THE SIZE OF \((L^2, L^p)\) MULTIPLIERS

by

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0. Introduction. A complex valued function \(\varphi\) defined on the dual \(\Gamma\) of an infinite compact abelian group \(G\) is called an \((L^p, L^q)\) multiplier if for all \(f \in L^p(G)\), \(M_{\varphi}f \in L^q(G)\) where by \(M_{\varphi}f\) we mean the function whose Fourier transform is given by 
\[
\hat{M_{\varphi}f}(\chi) = \varphi(\chi) \hat{f}(\chi)
\]
for \(\chi \in \Gamma\). The space of \((L^p, L^q)\) multipliers will be denoted by \(M(p, q)\). When \(\mu\) is a bounded Borel measure on \(G\), then \(M_\mu \in M(p, p)\) (we will write \(\mu \in M(p, p)\)). If a multiplier \(\varphi \in M(2, p)\) for some \(p > 2\) then \(\varphi\) is called \(L^p\)-improving. For basic properties, and background information on \(L^p\)-improving multipliers we refer the reader to [5] and [8].

In this paper we investigate the relationship between the size of the function \(\varphi\) and membership in \(M(2, p)\) for certain types of multipliers, furthering the work of [2] and [5] in particular.

By a one-sided Riesz product we mean a multiplier \(\varphi\) given by
\[
\varphi(\chi) = \left\{ \begin{array}{ll}
\prod a_i^{\varepsilon_i} & \text{if } \chi = \prod \chi_i^{\varepsilon_i}, \varepsilon_i = 0, 1, \\
0 & \text{otherwise}
\end{array} \right.
\]
where \(\{\chi_i\}\) is a dissociate subset of \(\Gamma\) and \(\{a_i\}\) is a bounded sequence of complex numbers. We will write \(\varphi = \prod (1 + a_i \chi_i)\) for short. When \(\chi_i^2 = 1\) for all \(\chi_i\) then a one-sided Riesz product is actually the Fourier transform of a Riesz product; and like Riesz products, one-sided Riesz products exhibit interesting phenomena. Extending work of Bonami [2], in Section 2 we characterize certain (one-sided) Riesz products on \(T^\infty\), \(D^\infty\) and \(T\) which belong to \(M(2, p)\). This characterization shows that the necessary conditions on the size of \((L^2, L^p)\) multipliers which we obtain in Section 1 are best possible, but are not sufficient even for (one-sided) Riesz products, answering an open problem in [5].

In [8] \((L^2, L^p)\) multipliers are “almost” characterized. The necessary conditions we establish are combined with this result to sharpen the known estimates of the \(\Lambda(p)\) constants of sums of dissociate sets. The previously known best estimates were developed in [2] by mainly combinatorial methods.

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1. Necessary conditions. As a preliminary result we obtain lower bounds for $L^p$ norms of Riesz products and one-sided Riesz products.

**Lemma 1.1.** Let $\{\chi_i\}_{i=1}^\infty$ be a dissociate subset of $\Gamma$ such that $\chi_i^2 \neq 1$. For each $p > 0$ there are positive constants $k = k_p$ and $c = c_p < 1/2$ such that

(a) \[ \prod_{i=1}^N (1 + (p-1)|c_i|^2 - k|c_i|^3) \leq \left\| \prod_{i=1}^N (1 + c_i\chi_i + \overline{c_i}\chi_i) \right\|_p \]

\[ \geq \prod_{i=1}^N (1 + (p-1)|c_i|^2 + k|c_i|^3) , \]

and

(b) \[ \prod_{i=1}^N (1 + |c_i|^2p/4 - k|c_i|^3) \leq \left\| \prod_{i=1}^N (1 + c_i\chi_i) \right\|_p \]

\[ \leq \prod_{i=1}^N (1 + c_i^2p/4 + k|c_i|^3) \]

whenever $N \in \mathbb{N}$ and $\{c_i\}$ is a sequence of complex numbers with $|c_i| \leq c$ for all $i$.

**Proof.** In what follows the constant $k = k_p$ may vary from one line to another.

(a) The Taylor series expansion of $(1 + x)^p$ for $|x|$ small yields that

\[ \left\| \prod_{i=1}^N (1 + c_i\chi_i + \overline{c_i}\chi_i) \right\|_p \]

\[ \geq \int \prod_{i=1}^N \left(1 + p(c_i\chi_i + \overline{c_i}\chi_i) + \frac{p(p-1)}{2}(c_i\chi_i + \overline{c_i}\chi_i)^2 - k|c_i|^3 \right) . \]

As $\{\chi_i\}$ is a dissociate set this integral equals $\prod_{i=1}^N (1 + p(p-1)|c_i|^2 - k|c_i|^3)$. By taking $p$th roots and another application of Taylor series we obtain the first inequality in (a). The other is similar.

