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FACTORISATION WITHOUT BOUNDED APPROXIMATE IDENTITIES

BY

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0. Introduction. The purpose of this paper is to address some questions related to the factorisation problem in Harmonic Analysis. The problem can be stated very generally, but we prefer to state it within the context in which we will work. Therefore, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be spaces of functions (or measures or pseudomeasures) defined on a locally compact group G, and let $*$ denote the usual operation of convolution. Suppose that for each $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_2$, $f * g$ is well-defined (in some sense). The problem is: when is $\mathcal{L}_1 * \mathcal{L}_2 = \mathcal{L}_2$? If $\mathcal{L}_1$ is closed under convolution then we obtain the interesting case: when is $\mathcal{L}_1 * \mathcal{L}_1 = \mathcal{L}_1$? If $\mathcal{L}_1 * \mathcal{L}_2 = \mathcal{L}_2$, then we say that $\mathcal{L}_2$ *factors over* $\mathcal{L}_1$, while if $\mathcal{L}_1 * \mathcal{L}_1 = \mathcal{L}_1$ then we say that $\mathcal{L}_1$ *factors*. Almost all the known results concerning this problem require the additional assumptions that $\mathcal{L}_1$ is a Banach algebra and that $\mathcal{L}_2$ is a normed space.

Salem proved in [15] that for the circle group $\mathbb{T}$, the group algebra $L^1(\mathbb{T})$ factors, and the Banach algebra $C(\mathbb{T})$ of continuous functions factors over $L^1(\mathbb{T})$. In [12] and [13], Rudin proved that if G is $\mathbb{R}$, or more generally any locally compact abelian Euclidean group, then $L^1(G)$ factors. The most significant step on the general factorisation problem was taken by Cohen in 1959 ([1]) when he proved that a Banach algebra with bounded left approximate identity factors. Further he deduced various extra properties of the factors. It follows from Cohen's Theorem that $L^1(G)$ factors for every locally compact group. However, Cohen's Theorem fails to give any information about $L^p(G)$ (the space of pth-power integrable functions) when $p > 1$ because it is known that for these values of p, $L^p(G)$ does not contain a bounded left approximate identity (see (34.40b) of [8]). In fact, for $p > 1$, $L^p(G)$ fails to factor if G is infinite. This has been established for compact groups in (34.40) of [8], and for non-compact groups as a consequence of Theorem 1 of [14]. From Cohen's Theorem, Hewitt [7] and Curtis and

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Figà-Talamanca [2] deduced the Module Factorisation Theorem, which can be used to prove that $L^p(G)$ and $C_0(G)$ (the space of continuous functions vanishing at infinity) factor over $L^1(G)$ for any locally compact group.

The reader is referred to Section 32 of [8], to Chapter 8 of [16] and to Section 7.5 of [5] for discussions of many of the factorisation and non-factorisation results that have proliferated. It is worthwhile pointing out the recent papers of Dixon [3], Feichtinger and Leinert [6] and Willis [18]. In [3] and [18], the authors study the relationships between the factorisation properties of various Banach algebras and the existence of approximate identities. (See also the many references in [3] and [18].) On the other hand, in [6] the authors consider the factorisation problem from a somewhat different perspective. They first show that the hypothesis of Cohen's Theorem cannot be relaxed to allow unbounded left approximate identities, no matter how slowly they "grow". Under even weaker hypotheses, they then establish several local factorisation formulae in the sense that they identify some elements of an algebra which factor, even when the algebra itself does not. (Their results are actually obtained in the more general module setting.)

In this paper we study function and measure spaces $\mathcal{L}$ which factor, with respect to the usual convolution, over linear spaces like $L^p(G)$ or $A^p(G)$ (the latter denoting the space of L^1-functions with Fourier transform in ℓ^p) for some $p > 1$. We do not, in fact, initially use the norm structure on these spaces, nor do we assume any on $\mathcal{L}$, but rather depend on the summability properties of the Fourier transforms of their elements. We restrict our attention to compact abelian groups G, which have discrete dual groups X. Given the duality that exists between convolution over G and pointwise multiplication over X, it is perhaps not surprising that we begin by considering certain sequence spaces.

