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HARMONIC FUNCTIONS AND HARDY SPACES ON TREES WITH BOUNDARIES

BY

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1. Introduction. The theory of harmonic functions on trees with respect to nearest neighbours transition operators (see [1], [4] and references there) resembles the classical theory on the unit disk. In the present paper we study harmonic functions on special subtrees, called *horotrees* because they correspond rather to horodisks than to the whole disk. In the classical theory of harmonic functions there is no difference between disks and horodisks, but in the case of trees the difference becomes essential. For the whole tree, thanks to some additional restrictions on the transition operator, the Poisson boundary coincides with the usual boundary Ω of the tree, and hence it is a compact totally disconnected space. In our case the Poisson boundary is a discrete topological space. For the case of horotrees the related random walks are always transient. There is more freedom in the choice of transition operators, and the guarantee that the space of harmonic functions is nontrivial. The "full tree" case rather imitates Riemannian manifolds whose sectional curvature always stays between two negative constants, while our case corresponds to a manifold with boundary whose sectional curvature is still negative but can tend to zero far away from the boundary.

The paper is organized as follows: In §3 we prove the maximum principle for the transition operator. In §4 we prove the Poisson representation theorem for harmonic functions and give an explicit formula for the Poisson kernel. Then in §5 we study the Hardy–Littlewood and the harmonic maximal operators and in the final section we give the atomic characterization of the H^p spaces.

I wish to thank the authors of [3] who have suggested to me the study of harmonic functions on horotrees and asked for a parallel theory to that presented in [3].

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2. Harmonic functions. Let X be a homogeneous tree of degree $q+1$. Fix a reference vertex x_0 in X and a point ω in the boundary Ω of X. For $x \in X$ put

$$\delta_{x_0,\omega}(x) = d(x,x') - d(x_0,x'),$$

where x' denote the confluence point of the geodesics $[x,\omega]$ and $[x_0,\omega]$ and $d(\cdot,\cdot)$ is the usual distance on X. The sets $H_n(\omega) = \{x \in X : \delta_{x_0,\omega}(x) = n\}$ and $D_n(\omega) = \{x \in X : \delta_{x_0,\omega}(x) \le n\}$ are called respectively the *horocycles* and *horotrees* with center ω. Any two horotrees are isomorphic; what is more, there exists an automorphism of the tree X which maps one of them onto the other.

Fix a horotree D. Then $D = D_0(\omega)$ for some $x_0 \in X$ and $\omega \in \Omega$. Denote by $\operatorname{int} D$ and ∂D the *interior* $\operatorname{int} D = D_{-1}(\omega)$ and the *boundary* $\partial D = H_0(\omega)$ of D. Suppose we are given a *transition matrix* p on D by assigning a positive number $p(x,y)$ to each (oriented) edge (x,y) with $x \in \operatorname{int} D$. Let $p(x,x) = 1$ for $x \in \partial D$ and $p(x,y) = 0$ for all other pairs x,y in D. The transition matrix p gives rise to an operator Δ, called the *Laplace operator*, which assigns to a function F on D a function ΔF by

$$\Delta F(x) = \sum_{y \in D} p(x,y)F(y) - F(x)\,.$$

A function F on D will be called *harmonic* if $\Delta F = 0$. The general theory of harmonic functions on trees, as presented in [1], does not apply to the case of a horotree because the latter contains, up to equivalence, only one infinite geodesic.

The aim of this paper is to study harmonic functions on D when the transition matrix $p = \{p(x,y)\}$ satisfies the following assumptions:

(A1) p is stochastic, i.e. $\sum_y p(x,y) = 1$ for any $x \in D$.

(A2) p is isotropic, i.e. $p(gx,gy) = p(x,y)$ for any $x \in \operatorname{int} D$, $y \in D$ and $g \in \operatorname{Aut}(D)$.

The first assumption implies that any constant function is harmonic and the second that p has a very simple form.

LEMMA 1. *The transition matrix $p = \{p(x,y)\}$ satisfies the assumptions* (A1) *and* (A2) *if and only if there exists a sequence $\alpha_1, \alpha_2, \ldots$ of positive numbers such that*

$$p(x,y) = \frac{1}{1+\alpha_n} \tag{1a}$$

whenever $x \in H_{-n}(\omega)$ for some $n \ge 1$ and $y = \overline{x}$ is the unique neighbour of x in $H_{-n-1}(\omega)$, and

$$p(x,y) = \frac{q^{-1}\alpha_n}{1+\alpha_n} \tag{1b}$$

when y is one of the other neighbours of x ($y \in H_{-n+1}(\omega)$ in that case).