For (b) first we observe that

\[ \left\| \prod_{i=1}^N (1 + c_i\chi_i) \right\|_p = \left[ \int \prod_{i=1}^N ((1 + c_i\chi_i)(1 + \overline{c_i}\chi_i))^{p/2} \right]^{1/p} \]

\[ = \prod_{i=1}^N (1 + |c_i|^2)^{1/2} \left[ \int \prod_{i=1}^N \left(1 + \frac{c_i\chi_i + \overline{c_i}\chi_i}{1 + |c_i|^2} \right)^{p/2} \right]^{1/p} . \]

Using part (a) it follows that if constants $|c_i|$ are sufficiently small, than the
integral in the line above dominates
\[
\prod_{i=1}^{N} \left( 1 + \left( \frac{p}{2} - 1 \right) \frac{|c_i|^2}{(1 + |c_i|^2)^2} - k|c_i|^3 \right)^{p/2}.
\]
This estimate together with another application of Taylor series establishes the lower bound for \( \| \prod_{i=1}^{N} (1 + c_i \chi_i) \|_p \), and similar arguments give the upper bound.

Remark. Of course, for any sequence \( \{c_i\} \), the \( L^2 \) norms of \( \prod_{i=1}^{N} (1 + c_i \chi_i + c_i \overline{\chi_i}) \) and \( \prod_{i=1}^{N} (1 + c_i \chi_i) \) are \( \prod_{i=1}^{N} (1 + 2|c_i|^2)^{1/2} \) and \( \prod_{i=1}^{N} (1 + |c_i|^2)^{1/2} \) respectively.

With the estimates of this lemma we can now obtain necessary quantitative estimates for certain \((L^2, L^p)\) multipliers. First we consider the case when the multiplier arises from a measure. Recall that a measure \( \mu \) is tame if for each \( \varphi \in \Delta M(G) \) there exists \( a \in \mathbb{C} \) and \( \gamma \in \Gamma \) such that
\[
\varphi \mu = a\gamma \text{ a.e. } d\mu \quad ([6, 6.1]).
\]
A Riesz product is an example of a tame measure.

**Theorem 1.2.** Let \( \mu \) be a tame measure on a compact abelian group \( G \) and assume \( \mu \in M(2, p) \) for some \( p > 2 \). Suppose that \( \Gamma \) has no elements of order 2. Then
\[
|\varphi\mu|^2 \leq \frac{1}{(p-1)} \| \mu \|^2_{M(G)}.
\]

Before proving this we state an immediate corollary and make some initial remarks.

**Corollary 1.3.** If tame \( \mu \in M(2, p) \) for \( p > 2 \) then
\[
\limsup_{\chi \in \Gamma} |\hat{\mu}(\chi)|^2 \leq \frac{1}{p-1} \| \mu \|^2_{M(G)}.
\]

**Remarks.** (1) For background information on \( \Delta M(G) \) see [6].
(2) This result improves the estimate in [5] and [7] for tame measures, and was shown by Bonami to be both necessary and sufficient for certain Riesz products ([2, p. 376, 385]).

**Proof of Theorem.** Let \( \varphi \in \Gamma \setminus \Gamma \) and suppose \( \varphi \mu = z\chi \mu \) a.e. where, without loss of generality, we may assume \( z \neq 0 \). Replacing \( \mu \) by \( \gamma \mu \) if necessary we may assume \( \hat{\mu}(1) \neq 0 \). Fix \( 0 < \delta < |z| \). Observe that
\[
|\hat{\mu}((\varphi\chi)^k)| = |\hat{\mu}(\varphi^k\chi^k)| = |z^k\hat{\mu}(1)|
\]
for all non-negative integers \( k \), thus we may choose a dissociate set \( \{\chi_i\}_{i=1}^{\infty} \) such that
\[
|\hat{\mu}\left( \prod_{i=1}^{N} \chi_i \right)| \geq (|z| - \delta)\sum_{\varepsilon} |\varepsilon_i| |\hat{\mu}(1)|
\]
whenever \( \varepsilon_i = 0, \pm 1 \).