1. The space $\ell^+(X)$. For each $r > 0$, $\ell^r(X)$ denotes the set of all complex-valued functions on X which are rth-power summable. Each function in $\ell^r(X)$ is supported by a countable subset of X and so we will speak of *sequences* in $\ell^r(X)$, even when X is uncountable. The *quasinorm* of $\mathbf{a} = (a_\chi) \in \ell^r(X)$ is given by

$$\|\mathbf{a}\|_r = \|(a_\chi)\|_r = \Big\{ \sum_{\chi \in X} |a_\chi|^r \Big\}^{1/r}.$$

For each $r > 0$, $\ell^r(X)$ is closed under the pointwise addition, pointwise multiplication and scalar multiplication of sequences. For $r \geq 1$, $\| \, \|_r$ is a norm on $\ell^r(X)$, and $(\ell^r(X), \| \, \|_r)$ is a Banach algebra, while for $r \in (0,1)$, $\| \, \|_r$ fails to satisfy the triangle inequality. However, if we put $d_r(\mathbf{a}, \mathbf{b}) = \|(a_\chi - b_\chi)\|_r^r$ then $(\ell^r(X), d_r)$ is a complete linear metric or Fréchet space. Finally, we denote by $\ell^\infty(X)$ the linear space of all bounded sequences on X, by $c_0(X)$ those which decay to 0 at infinity (each of $\ell^\infty(X)$ and $c_0(X)$ can be normed by $\|\mathbf{a}\|_\infty = \|(a_\chi)\|_\infty = \sup\{|a_\chi| : \chi \in X\}$), and by $c_{00}(X)$ the space of sequences which are finitely supported. Then for $0 < r \le s$,

$$c_{00}(X) \subset \ell^r(X) \subseteq \ell^s(X) \subset c_0(X) \subset \ell^\infty(X)$$

and $\| \ \|_r \ge \| \ \|_s > \| \ \|_\infty$. Two well-known facts that we will use are:

(i) $\ell^r(X)\ell^\infty(X) = \ell^r(X)$ for all $r > 0$, and
(ii) $\ell^r(X)\ell^s(X) \subseteq \ell^{rs/(r+s)}(X)$ for all $r, s > 0$.

If the function space $\mathcal{L}$ satisfies the inclusion $\mathcal{L} \subseteq \mathcal{L} * A^p(G)$ then, as a consequence of the Convolution Theorem, we have $\widehat{\mathcal{L}} \subseteq \widehat{\mathcal{L}}.\ell^p(X)$, where termwise multiplication of sequences is used in the second inclusion. The first theorem will be used to determine when $\mathcal{L}$ satisfies such an inclusion.

1.1. THEOREM. *If A is a subset of $\ell^\infty(X)$ and if there exists a positive number s such that $A \subseteq A\ell^s(X)$ then $A \subseteq \bigcap_{r>0} \ell^r(X)$.*

Proof. By repeatedly using property (ii) we see that

$$\begin{aligned} A &\subseteq A\ell^s(X) \subseteq A(\ell^s(X))^2 \subseteq A\ell^{s/2}(X) \subseteq A\ell^s(X)\ell^{s/2}(X) \\ &\subseteq \ldots \subseteq A\ell^{s/n}(X) \subseteq \ell^\infty(X)\ell^{s/n}(X) = \ell^{s/n}(X) \end{aligned}$$

for each positive integer n. Then since the ℓ^r-spaces are nested the conclusion follows. ∎

The intersection of the ℓ^r-spaces has been discussed by several authors; see for example [10] where it is denoted by $\ell^{0^+}(X)$. We modify this notation to $\ell^+(X)$. We will deduce from the next proposition that $\ell^+(X)$ factors over each $\ell^s(X)$ for $s > 0$, when termwise multiplication of sequences is used.

1.2. PROPOSITION. $\ell^+(X).\ell^+(X) = \ell^+(X)$.

Proof. Let $\mathbf{x} = (x_\chi)$ denote a sequence in $\ell^+(X)$. For each $\chi \in X$, put $y_\chi = |x_\chi|^{1/2}$ and $z_\chi = x_\chi|x_\chi|^{-1/2}$ when $x_\chi \ne 0$ and $z_\chi = 0$ otherwise. Now write $\mathbf{y} = (y_\chi)$ and $\mathbf{z} = (z_\chi)$. Then $\mathbf{y}.\mathbf{z} = (y_\chi z_\chi) = (x_\chi) = \mathbf{x}$ and for each $r > 0$,

$$\|\mathbf{y}\|_r = \Big\{ \sum_{\chi\in X} |y_\chi|^r \Big\}^{1/r} = \Big\{ \sum_{\chi\in X} |x_\chi|^{r/2} \Big\}^{1/r} = \|\mathbf{x}\|_{r/2}^{1/2} < \infty\,, \quad \text{and}$$

$$\|\mathbf{z}\|_r = \Big\{ \sum_{\chi\in X} |z_\chi|^r \Big\}^{1/r} = \Big\{ \sum_{\chi\in X} |x_\chi|^{r/2} \Big\}^{1/r} = \|\mathbf{x}\|_{r/2}^{1/2} < \infty\,. \ ∎$$

1.3. COROLLARY. *For all $s \in (0,\infty)$, $\ell^+(X).\ell^s(X) = \ell^+(X)$.*

Proof. Since $\ell^+(X) \subset \ell^s(X)$, it follows from Proposition 1.2 that $\ell^+(X).\ell^s(X) \supseteq \ell^+(X)$. The reverse inclusion is a trivial consequence of the fact that $\ell^s(X) \subset \ell^\infty(X)$ and property (i) above. ∎