P r o o f. Assume (A1) and (A2). The group $\mathrm{Aut}(D)$ acts transitively on each of the horocycles $H_n(\omega)$, $n \le 0$, and also $\overline{gx} = g\overline{x}$ for $g \in \mathrm{Aut}(D)$. It follows that $p(x, \overline{x})$ depends only on the index of the horocycle to which x belongs. Moreover, any permutation of the set of neighbours $\neq \overline{x}$ of x can be extended to an automorphism of D (which stabilizes x) so $p(x, y) = \mathrm{const}$ when y varies, $y \neq \overline{x}$.

On the other hand, any $g \in \mathrm{Aut}(D)$ preserves each $H_n(\omega)$, $n \le 0$. Thus if p is defined by (1) it is g-invariant. ∎

EXAMPLE 1. Let the transition matrix be as in Lemma 1. Let $\beta_0 = 0$ and $\beta_{n+1} = 1 + \alpha_1 + \alpha_1\alpha_2 + \ldots + \alpha_1 \cdot \ldots \cdot \alpha_n = \sum_{k=0}^{n} \prod_{i=0}^{k} \alpha_i$ (with $\alpha_0 = 1$) for $n = 0, 1, 2, \ldots$ It is obvious that the following recurrence is then satisfied:

$$\beta_n = \frac{1}{1+\alpha_n}\beta_{n+1} + \frac{\alpha_n}{1+\alpha_n}\beta_{n-1}\,.$$

It follows that the function F_0 defined on D by $F_0(x) = \beta_n$ for $x \in H_{-n}(\omega)$ is harmonic.

3. Maximum principle. From now on we always assume that the transition matrix p satisfies (A1) and (A2).

THEOREM 1 (Maximum principle). *Assume additionally that*

$$\text{(A3)} \qquad 1 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \ldots = \sum_{k=0}^{\infty} \prod_{i=0}^{k} \alpha_i = \infty.$$

Let F be a real function bounded (from above) on D and suppose that $\Delta F \ge 0$ but $F|_{\partial D} \le 0$. Then $F \le 0$.

If (A3) is not satisfied then there exists a real bounded harmonic function F_0 on D such that $F_0(x) = 0$ for $x \in \partial D$ and $F(x) > 0$ for $x \in \mathrm{int}\, D$.

P r o o f. To prove the second part of the theorem take F_0 from Example 1.

Let $x_0, x_1, x_2, \ldots$ denote the sequence of vertices of the geodesic $[x_0, \omega]$ and let E_n for $n = 0, 1, 2, \ldots$ denote the (finite) subtree in D of those x for which $x_n \in [x, \omega]$. The boundary ∂E_n is contained in $\partial D \cup \{x_n\}$. Let $F_n = F - \beta_n^{-1} F_0$, where β_n, F_0 are as in Example 1 and F satisfies the assumptions of the first part of the theorem (we may additionally assume that $F \le 1$). Then $\Delta F_n \ge 0$ on E_n and $F_n|_{\partial E_n} \le 0$. The usual maximum principle applied to the operator Δ on the finite set E_n implies that $F_n \le 0$ on E_n. Fix x in D. Since $x \in E_n$ for n large enough, say $n \ge n_0$, we have

$$0 \ge F(x) - \beta_n^{-1} F_0(x) \ge F(x) - \beta_{n_0}/\beta_n\,.$$

But β_{n_0}/β_n tends to zero as $n \to \infty$. Thus $F(x) \le 0$. ∎

4. The Poisson formula. The maximum principle implies that if (A1)–(A3) are satisfied then any bounded harmonic function F on D is uniquely determined by its restriction to the boundary ∂D. This allows us to hope that there exists a reproducing kernel $P : D\times\partial D \to \mathbb{R}$ (independent of F), called here the *Poisson kernel*, so that

$$F(x) = \sum_{y\in\partial D} P(x,y)F(y)\,, \tag{2}$$

the series being absolutely convergent. We give an explicit formula for P and we show that (2) can also be used to produce harmonic functions.