For \( \varepsilon > 0 \) (and small), define the trigonometric polynomial \( f_{N, \varepsilon} \) by
\[
f_{N, \varepsilon}(\chi) = \begin{cases} 
(\varepsilon(|z| - \delta))^k / \hat{\mu}(\chi) & \text{if } \chi = \prod_{j=1}^{N} \chi_j \varepsilon_j, \ varepsilon_j = 0, \pm 1 \text{ and } \sum_{j=1}^{N} |\varepsilon_j| = k, \\
0 & \text{otherwise}.
\end{cases}
\]
Then
\[ \mu * f_{N, \varepsilon} = \prod_{j=1}^{N} (1 + \varepsilon(|z| - \delta)(\chi_j + \overline{\chi}_j)). \]
Thus
\[ |\hat{f}_{N, \varepsilon}(\chi)| \leq \begin{cases} \frac{\varepsilon^k}{|z| - \delta} & \text{if } \chi = \prod_{j=1}^{N} \chi_j^{\varepsilon_j}, \varepsilon_j = 0, \pm 1 \text{ and } \sum_{j=1}^{N} |\varepsilon_j| = k, \\ 0 & \text{otherwise} \end{cases} \]
so
\[ \|f_{N, \varepsilon}\|_2 \leq \frac{1}{|z| - \delta} (1 + 2\varepsilon^2)^{N/2}. \]

An application of the closed graph theorem shows that there is a constant \( C \) such that \( \|\mu * f\|_p \leq C\|f\|_2 \) for all \( f \in L^2 \). Together with Lemma 1.1 this shows that for all \( N \) and for all sufficiently small \( \varepsilon \),
\[ C \geq \frac{\|\mu * f_{N, \varepsilon}\|_p}{\|f_{N, \varepsilon}\|_2} \geq \frac{1}{|z| - \delta} \left[ \frac{1 + (p - 1)\varepsilon^2(|z| - \delta)^2 - k\varepsilon^3(|z| - \delta)^3}{(1 + 2\varepsilon^2)^{1/2}} \right]^N. \]
Hence for all small \( \varepsilon \),
\[ 1 + (p - 1)\varepsilon^2(|z| - \delta)^2 - k\varepsilon^3(|z| - \delta)^3 \leq (1 + 2\varepsilon^2)^{1/2}. \]
Letting \( \varepsilon \to 0 \) we see that this can occur only if \( (p - 1)(|z| - \delta)^2 \leq 1 \), but as \( \delta > 0 \) was arbitrary this implies that \( |z|^2 \leq 1/(p - 1) \) as desired.

Unlike measures, for general \((L^2, L^p)\) multipliers \( \varphi \) it is not necessary that \( \limsup |\varphi(\chi)| < |\varphi|_{L^\infty} \). Indeed, it is easy to see that the characteristic function of a Sidon set is an \((L^2, L^p)\) multiplier for all \( p > 2 \) (cf. [8] or [12]). However, in the next proposition we will prove that for one-sided Riesz products a better estimate can be obtained, and we will prove an estimate sharper than Corollary 1.3 for Riesz products.

**Proposition 1.4.** Let \( \{\chi_n\} \) be a dissociate set in \( \Gamma \) with \( \chi_n^2 \neq 1 \) and let \( 1 < p < q < \infty \). Suppose \( \{r_n\} \) and \( \{t_n\} \) are sets of complex numbers and let
\[ \varepsilon_n^{(1)} = \max \left( |r_n|^2 - \frac{p - 1}{q - 1}, 0 \right), \quad \varepsilon_n^{(2)} = \max \left( |t_n|^2 - \frac{p}{q}, 0 \right). \]
If either \( \varphi_1 = \prod (1 + r_n \chi_n + r_n \overline{\chi}_n) \) or \( \varphi_2 = \prod (1 + t_n \chi_n) \) belong to \( M(p, q) \), then \( \sum_n (\varepsilon_n^{(1)})^3 < \infty \) for \( i = 1, 2 \).
If, in addition, \( \{\chi_n\} \) satisfies the further independence condition
\[ \prod \chi_j^{\delta_n} = 0 \text{ for } \delta_n = 0, \pm 1, \pm 2, \pm 3 \text{ implies } \delta_n = 0, \]
then \( \sum (\varepsilon_n^{(i)})^2 < \infty \) is a necessary condition.

