $Ax \subset (1-p)A$ is the set union of all (two-sided) ideals of A contained in $(1-p)A$. Now

$$(1-e)x = (1-p)(x-ex)$$

so that $(1-e)A \subset (1-p)A$. As $(1-e)A$ is a maximal ideal we have

$$(1-e)A = \{x \in A : Ax \subset (1-p)A\}\,.$$

NOTATION. For convenience we denote the set of non-zero simple idempotents of A which lie in $S \cap Z$ by Γ.

THEOREM 2. *F is the algebraic direct sum of the ideals eA for $e \in \Gamma$.*

P r o o f. Let $x \in F$. We can, by Lemma 3, write

$$x = \sum_{j=1}^{r} p_j x_i$$

where each p_j is a minimal idempotent in F and $x_j \in A$. By Lemma 5 each $p_j x_j$ can be expressed as some $e_j w_j$ where $e_j \in \Gamma$ and $w_j \in A$. Thus F is contained in the algebraic sum of the eA, $e \in \Gamma$. Next we know by Lemma 4 that each such eA lies in F. Inasmuch as $e_1 e_2 = 0$ for two different elements of Γ, the algebraic sum of the eA, $e \in \Gamma$, is direct.

COROLLARY 1. *For an infinite-dimensional primitive Banach algebra A we have $F = (0)$.*

Proof. By [8, Cor. 2.4.5] the center Z of A is either (0) or is the set of scalar multiples of non-zero idempotent p. If $Z = (0)$ then $F = (0)$ by Theorem 2. Suppose $p \neq 0$. Let $I_1 = pA$, $I_2 = (1 - p)A$. These are ideals in A and $I_1 I_2 = (0)$. As A is primitive and $I_1 \neq (0)$ we have $I_2 = (0)$. But then p is the identity for A. As A is infinite-dimensional, $p \notin F$. Thus F cannot have any non-zero central idempotent and, by Theorem 2, $F = (0)$ in this case also.

COROLLARY 2. *Any ideal K of A which does not contain F is contained in a modular maximal ideal of A.*

Proof. Since K does not contain F there is an idempotent $p \in \Gamma$ where $p \notin K$ by Theorem 2. As p is a simple idempotent ($pAp = pA$ is a simple algebra) we have $p \in K^a$ by [13, Lemma 5.1]. Therefore $K \subset (1 - p)A$. However, from $A = pA \oplus (1 - p)A$ we see that $(1 - p)A$ is a modular maximal ideal.

In particular, if F is dense then any proper closed ideal is contained in a modular maximal ideal. This is the case, for example, for the group algebra of a compact group, where the multiplication is convolution (see [6, Theorem 15]).

As in [8, p. 59] by the *strong radical* of an algebra we mean the intersection of its modular maximal ideals.

THEOREM 3. *Suppose that A has dense socle. Let B be the completion of A in the normed algebra norm $|x|$. Then the modular maximal ideals of B are the ideals $(1 - q)B$ for $q \in \Gamma$. Moreover, the strong radical of B is the left annihilator in B and also the right annihilator in B of F.*

Proof. To avoid confusion we state that the sets Γ and F of Theorem 3 refer to the Banach algebra A. Let $p \in \Gamma$. As pA is finite-dimensional, $pA = pB$. Also p lies in the center of B. From $B = pB \oplus (1 - p)B$ and the fact that pB is a simple algebra we see that $(1 - p)B$ is a modular maximal ideal of B.