So far, we have not defined a topology on $\ell^+(X)$. We know that for each $n \in \mathbb{N}$, $c_{00}(X)$ is dense in $(\ell^{1/n}(X), d_{1/n})$ and so $\ell^+(X)$ cannot be complete with respect to any of the metrics obtained by restricting $d_{1/n}$ to $\ell^+(X)$. We can, however, take the Fréchet combination of these restrictions to obtain a new metric d on $\ell^+(X)$ given by

$$d(\mathbf{x},\mathbf{y}) = \sum_{n\in\mathbb{N}} 2^{1-n} \frac{d_{1/n}(\mathbf{x},\mathbf{y})}{1+d_{1/n}(\mathbf{x},\mathbf{y})}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $\ell^+(X)$. (It will be convenient to write the nth term of the sum as $2^{1-n}\varrho_n(\mathbf{x},\mathbf{y})$.) It follows from Theorem 3 in Section 11.3 of [17] that $(\ell^+(X), d)$ is complete, and from Lemma 2.2-9 of [11] that it is not normable. In fact, it is not even locally convex.

1.4. Theorem. *The metric space $(\ell^+(X), d)$ is not locally convex.*

Proof. We argue by contradiction. If $(\ell^+(X), d)$ is locally convex then it has a local base $\{U_k : k \in \mathbb{N}\}$ of convex neighbourhoods at $\mathbf{0}$. On the other hand, the metric topology has a subbase at $\mathbf{0}$ given by

$$\{\{\mathbf{x} \in \ell^+(X) : \|\mathbf{x}\|_{1/n}^{1/n} < \delta\},\ n \in \mathbb{N},\ \delta > 0\}\,.$$

From this we can deduce that there exists an increasing sequence (n_k) of integers greater than 1 and a decreasing sequence (δ_k) of positive real numbers such that

$$\{\mathbf{x} \in \ell^+(X) : \|\mathbf{x}\|_{1/n_k}^{1/n_k} < \delta_k\} \subseteq U_k\,.$$

To see this, first fix a positive integer k. There exist positive integers $p_1, \ldots, p_m \in \mathbb{N}$ and positive real numbers $\gamma_1, \ldots, \gamma_m$ such that

$$\bigcap_{j=1}^{m} \{\mathbf{x} \in \ell^+(X) : \|\mathbf{x}\|_{1/p_j}^{1/p_j} < \gamma_j\} \subseteq U_k\,.$$

Put $n_k = \max\{p_1, \ldots, p_m, n_{k-1}\}$ and $\delta_k = \min\{\gamma_1, \ldots, \gamma_m, \delta_{k-1}\}$, where for completeness we let $n_0 = 2$ and $\delta_0 = 1$.

We obtain a contradiction by constructing a sequence $(\mathbf{x}^{(k)})$ of sequences which converges to $\mathbf{0}$ in $(\ell^+(X), d)$, but for which $\lim_{k\to\infty} \|\mathbf{x}^{(k)}\|_{1/n_k}^{1/n_k} = \infty$. For each $\xi \in X$ let $\mathbf{e}^\xi$ denote the sequence in which the "ξth term" is 1 whilst all other terms are 0. Then $\frac{1}{2}\delta_k\mathbf{e}^\xi \in U_k$ for every $\xi \in X$. Let $\{\xi_i : i \in \mathbb{N}\}$ be a set of distinct elements of X. Then for each pair, k and m, of positive integers, let $\mathbf{x}^{(k,m)}$ be the finitely supported sequence defined by

$$\mathbf{x}^{(k,m)} = \frac{\delta_k}{2} \sum_{i=1}^{m} \frac{1}{m} \mathbf{e}^{\xi_i}\,.$$

Since U_k is convex and $\sum_{i=1}^{m} 1/m = 1$, $\mathbf{x}^{(k,m)} \in U_k \cap c_{00}(X)$. Moreover, for each $n \geq 1$,

$$\|\mathbf{x}^{(k,m)}\|_{1/n}^{1/n} = (\delta_k/2)^{1/n} \sum_{i=1}^{m} (1/m)^{1/n} = (\delta_k/2)^{1/n} m^{1-1/n} .$$

Thus, for each k and n,

$$\lim_{m\to\infty} \|\mathbf{x}^{(k,m)}\|_{1/n}^{1/n} = \infty . \tag{1}$$

Now choose m' (depending on k) such that $\|\mathbf{x}^{(k,m')}\|_{1/n_k}^{1/n_k} > k$ and write $\mathbf{x}^{(k)} = \mathbf{x}^{(k,m')}$. Then, since $\{U_k\}$ is a neighbourhood base at 0, the sequence $(\mathbf{x}^{(k)})$ converges to $\mathbf{0}$ in $(\ell^+(X), d)$; but it follows from (1) that $\lim_{k\to\infty} \|\mathbf{x}^{(k)}\|_{1/n_k}^{1/n_k} = \infty$. ∎

The metric topology on $(\ell^+(X), d)$ is stronger than that induced on $\ell^+(X)$ as a subspace of $\ell^1(X)$. Hence $\ell^\infty(X)$ is (isomorphic to) a subspace of the continuous dual, $(\ell^+(X))^*$, of $\ell^+(X)$. In fact, the reverse inclusion also holds.