**Remark.** If \( |r_n| \leq 1/2 \) then \( \varphi_1 \) is a measure, otherwise by \( \varphi_1 \) we simply mean the obvious multiplier.
If we take $g$ and Taylor series expansion, show that for thus arguments similar to Lemma 1.1, but taking the first four terms of the trigonometric polynomials $f_N^{(2)} = N \prod_{n=1}^N (1 + c\varepsilon_n^{(2)} \chi_n)$ where $c \geq 0$ is a small constant.

As $\varphi_1, \varphi_2 \in M(p,q)$ the usual closed graph theorem argument shows that for $i = 1, 2$, $\sup_N ||M_{\varphi_1} f_N^{(i)}||_q ||f_N^{(i)}||_p < \infty$. Thus for $c$ chosen sufficiently small, Lemma 1.1 implies that

$$\sup_N \prod_{n=1}^N \left( 1 + (q-1)|\varepsilon_n^{(1)} r_n|^2 - k|\varepsilon_n^{(1)} r_n|^3 \right)$$

which forces $\sum (\varepsilon_n^{(1)})^3 < \infty$. Similar arguments apply to $\sum (\varepsilon_n^{(2)})^3$.

If $\{\chi_i\}$ satisfies the stronger independence property, then Lemma 1.1 can be improved. The stronger property implies that for every $i$,

$$\int \chi_i = 0 \quad \text{for } \delta = 0, 1, 2, 3 \text{ and } \delta_j = 0, \pm 1, \pm 2, \pm 3,$$

thus arguments similar to Lemma 1.1, but taking the first four terms of the Taylor series expansion, show that for $c_i$ sufficiently small

$$\prod_{i=1}^N (1 + (p-1)|c_i|^2 - k|c_i|^4) \leq \left\| \prod_{i=1}^N \left( 1 + c_i \chi_i + \varepsilon_i \right) \right\|_p$$

and

$$\prod_{i=1}^N \left( 1 + |c_i|^2 p/4 - k|c_i|^4 \right) \leq \left\| \prod_{i=1}^N (1 + c_i \chi_i) \right\|_p \leq \prod_{i=1}^N \left( 1 + |c_i|^2 p/4 + k|c_i|^4 \right).$$

If we take $g_N^{(i)} = N \prod_{n=1}^N (1 + c\varepsilon_n^{(i)} \chi_n)$ and $g_N^{(2)} = N \prod_{n=1}^N (1 + c\varepsilon_n^{(2)} \chi_n)$, then by estimating $||M_{\varphi, g_N^{(i)}}||_q ||g_N^{(i)}||_p$ with these sharper estimates we get the necessary condition $\sum (\varepsilon_n^{(i)})^2 < \infty$ for $i = 1, 2$. ■

**Corollary 1.5.** Let $\{\chi_i\}$ be a dissociate subset of $\Gamma$ and let $\varphi = \prod (1 + a_i \chi_i)$ be a one-sided Riesz product. If $\varphi \in M(2,p)$ then $\lim sup |a_i|^2 \leq 2/p$. 

**Proof.** Note that a necessary condition for $\varphi_1$ or $\varphi_2$ to be an element of $M(p,q)$ is that $\{\varepsilon_n^{(i)}\}$ is a bounded sequence for $i = 1, 2$. Define
Remark. This condition is both necessary and sufficient for certain one-sided Riesz products (see [2, p. 389] and §2).

Proof. Assume \( \{ \chi_i \} = \{ \chi_i \}_{i \in J} \cup \{ \chi_i \}_{i \in K} \) where \( \chi_i^2 = 1 \) for \( i \in J \) and \( \chi_i^2 \neq 1 \) for \( i \in K \). Let \( \alpha = \prod_{i \in J} (1 + a_i \chi_i) \) and \( \beta = \prod_{i \in K} (1 + a_i \chi_i) \). By duality \( M(2, p) = M(p', 2) \) and as \( |\alpha(\chi)| \) and \( |\beta(\chi)| \) are both dominated by \( |\varphi(\chi)| \) for all \( \chi \in \Gamma \) it follows that \( \alpha \) and \( \beta \) belong to \( M(2, p) \). But \( \alpha \) is actually a Riesz product so, by [5] or [7], \( \text{lim sup}_{i \in J} |a_i|^2 \leq 2/p. \) By the previous proposition \( \text{lim sup}_{i \in K} |a_i|^2 \leq 2/p. \) ■