LEMMA 2. *Let F be a harmonic function on D and let $x \in H_{-k}(\omega)$, $k > 0$. Then*

$$F(x) = (\beta_k/\beta_{k+1})F(\overline{x}) + (1-\beta_k/\beta_{k+1})q^{-k} \sum_{\substack{y\in\partial D\\ d(y,x)=k}} F(y)\,,$$

where $\overline{x}$ is the predecessor of x, i.e. the unique neighbour of x in the geodesic $[x,\omega]$.

Proof. Let E be the finite subtree in D consisting of $\overline{x}$ and all y in D such that $x \in [y,\omega]$. Consider a discrete parameter Markov chain (random walk) $X_0, X_1, X_2, \ldots$ on E with one-step transition probabilities $\{p_{uv}\}_{u,v\in E}$, where $p_{uv} = p(u,v)$ if $u \neq \overline{x}$ and where $\overline{x}$ is an absorbing barier; $p_{\overline{x}\overline{x}} = 1$ and $p_{\overline{x}v} = 0$ for $v \neq \overline{x}$ (we refer to [2] for basic notations and properties of Markov chains). For $m = 1,2,\ldots$ denote by $p^{(m)}_{uv}$ the m-step transition probabilities

$$p^{(m)}_{uv} = \mathbb{P}\{X_m = v \mid X_0 = u\}.$$

Then since F is harmonic we have

$$F(x) = \sum_{y\in E} p^{(m)}_{xy}F(y), \qquad m = 1,2,\ldots$$

The limit $\lim_{m\to\infty} p^{(m)}_{xy}$ exists. Indeed, let p^*_{uv} denote the *hitting probability*

$$\begin{aligned} p^*_{uv} &= \mathbb{P}\{X_m = v \text{ for some } m > 0 \mid X_0 = u\} \\ &= \sum_{m=1}^{\infty} \mathbb{P}\{X_k \neq v,\ 0 < k < m;\ X_m = v \mid X_0 = u\}. \end{aligned}$$

If $v = \overline{x}$ or $v \in E\cap\partial D = \{y \in \partial D : d(y,x) = k\}$ then v is an absorbing state for the random walk, $p^{(m)}_{uv}$ increases in that case and

$$\lim_{m\to\infty} p^{(m)}_{uv} = p^*_{uv}.$$

Any other state v in E is nonrecurrent (i.e. $p^*_{uv} < 1$), hence $\sum_{m=1}^{\infty} p^{(m)}_{uv} < \infty$

and in particular, $\lim_{m\to\infty} p_{uv}^{(m)} = 0$. All this together gives

$$F(x) = p^*_{x\bar{x}}F(\bar{x}) + \sum_{y\in E\cap\partial D} p^*_{xy}F(y).$$

The numbers p^*_{xy}, $y \in E\cap\partial D$, are all equal by (A2) and their sum is $1-p^*_{x\bar{x}}$. To finish the proof we only have to show that

$$p^*_{x\bar{x}} = \beta_k/\beta_{k+1}. \tag{3}$$

Write

$$p^*_{x\bar{x}} = p_{x\bar{x}} + \sum_{y\neq\bar{x}} p^*_{xy}p^*_{y\bar{x}}.$$

The sum here can be taken only over the neighbours of x. Now $p^*_{y\bar{x}} = p^*_{yx}p^*_{x\bar{x}}$ because a random walk starting at $y \neq \bar{x}$ and visiting $\bar{x}$ has to visit x in between. Thus

$$p^*_{x\bar{x}} = p_{x\bar{x}}\Big/\Big(1 - \sum_{y\neq x} p_{xy}p^*_{yx}\Big). \tag{4}$$

But $x = \bar{y}$ with $y \in H_{-(k-1)}(\omega)$ and (3) follows from (4) by induction on k. ∎

LEMMA 3. *The Poisson kernel has the form*

$$P(x,y) = \sum_{n=k}^{\infty} \beta_k(\beta_n^{-1} - \beta_{n+1}^{-1})m_{n,x}(y), \qquad x \in H_{-k}(\omega),\ y \in \partial D, \tag{5}$$

where

$$m_{n,x} = \begin{cases} q^{-n} & \text{if } d(y,x) \le 2n-k, \\ 0 & \text{otherwise.} \end{cases}$$