1.5. THEOREM. $(\ell^+(X))^* \simeq \ell^\infty(X)$.

Proof. It is sufficient to prove that $(\ell^+(X))^*$ is isomorphic to a subspace of $\ell^\infty(X)$. Take $L \in (\ell^+(X))^*$ and define φ on X by $\varphi(\xi) = L(\mathbf{e}^\xi)$ (using the same notation as in the previous proof). Then for each $\mathbf{x} \in \ell^+(X)$,

$$L(\mathbf{x}) = \sum_{\xi\in X} x_\xi L(\mathbf{e}^\xi) = \sum_{\xi\in X} x_\xi \varphi(\xi) ,$$

where the sums converge since $\mathbf{x} \in \ell^1(X)$. Now using the same technique employed in the proof of Theorem 1.4, we see that there must exist an integer $N > 1$ and a real number $\delta > 0$ such that $\|\mathbf{x}\|_{1/N} < \delta^N \Rightarrow |L(\mathbf{x})| < 1$. Thus, for any $\mathbf{x} \in \ell^+(X)$,

$$|L(\mathbf{x})| < \delta^{-1}(\|\mathbf{x}\|_{1/N})^{1/N} ,$$

and in particular, if $\mathbf{x} = \mathbf{e}^\xi$ for some $\xi \in X$ then

$$|L(\mathbf{e}^\xi)| = |\varphi(\xi)| < \delta^{-1}(\|\mathbf{e}^\xi\|_{1/N})^{1/N} = \delta^{-1}$$

so that $\varphi \in \ell^\infty(X)$. ∎

2. Factorisation problems. Let G denote an infinite compact abelian group, with dual group X. We let $L^1(G)$ denote the Banach space of (equivalence classes of) functions which are absolutely integrable with respect to the normalised Haar measure on G; its norm is given by

$$\|f\|_{L^1} = \int_G |f(x)|\, dx .$$

The Fourier transform and convolution product are defined for L^1-functions f and g by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\,dx \quad \text{and} \quad f * g(x) = \int_G f(u)g(x-u)\,du\,.$$

Then $\widehat{f} \in c_0(X)$, $f * g \in L^1(G)$ and the Convolution Theorem holds (that is $(f*g)^\wedge = \widehat{f}\widehat{g}$).

We denote by $L^p(G)$ the Banach space of functions which are pth-power integrable, with $\|f\|_{L^p} = \|\,|f|^p\|_{L^1}^{1/p}$ and by $A^p(G)$ the Banach space of integrable functions with Fourier transforms in $\ell^p(X)$, normed by $\|f\|_{A^p} = \|f\|_{L^1} + \|\widehat{f}\|_p$. In each case p is any number greater than or equal to 1. Each of $L^p(G)$ and $A^p(G)$ is a Banach algebra when convolution is used as multiplication. The continuous dual of $A^1(G)$ is denoted by $PM(G)$ and its elements are called *pseudomeasures*. The definitions of Fourier transform and convolution can be extended to pseudomeasures in such a way that $\widehat{S} \in \ell^\infty(X)$ and $S*T \in PM(G)$ for each $S, T \in PM(G)$. (See, for example, [9].)

Let $\mathcal{L}$ denote a subset of $PM(G)$ and $\widehat{\mathcal{L}} = \{\widehat{S} : S \in \mathcal{L}\}$ the corresponding subset of $\ell^\infty(X)$; and let E denote a linear space of pseudomeasures for which there exists an $s \in (0,\infty)$ satisfying $\widehat{E} \subseteq \ell^s(X)$. (Examples of such linear spaces include $A^p(G)$ for all $p \in [1,\infty)$ — in which case $s = p$ — and $L^p(G)$ for $p \in (1,\infty]$ — here $s = p'$, where $p^{-1} + (p')^{-1} = 1$, when $p \in (1,2]$ and $s = 2$ otherwise.) It is an immediate consequence of the Convolution Theorem and Theorem 1.1 that if $\mathcal{L} \subseteq E * \mathcal{L}$ then $\widehat{\mathcal{L}} \subseteq \ell^+(X)$. Thus the elements of $\mathcal{L}$ must be continuous functions on G.