We shall show that every modular maximal ideal M of B is of the form $(1 - q)B$ for some $q \in \Gamma$. Let ν be the embedding map of A into B, let π be the natural homomorphism of B onto B/M and let α be the composite map of ν followed by π. Note that ν need not be continuous. However, $\nu(A)$ is dense in B and π is continuous so that $\alpha(A)$ is dense in B/M. Consider the separating set Σ in B/M corresponding to the map α. That is, Σ is the set of elements $\pi(w)$ in B/M, $w \in B$, for which there is a sequence $\{x_n\}$ in A where

$$\|x_n\| \to 0 \quad \text{and} \quad |\alpha(x_n) - \pi(w)| \to 0\,.$$

As Σ is an ideal in B/M which is simple then either $\Sigma = (0)$ or $\Sigma = B/M$. We cannot have $\Sigma = B/M$ for it is known [2, Theorem 1] that Σ cannot possess a non-zero idempotent but B/M has an identity. Consequently, $\Sigma = (0)$ and so α is a continuous homomorphism of A onto a dense subset of B/M. By hypothesis the socle S of A is dense in A so that $\alpha(S)$ is dense in B/M. Hence there is a minimal idempotent f of A where $\alpha(f) \neq 0$. As fAf is the set of scalar multiples of f, $\alpha(f)(B/M)\alpha(f)$ is the set of scalar multiples of $\alpha(f)$. As B/M is simple and $\alpha(f)$ is a minimal idempotent in B/M, it follows that B/M is equal to its socle. Therefore by [10, Theorem 5] we see that B/M is finite-dimensional.

By the proof of Lemma 5, AfA is a simple algebra. Now $(AfA)\cap M$ is an ideal in AfA which cannot be AfA since $f \notin M$. Therefore $(AfA)\cap M = (0)$ so that α is a one-to-one mapping when its domain is restricted to AfA. But $\alpha(AfA)$ is a linear subspace of the finite-dimensional B/M. Hence AfA is finite-dimensional and, in particular, $f \in F$. By Lemma 5 there is some $q \in \Gamma$ with $AfA = qA$. As qA is finite-dimensional, $qA = qB$. Also $qM = (0)$ or $qM = qB$. In the latter case we would have $q \in M$, which is not so. Therefore $qM = (0)$ and so $M \subset (1-q)B$. Note that $(1-q)B$ is a proper modular ideal of B and M is a modular maximal ideal of B. Hence $M = (1-q)B$.

The strong radical R of B is the intersection of the ideals $(1-p)B = B(1-p)$ for $p \in \Gamma$. As F is the direct sum of the $pA = pB$ for $p \in \Gamma$, by Theorem 2, we get $FR = RF = (0)$. Suppose $w \in B$ and $wF = (0)$. Then, for any $p \in \Gamma$, $wpA = (0)$. But $wp \in Bp = Ap \subset A$ and A is semi-prime. Therefore $wp = 0$ and so $w \in (1-p)B$. Hence $w \in R$. This concludes the proof of Theorem 3.

COROLLARY 3. *Suppose that A has a dense socle that B is its completion in some normed algebra norm on A. An ideal W of B is contained in a modular maximal ideal of B if and only if W does not contain F.*

Proof. Suppose that W fails to contain F. Then, by Theorem 2, there is some $p \in \Gamma$ with $p \notin W$. As $pA = pB$ is simple and $pW \neq pB$ we get $pW = (0)$ and $W \subset (1-p)B$. But $(1-p)B$ is a modular maximal ideal of B.

Conversely, if W is contained in a modular maximal ideal of B then, by Theorem 3, there is some $q \in \Gamma$ so that $W \subset (1-q)B$. Then $qW = (0)$ so that $q \notin W$ and so W does not contain F.

In particular, if $F = (0)$ in A then B cannot have a modular maximal ideal.

We point out that the conclusion of Theorem 4 can fail if the hypothesis of a dense socle is dropped. For let A be the commutative Banach algebra of all continuous functions on [0,1]. Then $F = (0)$ yet A has modular maximal ideals.

LEMMA 6. *The following statements are equivalent.* (1) $F = F^{aa}$, (2) F *is closed and* (3) F *is finite-dimensional.*

Proof. Clearly (3) and (1) each imply (2). Inasmuch as F is equal to its socle, by [10, Theorem 3], (2) implies (3). Assume (3). By [5, p. 30], F has an identity element w which lies in the center of A. Then as $A = wA \oplus (1-w)A$ and $F^a = (1-w)A$ we get $A = F \oplus F^a$. Suppose $z \in F^{aa}$ and $z = u + v$ where $u \in F$ and $v \in F^a$. Then $z - u = v$ where $z - w \in F^{aa}$ and $v \in F^a$. Hence $v = 0$ and $z \in F$. Thus (3) implies (1).