For $k=0$, (5) has to be read

$$P(x,y) = m_{0,x}(y) = \delta_{x,y}.$$

P r o o f. Let $x_k, x_{k+1}, \ldots$ be the sequence of vertices in the geodesic $[x,\omega]$, i.e. $x_k = x$ and $x_{n+1} = \bar{x}_n$ for $n \ge k$. Let F be a bounded harmonic function on D. Lemma 2 applied successively to F and x_n, $n = k, k+1, \ldots, N$, gives

$$F(x) = \sum_{n=k}^{N} \beta_k(\beta_n^{-1} - \beta_{n+1}^{-1})m_{n,x}(y)F(y) + (\beta_k/\beta_{N+1})F(x_N).$$

This is because $\{y \in \partial D : d(y,x) \le 2n-k\} = \{y \in \partial D : d(y,x_n) = n\}$. Since β_{N+1}^{-1} tends to zero (assumption (A2)), the series converges. ∎

THEOREM 2 (Poisson formula). *Let f be a bounded function on the boundary ∂D. Define a function F on D by the formula*

$$F(x) = \sum_{y\in\partial D} P(x,y)f(y),$$

where $P(x,y)$ is the Poisson kernel given by (5). *Then F is a bounded harmonic function. Any bounded harmonic function F on D is of that form.*

Proof. The second part of the theorem is just Lemma 3. To prove the first it suffices to observe that any of the functions $P(\cdot,y)$, $y\in\partial D$, is harmonic. This can be shown by general arguments but it is much simpler to read it off from (5) just because $m_{n,\bar{x}}=m_{n,x}$ for $n>k$ and $\sum_{v\neq\bar{x}} m_{n,v} = qm_{n,x}$. ■

5. Maximal functions. The Poisson formula implies that any bounded function f on ∂D has a unique extension to a bounded harmonic function F on D. Define the *maximal function Mf* of f on ∂D by

$$Mf(y)=\sup_{x\in[y,\omega]}|F(x)|,$$

where, as before, $[y,\omega]$ stands for the geodesic from y to ω. We will prove that the operator $f\to Mf$ is bounded on each of the spaces $\ell^p(\partial D)$, $p>1$.

For $y\in\partial D$ put $B(y,n)=\{v\in\partial D : d(v,y)\le 2n\}$ (the distance of two vertices in ∂D is always an even number). Then $|B(y,n)|=q^n$. The sets of type $B(y,n)$ will be called *intervals*. For a locally bounded function f on ∂D let f^* denote the Hardy–Littlewood maximal function

$$f^*(y)=\sup_n\frac{1}{|B(y,n)|}\Big|\sum_{v\in B(y,n)} f(v)\Big|.$$

LEMMA 4. *The maximal operator $f\to f^*$ is of weak type* $(1,1)$.

Proof. Fix $s>0$ and put $A_s=\{y\in\partial D : f^*(y)>s\}$. We have to show that $|A_s|\le\|f\|_1/s$. If $y\in A_s$ then there exists an index n_y so that

$$\frac{1}{|B(y,n_y)|}\Big|\sum_{v\in B(y,n_y)} f(v)\Big|>s.$$

Since any two intervals in ∂D are either disjoint or included one in the other, we can find a (finite) family I of pairwise disjoint intervals so that $A_s\subseteq\bigcup_{y\in I}B(y,n_y)$. But then

$$\begin{aligned} s|A_s| &\le \sum_{y\in I} s|B(y,n_y)| \le \sum_{y\in I}\sum_{v\in B(y,n_y)}|f(v)| \\ &\le \sum_{v\in\partial D}|f(v)| = \|f\|_1\,. \ ■ \end{aligned}$$

THEOREM 3. *Let f be a bounded function on ∂D. Then*

$$Mf(y)\le f^*(y),\qquad y\in\partial D.$$

It follows that the maximal operator $f\to Mf$ is of weak type $(1,1)$ *and strong type* (p,p) *for any* $p>1$.