2.1. THEOREM. *Let E be a set of pseudomeasures on G for which there exists $s \in (0,\infty)$ satisfying $\widehat{E} \subseteq \ell^s(X)$, and $\mathcal{L}$ another set of pseudomeasures with $\mathcal{L} \subseteq E * \mathcal{L}$. Then $\widehat{\mathcal{L}} \subseteq \ell^+(X)$.*

Let H be the subspace of $C(G)$ which is isomorphic, under the Fourier transformation, to $\ell^+(X)$. It follows from Proposition 1.2 that $H * H = H$. Hence we have the following corollary to Theorem 2.1.

2.2. COROLLARY. *With the notation of Theorem* 2.1, *if E contains the linear space H then $H * E = H$. In particular*, *$H * A^p(G) = H$ and $H * L^p(G) = H$ for all $p \geq 1$.*

Proof. Since $H \subseteq E$, it is obvious that $H = H * H \subseteq H * E$. On the other hand,

$$(H*E)^\wedge \subseteq \widehat{H}\widehat{E} \subseteq \ell^+(X)\ell^\infty(X) \subseteq \ell^+(X) = \widehat{H}$$

so that the reverse inclusion also holds. ∎

It is easy to see that we cannot relax the requirement that $\widehat{E} \subseteq \ell^s(X)$ for some $s \in (0,\infty)$ in Theorem 2.1 to say $\widehat{E} \subseteq c_0(X)$ since, for example, $E = L^1(G)$ satisfies this weaker condition but $\widehat{E}$ is not in $\ell^s(X)$ for any s. However, $A^1(G)$ is a linear space which is strictly larger than $\ell^+(X)$ and yet factors over $L^1(G)$. (See (34.39) of [8].)

Obviously, H contains the set of trigonometric polynomials $T(G)$ (which is isomorphic to $c_{00}(X)$). It is both character- and translation-invariant, in the sense that the functions χf and $\tau_a f$ (defined by $(\chi f)(x) = \chi(x)f(x)$ and $\tau_a f(x) = f(x-a)$ for each $x \in G$) are in H whenever $f \in H$, $\chi \in X$ and $a \in G$. H naturally inherits a metric topology from $(\ell^+(X), d)$. We denote the corresponding metric on H by d', so that for $f, g \in H$,

$$d'(f,g) = d(\widehat{f},\widehat{g}) = \sum_{n\in\mathbb{N}} 2^{1-n}\frac{\|\widehat{f}-\widehat{g}\|_{1/n}^{1/n}}{1+\|\widehat{f}-\widehat{g}\|_{1/n}^{1/n}} = \sum_{n\in\mathbb{N}} 2^{1-n}\varrho_n(\widehat{f},\widehat{g})\,.$$

Then (H,d') is a complete metric space which cannot be normed, and so is a Fréchet space which is not a Banach space.

We end this section by proving a theorem which will be used in Section 3 to identify all of the closed subspaces of H for which analogues of Theorem 2.1 and Corollary 2.2 hold. Essentially it establishes that (H,d') behaves like a *homogeneous* Banach space. (We refer the reader to [9] for the basic facts about homogeneous spaces.)

2.3. THEOREM. *For each $f \in H$, the shift map $a \to \tau_a f$ is continuous from G to (H,d').*

Before proving this theorem, we prove the following lemma which is well-known for the case $r \geq 1$.

2.4. LEMMA. *For each $r \in (0,1]$ and sequence $(x_\chi) \in \ell^r(X)$, the map $a \to (\overline{\chi(a)}x_\chi)$ is continuous from G to $(\ell^r(X), d_r)$.*

Proof. It is sufficient to prove continuity at the identity e of G. Write $\mathbf{x} = (x_\chi)$ and $\tau_a\mathbf{x} = (\overline{\chi(a)}x_\chi)$. Then

$$d_r(\mathbf{x},\tau_a\mathbf{x}) = \sum_{\chi\in X} |\chi(a)-1|^r|x_\chi|^r\,.$$

Let $\varepsilon > 0$ be given. Then there exists a finite subset K of X for which

$$\sum_{\chi\in X\setminus K} |x_\chi|^r < 2^{-(1+r)}\varepsilon\,.$$

Moreover, for each $\chi \in K$ there exists a zero neighbourhood U_χ satisfying

$$a \in U_\chi \;\Rightarrow\; |\overline{\chi(a)}-1| < (\varepsilon/2)^{1/r}\|\mathbf{x}\|_r^{-1}\,.$$