For each $x \in A$ let L_x (R_x) be the operator on A defined by $L_x(y) = xy$ ($R_x(y) = yx$). Set

$$N_l = \{x \in A : L_x \text{ is a compact operator}\},$$
$$N_r = \{x \in A : R_x \text{ is a compact operator}\}.$$

In [14, Theorem 4.3] the author showed that if A has dense socle then $N_l = A$ if and only if $N_r = A$. Later Smyth [9] gave an independent proof of this result. Moreover, he gave an example where $N_l = A$ and $N_r \neq A$. An open question is to determine just when $N_l = N_r$. We make a small advance in the following result.

THEOREM 4. *Suppose either* $S^a = (0)$ *or* A *is semi-simple. If* F *is finite-dimensional then* $N_l = N_r = F$.

Proof. By the Riesz–Schauder theory each of N_l and N_r have F as its socle. Suppose $S^a = (0)$. Then every non-zero left or right ideal of A contains a minimal idempotent of A [13, Lemma 3.1]. In particular, this shows that $N_l F^a = (0) = N_r F^a$. Hence $N_l \subset F^{aa}$ and $N_r \subset F^{aa}$. Thus if F is finite-dimensional we have $F = N_l = F^{aa} = N_r$ by Lemma 6.

Suppose that A is semi-simple. By [1, Theorem 7.2] each of N_l and N_r is a modular annihilator algebra. As N_l and N_r are also semi-simple it follows from [12, p. 38] that the annihilator of the socle of N_l (N_r) in N_l (N_r) is (0). Hence, arguing as above we see that $N_l \subset F^{aa}$ and $N_r \subset F^{aa}$. Thus, in this case also, the conclusion follows.

REFERENCES

[1] B. Barnes, *Modular annihilator algebras*, Canad. J. Math. 18 (1966), 566–578.

[2] —, *Some theorems concerning the continuity of algebra homomorphisms*, Proc. Amer. Math. Soc. 18 (1967), 1035–1037.

[3] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, New York 1973.

[4] L. Dalla, S. Giotopoulos and N. Katseli, *The socle and finite-dimensionality of a semiprime Banach algebra*, Studia Math. 92 (1989), 201–204.

[5] I. N. Herstein, *Noncommutative Rings*, Carus Math. Monographs 15, Math. Assoc. America, 1968.

[6] I. Kaplansky, *Dual rings*, Ann. of Math. 49 (1948), 689–701.
[7] J. Puhl, *The trace of finite and nuclear elements in Banach algebras*, Czechoslovak Math. J. 28 (1978), 656–676.
[8] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, Princeton 1960.
[9] M. R. F. Smyth, *On problems of Olubummo and Alexander*, Proc. Royal Irish Acad. 80A (1980), 69–74.
[10] A. W. Tullo, *Conditions on Banach algebras which imply finite-dimensionality*, Proc. Edinburgh Math. Soc. 20 (1976), 69–74.
[11] K. Vala, *Sur les éléments compacts d'une algèbre normée*, Ann. Acad. Sci. Fenn. Ser. A I Math. 407 (1967), 1–7.
[12] B. Yood, *Ideals in topological rings*, Canad. Math. J. 16 (1964), 28–45.
[13] —, *Closed prime ideals in topological rings*, Proc. London Math. Soc. (3) 24 (1972), 307–323.
[14] —, *On the strong radical of certain Banach algebras*, Proc. Edinburgh Math. Soc. 21 (1978), 81–85.

DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802
U.S.A.

Reçu par la Rédaction le 4.1.1991;
en version modifiée le 24.7.1991