Proof. Let $x \in [y,\omega] \cap H_{-k}(\omega)$. Then

$$(6)\quad F(x) = \sum_{v\in\partial D} P(x,v)f(v) = \sum_{n=k}^{\infty} \beta_k(\beta_n^{-1} - \beta_{n+1}^{-1}) \sum_{v\in\partial D} m_{n,x}(v)f(v)$$
$$= \sum_{n=k}^{\infty} \beta_k(\beta_n^{-1} - \beta_{n+1}^{-1}) \frac{1}{|B(y,n)|} \sum_{v\in B(y,n)} f(v).$$

Hence

$$|F(x)| \le \sum_{n=k}^{\infty} \beta_k(\beta_n^{-1} - \beta_{n+1}^{-1}) f^*(y) = f^*(y).$$

The second part of the theorem follows from Lemma 4 and the Marcinkiewicz interpolation theorem. ∎

Remarks. 1. The maximal operator $f \to Mf$ is not ℓ^1 bounded. If $v, w \in \partial D$ then

$$M\delta_v(w) = \sum_{n=k}^{\infty} \beta_k\big(\beta_n^{-1} - \beta_{n+1}^{-1}\big) q^{-n} \ge q^{-k} \frac{\beta_{k+1} - \beta_k}{\beta_{k+1}},$$

where $k = d(v,w)/2$. This gives

$$\|M\delta_v\|_1 \ge \frac{q-1}{q} \sum_{k=1}^{\infty} \frac{\beta_{k+1} - \beta_k}{\beta_{k+1}} = \infty.$$

2. In fact, a stronger result is true. If $f \in \ell^1(\partial D)$ and $\sum_{v\in\partial D} f(v) \ne 0$ then $\|Mf\|_1 = \infty$. To see this use (6) with k large enough.

3. Assume that $\inf_n \alpha_n > 1$. Then the maximal operators $f \to f^*$ and $f \to Mf$ are equivalent (cf. [3, Theorem 4]). Indeed, let $y \in \partial D$ and let $[y,\omega] = \{y_0, y_1, \ldots\}$. Then

$$\beta_k^{-1} F(y_k) - \beta_{k+1}^{-1} F(y_{k+1}) = \big(\beta_k^{-1} - \beta_{k+1}^{-1}\big) \frac{1}{|B(y,k)|} \sum_{v\in B(y,k)} f(v).$$

Hence

$$f^*(y) \le \sup_k \left| \frac{\beta_{k+1}}{\beta_{k+1} - \beta_k} F(y_k) - \frac{\beta_k}{\beta_{k+1} - \beta_k} F(y_{k+1}) \right|$$
$$\le \sup_k \frac{\beta_{k+1} + \beta_k}{\beta_{k+1} - \beta_k} Mf(y) \le \frac{1+r}{1-r} Mf(y)$$

with $1/r = \inf_n \alpha_n$.

4. If $\alpha_n \equiv 1$ then the maximal functions δ_v^* and $M\delta_v$ are not equivalent.

Indeed, if $w \in \partial D$ and $d(w,v) = 2m$ then $\delta_v^*(w) = q^{-m}$ but

$$M\delta_v(w) = \sup_k \sum_{n\ge m, n\ge k} \frac{k}{n(n+1)} q^{-n} = \sum_{n\ge m} \frac{m}{n(n+1)} q^{-n} < \frac{q}{(m+1)(q-1)} \cdot q^{-m}.$$

6. Hardy spaces H^p. Consider the following question: what conditions on a bounded function f on ∂D ensure that the restriction $F|_{H_{-k}}$ of its harmonic extension F to any horocycle H_{-k}, $k = 0, 1, 2, \ldots$, is a p-summable function? Clearly, since $f = F|_{H_0}$, f itself must be p-summable. For $p \ge 1$ a necessary and sufficient condition is $f \in \ell^p(\partial D)$, but the case $0 < p < 1$ is much more subtle. Generally, the answer depends on the choice of the transition matrix p. But even the Dirac functions δ_v may not be admissible (see Remark 4). Nevertheless, there is still a very large class of positive examples. By using the inequalities

$$\sup_n \big\|F|_{H_{-k}}\big\|_p \le \big\|Mf\big\|_p \le \big\|f^*\big\|_p$$

one can construct many of them, even without looking at the Poisson formula.

Following the classical definition, for $0 < p < 1$, let

$$H^p(\partial D) = \{f : f^* \in \ell^p(\partial D)\}.$$

It is a proper linear subspace in $\ell^p(\partial D)$ and consists of functions with mean value zero (cf. Remark 2). Continuing the analogy to the classical case, define a *p-atom* to be a function a on ∂D such that there exists an interval $B = B(y,n) = \{v \in \partial D : d(v,y) \le 2n\}$ with $\operatorname{supp} a \subset B$, $\sum_{v\in\partial D} a(v) = 0$ and $\sup_v |a(v)| \le |B|^{-1/p}$.