Put $U=\bigcap\{U_\chi:\chi\in K\}$. Then for $a\in U$ we have

$$\begin{aligned}d_r(\tau_a\mathbf{x},\mathbf{x})&=\Big(\sum_{\chi\in K}+\sum_{\chi\in X\setminus K}\Big)|\chi(a)-1|^r|x_\chi|^r\\&\le\max_{\chi\in K}|\overline{\chi(a)}-1|^r\sum_{\chi\in K}|x_\chi|^r+2^r2^{-(1+r)}\varepsilon\\&\le(\varepsilon/2)\|\mathbf{x}\|_r^{-r}\|\mathbf{x}\|_r^r+\varepsilon/2=\varepsilon\,.\end{aligned}$$

Hence, the map $a\to(\overline{\chi(a)}x_\chi)$ is continuous. ∎

Proof of Theorem 2.3. For each $\chi\in X$, $(\tau_af)^\wedge(\chi)=\overline{\chi(a)}\widehat{f}(\chi)$. Therefore, it follows immediately from Lemma 2.4 that for each $r\in(0,1)$, the map $a\to(\tau_af)^\wedge$ is continuous from G to $\ell^r(X)$.

We know that $f-\tau_af\in H$, and so for any $\varepsilon>0$ there exists a positive integer N for which

$$\sum_{n>N}2^{1-n}\varrho_n((\tau_af)^\wedge,\widehat{f})<\varepsilon/2\,.$$

Further, for $n=1,\ldots,N$ there is a zero neighbourhood U_n satisfying

$$a\in U_n\ \Rightarrow\ \|(\tau_af)^\wedge-\widehat{f}\|_{1/n}^{1/n}<\varepsilon/(2N)\,.$$

Put $U=\bigcap\{U_n:n=1,\ldots,N\}$. It follows that if $a\in U$ then

$$d'(\tau_af,f)\le(\varepsilon/(2N))N+\varepsilon/2=\varepsilon$$

as required. ∎

3. More factorisation results. For each subset F of X, we denote by H_F the set of functions in H with Fourier transforms supported by F. Clearly H_F is closed in (H,d'); further $H_F*E=H_F$ whenever $H*E=H$ since

$$H_F*E=(H_F*H)*E=H_F*(H*E)=H_F*H=H_F\,.$$

In this section, we prove that if $H\subseteq E$ then the subspaces H_F are the only closed subspaces of H which factor over E. To do this, we first note that if a subspace K of H factors over E then $K*H\subseteq K*E=K$ and so K is a closed ideal of H. In Theorem 3.2 we prove that the closed ideals of H are precisely its closed translation-invariant subspaces, which we know to be the sets H_F since $T(G)$ is dense in H. This leads to our conclusion.

Before stating and proving Theorem 3.2, however, we verify that H shares another important property of homogeneous spaces; namely, that if (k_λ) is a bounded approximate identity in $L^1(G)$ then

$$\lim_\lambda d'(k_\lambda*f,f)=0$$

for any $f \in H$. (Recall that a *bounded approximate identity* in $L^1(G)$ is a net (k_λ) of absolutely integrable functions which satisfies

(i) $\sup_\lambda \|k_\lambda\|_{L^1} < \infty$ and (ii) $\lim_\lambda \|k_\lambda * g - g\|_{L^1} = 0$

for all $g \in L^1(G)$.)

3.1. THEOREM. *For any $f \in H$, and bounded approximate identity (k_λ) in $L^1(G)$, $\lim_\lambda d'(k_\lambda * f, f) = 0$.*

Proof. We begin by proving that for each $r \in (0,1]$ and $\mathbf{x} \in \ell^r(X)$

$$\lim_\lambda \|(\widehat{k}_\lambda(\chi) - 1)x_\chi\|_r^r = 0\,.$$

Once this has been established, an argument similar to that given in the final paragraph of the proof to Theorem 2.3 shows that this theorem holds.

To prove the limit statement we note that for any $\varepsilon > 0$ there exists a finite subset K of X and indices λ_χ for each $\chi \in K$ such that

$$\text{(i)} \qquad \sum_{\chi \in X\setminus K} |x_\chi|^r < \varepsilon/(2B^r), \quad \text{where } B = 1 + \sup_\lambda \|k_\lambda\|_{L^1}\,, \quad \text{and}$$

$$\text{(ii)} \qquad \lambda > \lambda_\chi \ \Rightarrow\ |\widehat{k}_\lambda(\chi) - 1| < (\varepsilon/2)^{1/r}\|\mathbf{x}\|_r^{-1}\,.$$

Let λ_0 be any index for which $\lambda_0 > \lambda_\chi$ for all $\chi \in K$. Then we can use the estimates obtained in (i) and (ii) to prove that for $\lambda > \lambda_0$

$$\begin{aligned}
&\|(\widehat{k}_\lambda(\chi) - 1)x_\chi\|_r^r \\
&\quad \le \Big\{\max_{\chi \in K} |\widehat{k}_\lambda(\chi) - 1|^r \sum_{\chi \in K} |x_\chi|^r\Big\} + \Big\{\max_{\chi \in X\setminus K} |\widehat{k}_\lambda(\chi) - 1|^r \sum_{\chi \in X\setminus K} |x_\chi|^r\Big\} \\
&\quad \le \varepsilon/2 + B^r(\varepsilon/(2B^r)) = \varepsilon\,. \ \blacksquare
\end{aligned}$$