LEMMA 5. *Let a be a p-atom. Then $a^* \in \ell^p(\partial D)$ and*

$$\|a^*\|_p \le 1.$$

Proof. It is clear that $a^*(y) \le \|a\|_\infty < |B|^{-1/p}$. But if $y \notin B$ then each of the intervals $B(y,n)$ is either disjoint from B or contains it. However, in both cases $\sum_{v\in B(y,n)} a(v) = 0$. Therefore $a^*(y) = 0$ outside B. This proves the claim. ∎

THEOREM 4 (Characterization of H^p). *Let $0 < p \le 1$. A function f on ∂D belongs to H^p if and only if it has an atomic decomposition*

$$f = \sum_n \lambda_n a_n, \tag{7}$$

where $a_1, a_2, \ldots$ are p-atoms and $\sum_n |\lambda_n|^p < \infty$.

Proof. We construct a sequence of functions $f_0, f_1, f_2, \ldots$ on ∂D as follows. Put $f_0 = f$. If f_k is already known let $s_k = \|f_k\|_\infty$ and let

$$A_k = \{y \in \partial D : f_k^*(y) > s_k/2\}.$$

For y in A_k denote by n_y the greatest natural number n so that

$$\frac{1}{|B(y,n)|}\Big|\sum_{v\in B(y,n)} f(v)\Big| > s_k/2.$$

Then $A_k = \bigcup_{y\in A_k} B(y, n_y)$. Consider the set

$$A_k' = \bigcup_{y\in A_k} B(y, n_y+1).$$

It is clear that $A_k' \supset A_k$, but note that $|A_k'| \le q|A_k|$. Following the idea of the proof of Lemma 4 we find a (unique) finite subset I_k in A_k such that

$$\bigcup_{y\in I_k} B(y, n_y+1) = A_k'$$

and the intervals are mutually disjoint. Put $f_{k+1} = f_k$ outside A_k', and on each of the intervals $B(y, n_y+1)$, $y \in I_k$, let f_{k+1} take the constant value

$$\gamma_y = \frac{1}{|B(y,n_y+1)|}\sum_{v\in B(y,n_y+1)} f(v).$$

Note, and this is important for the construction, that $\gamma_y \le s_k/2$. To any y in I_k we assign a function a_y on ∂D which, being zero outside, on $B(y, n_y+1)$ coincides with the function $\lambda_y^{-1}(f_k - \gamma_y)$, where

$$\lambda_y = |B(y, n_y+1)|^{1/p} 4 s_k.$$

It is easy to check that any a_y, $y \in I_k$, is a p-atom and that

$$f_k = f_{k+1} + \sum_{y\in I_k} \lambda_y a_y. \tag{8}$$

By the construction we have $s_{k+1} \le s_k/2$,

$$f_{k+1}^*(v) \le \min\{s_{k+1}, f_k^*\}, \qquad v \in \partial D, \tag{9}$$

and

$$\Big\|\sum_{y\in I_k} \lambda_y a_y\Big\|_p^p \le \sum_{y\in I_k} |\lambda_y|^p \le 4^p s_k^p \sum_{y\in I_k} |B(y, n_y+1)|$$
$$= 4^p s_k^p |A_k'| \le 4^p q s_k^p |A_k| < \infty.$$

It is now easy to deduce from (8) and (9) that

$$f = \sum_{k=0}^{\infty}\sum_{y\in I_k} \lambda_y a_y$$

and that

$$\sum_{k=0}^{\infty}\sum_{y\in I_k}|\lambda_y|^p \le 4^p q \sum_{k=0}^{\infty} s_k^p \mid \{v\in\partial D : f^*(v) > s_k/2\} \mid$$
$$\le 8^p q(2^p-1)^{-1}\|f^*\|_p^p.$$

The proof in the opposite direction is much easier. If f has the form (7) then $f^* \le \sum_n |\lambda_n| a_n^*$ and $\|f^*\|_p^p \le \sum_n |\lambda|^p \|a^*\|_p^p < \infty$ because the pth power is concave. Consequently, $f \in H^p$.

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