3.2. THEOREM. *A closed subspace I of H is an ideal in H if and only if it is translation-invariant.*

Proof. First, suppose that I is a closed ideal in H. Let (k_λ) be a bounded approximate identity in $L^1(G)$, consisting of trigonometric polynomials. (See (28.53) of [8].) Then, for each $a \in G$ and each index λ, the a-translate $\tau_a(k_\lambda) \in T(G) \subseteq H$, so that $\tau_a(k_\lambda) * f \in I$ whenever $f \in I$. But $\tau_a(k_\lambda) * f = \tau_a(k_\lambda * f)$ and so, by Theorems 2.3 and 3.1,

$$\lim_\lambda d'(\tau_a(k_\lambda * f), \tau_a f) = 0\,.$$

Since I is closed in H, this means that $\tau_a f \in I$.

The converse holds since $H * H_F = H_F$ for any subset F of X. $\blacksquare$

3.3. COROLLARY. *If $H \subseteq E$ and K is a closed subspace of H which factors over E then K is translation-invariant.*

Using the corollary we justify our earlier claim that if $H \subseteq E$ then the only closed subspaces of (H, d') that factor over E are those of the form H_F, $F \subseteq X$. There are, of course, other (non-closed) subspaces with this property — the most obvious example is $T(G)$ itself. In the next section we will briefly discuss another family of examples.

4. The case $G = \mathbb{T}$. In this section we obtain some results which apply specifically to the circle group $\mathbb{T}$, which we identify with the interval $[0, 2\pi)$.

Since the Fourier transforms of elements of H are in $\ell^r(\mathbb{Z})$ for each $r > 0$, we would expect the elements to satisfy strong smoothness conditions. An obvious one to investigate is the Lipschitz condition. Recall that a continuous function f on $\mathbb{T}$ satisfies a *Lipschitz condition of order* $\alpha \in (0, 1]$ if and only if there exists a constant K for which

$$\|\tau_a f - f\|_\infty \le K|a|^\alpha$$

for any $a \in (0, 2\pi)$. The linear space of all such functions is usually denoted by $\Lambda_\alpha(\mathbb{T})$. Clearly, if $\beta < \alpha$ then $\Lambda_\alpha(\mathbb{T}) \subset \Lambda_\beta(\mathbb{T})$. It is known that $\Lambda_\alpha(\mathbb{T})$ does not factor over $L^p(\mathbb{T})$ for any $p \ge 1$ (see [16]).

4.1. Proposition. *$H \setminus \bigcup\{\Lambda_\alpha : \alpha \in (0, 1]\}$ is non-empty.*

Proof. We use Theorem 1 of [4] to prove this proposition. Define f on $\mathbb{T}$ by $f(x) = \sum_{n\in\mathbb{N}} 2^{-n} \exp(i2^{2^n} x)$. Then for any $r > 0$ and any positive integer N, a simple calculation shows that

$$\sum_{n\ge N} |\widehat{f}(n)|^r = \sum_{2^{2^n} \ge N} 2^{-nr} = \frac{2^{-rc\ln\ln N}}{1 - 2^{-r}}$$

for some $c > 0$. For each choice of r, this clearly tends to zero as $N \to \infty$. Now $\{2^{2^n} : n > 0\}$ is a Sidon subset of $\mathbb{Z}$ and so Edwards's characterisation of elements in $\Lambda_\alpha(\mathbb{T})$ which have Fourier transforms supported by Sidon sets may be used. Putting $r = 1$ in the previous calculation and writing $2^c = e^{-K}$, where K is necessarily positive, gives

$$\sum_{|n|\ge N} |\widehat{f}(n)| = \sum_{n\ge N} |\widehat{f}(n)| = 2e^{-K\ln\ln N} = 2(\ln N)^{-K}$$

which is not $O(N^{-\alpha})$ for any $\alpha > 0$. Hence $f \notin \Lambda_\alpha(\mathbb{T})$ for any $\alpha \in (0, 1)$. ∎

We can also rule out the possibility that every element of H is *analytic*. Recall that a function f is analytic on $\mathbb{T}$ if, in a neighbourhood of each $t_0 \in \mathbb{T}$, $f(t)$ can be represented by a power series centred at t_0. If f is analytic then its Fourier transform $\widehat{f}$ decays exponentially; that is, there exist positive real numbers λ and C such that $|\widehat{f}(n)| \le Ce^{-\lambda|n|}$. The next example shows that H contains many non-analytic functions.

EXAMPLE. Let (p_n) be an increasing sequence of positive numbers for which there exists $\alpha \in (0,1)$ satisfying $n^{-\alpha}p_n \to 0$ as $n \to \infty$. Define the sequence $\mathbf{x}$ by

$$x_n = \{n\}^{-p_n} \quad \text{if } n \geq 1 \quad \text{and} \quad x_n = 0 \quad \text{otherwise}.$$

Then $\mathbf{x} \in \ell^r(\mathbb{Z})$ for any $r \in (0,1]$ since, given such an r, there exists $N_r > 1$ for which $n > N_r$ implies that $p_n > 2/r$ and so that $n^{-p_n} < n^{-2/r}$. However, $\mathbf{x}$ is not exponentially bounded. To see this, take $\lambda > 0$ and put $\omega_n = n^{-p_n}e^{\lambda n}$ so that $\ln \omega_n = -p_n \ln n + \lambda n$. Then

$$\lim_{n\to\infty} n^{-1}p_n \ln n = \lim_{n\to\infty} (n^{-\alpha}p_n)(n^{\alpha-1}\ln n) = 0,$$

and so for any $\varepsilon > 0$ there exists N_ε for which $n > N_\varepsilon$ implies that $0 < p_n \ln n < \varepsilon n$. Take $\varepsilon = \lambda/2$. Then for $n > N_{\lambda/2}$,

$$\ln \omega_n = \lambda n - p_n \ln n > \lambda n - \lambda n/2 = \lambda n/2,$$

ensuring that $\lim_{n\to\infty} \ln \omega_n = \infty = \lim_{n\to\infty} \omega_n$. ∎

For each $\lambda \geq 0$, let H^λ be the set defined by

$$H^\lambda = \{f \in H : (\widehat{f}(n)e^{\lambda|n|}) \in \ell^\infty(\mathbb{Z})\}.$$

Since H^0 is the whole of H, we will assume that $\lambda > 0$. The set of analytic functions in H is precisely $\bigcup\{H^\lambda : \lambda > 0\}$. It is easy to verify that H^λ is a linear subspace of H which contains $T(G)$ and so cannot be closed in H, and that if $\lambda > \sigma$ then H^λ is a proper subspace of H^σ. Each H^λ may be identified with the set Δ^λ, where

$$\Delta^\lambda = \{\mathbf{x} = (x_n) : (x_n e^{\lambda|n|}) \in \ell^\infty(\mathbb{Z})\},$$

via the Fourier transformation. Clearly Δ^λ is a subset of $\ell^s(\mathbb{Z})$ for every $s > 0$ and $\Delta^\lambda\ell^\infty(\mathbb{Z}) = \Delta^\lambda$. More generally, for every subset A of $\ell^\infty(\mathbb{Z})$, $\Delta^\lambda A \subseteq \Delta^\lambda$; however, in general, this will not be an equality. In particular, if $A \subseteq c_0(\mathbb{Z})$, it will never be an equality since the sequence $\mathbf{x} = (\exp(-\lambda|n|))$ is in Δ^λ but does not factor over any subset of $c_0(\mathbb{Z})$.

Consider instead the set $\Omega^\lambda = \bigcup\{\Delta^\eta : \eta > \lambda\}$, which is a proper subset of Δ^λ. It is a linear space since the Δ^η's are nested, and for any subset A of $\ell^\infty(\mathbb{Z})$, $\Omega^\lambda A \subseteq \Omega^\lambda$. For many choices of A, the reverse inclusion also holds. In particular, we can prove the following proposition.

4.2. PROPOSITION. *If $\ell^+(\mathbb{Z}) \subseteq A$ then $\Omega^\lambda A = \Omega^\lambda$ for each $\lambda > 0$.*

Proof. It remains only to prove that $\Omega^\lambda \subseteq \Omega^\lambda A$. Let $\mathbf{x} \in \Omega^\lambda$ and choose $\eta > \lambda$ such that $\mathbf{x} \in \Delta^\eta$. Now define the sequences $\mathbf{a}$ and $\mathbf{y}$ by

$$a_n = \exp\left(\frac{\lambda - \eta}{2}|n|\right) \quad \text{and} \quad y_n = x_n \exp\left(\frac{\eta - \lambda}{2}|n|\right)$$

for each $n \in \mathbb{Z}$. Then it is easy to see that $\mathbf{x} = \mathbf{ya}$, $\mathbf{a} \in \ell^+(\mathbb{Z})$, and $\mathbf{y} \in \Delta^{(\lambda+\eta)/2} \subseteq \Omega^\lambda$. ∎

4.3. COROLLARY. *For each* $\lambda > 0$ *and* $p \geq 1$, $\{f \in C(G) : \widehat{f} \in \Omega^\lambda\}$ *factors over* $L^p(\mathbb{T})$ *and* $A^p(\mathbb{T})$.

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