Corollary 1.6. A one-sided Riesz product \( \varphi \) maps \( L^2 \) to \( L^p \) for some \( p > 2 \) if and only if \( \text{lim sup} |\varphi(\chi)| < 1. \)

Proof. Necessity has already been established. For sufficiency, assume \( |\varphi(\chi_i)| \leq 1 - \delta < 1 \) for all \( i \geq k \) and let \( \varphi_1 = \prod_{i=k}^{\infty} (1 + \varphi(\chi_i) \chi_i) \). Let \( \varphi_1^N \) denote the composition of \( \varphi_1 \) with itself \( N \) times. If \( N \) is chosen sufficiently large, and \( \mu \) is the Riesz product \( \mu = \prod_{i=k}^{\infty} (1 + (\chi_i + \overline{\chi_i})/4) \) then \( |\varphi_1^N(\chi)| \leq |\mu(\chi)| \) for all \( \chi \in \Gamma \). By [13], \( \mu \in M(2, p) \) for some \( p > 2 \), hence \( \varphi_1^N \in M(2, p) \). An interpolation argument ([8, 1.3]) shows that \( \varphi_1 \in M(2, q) \) for some \( 2 < q \leq p \). As \( \varphi \) is a finite linear combination of translates of \( \varphi_1 \), the multiplier \( \varphi \in M(2, q). \) ■

2. \( L^p \)-Improving Riesz products and one-sided Riesz products.

Perhaps the most difficult problem in the study of \( (L^2, L^p) \) multipliers, and the one with the least satisfactory solutions, is of finding good (and practical) sufficient conditions to describe the \( p > 2 \) for which a multiplier \( \varphi \) maps \( L^2 \) to \( L^p \). Other than for monotonic functions ([5, 2.2]), optimal sufficient conditions are known only for certain (one-sided) Riesz products.

In Chapter 3 of [2], Bonami showed that the Riesz products \( \prod (1 + r \chi) \) on \( D^\infty \) and \( \prod (1 + 2r \cos x_j) \) on \( T^\infty \) belong to \( M(p, q) \) if and only if \( r^2 \leq (p-1)/(q-1) \), and for even integers \( p \) the one-sided Riesz products \( \prod (1 + r \chi) \) on \( T^\infty \) belong to \( M(2, p) \) if and only if \( r^2 \leq 2/p. \) In contrast, our Proposition 1.4 shows that there are Riesz products \( \mu \) satisfying \( \text{lim sup} |\mu|^2 \leq (p-1)/(q-1) \) but with \( \mu \notin M(p, q) \), answering [5, 3.2(vi)], and similarly that there are one-sided Riesz products \( \varphi \) with \( \text{lim sup} |\varphi|^2 \leq 2/p \) but with \( \varphi \notin M(2, p) \). In this section we characterize a more general class of \( L^p \)-improving (one-sided) Riesz products and as a corollary extend Bonami’s result on one-sided Riesz products to all \( p > 2. \)

Theorem 2.1. Let \( p > 2 \) and let \( \{ r_j \} \) and \( \{ t_j \} \) be sequences of complex numbers such that \( |t_j|^2 \geq 2/p \) and \( |r_j|^2 \geq 1/(p-1) \). Let \( \varphi = \prod (1 + t_j e^{i \tau_j}) \) be a one-sided Riesz product on \( T^\infty \), and \( \mu = \prod (1 + t_j e^{i \tau_j} + r_j e^{-i \tau_j}) \) be a Riesz product on \( T^\infty \). Then \( \varphi \in M(2, p) \) if and only if \( \sum |t_j|^2 - 2/p \) is finite, and \( \mu \in M(2, p) \) if and only if \( \sum |t_j|^2 - 1/(p-1) \) is finite.
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\[ \text{Proof.} \] Notice that the characters defined on \(T^\infty\) by \((x_k) \mapsto e^{ix_k}\) satisfy the “further independence condition” of Proposition 1.4, thus necessity is clear in both cases.

To prove sufficiency we need the following lemma which is a straightforward modification of [2, p. 374].

**Lemma 2.2.** Let \(\varphi = \prod(1 + a_j e^{ix_k} + b_j e^{-ix_j})\) be a multiplier on \(T^\infty\). For each \(n\) let \(\varphi_n = 1 + a_n e^{ix_n} + b_n e^{-ix_n}\) and let \(\|\varphi_n\|_{p,q}\) denote the norm of \(\varphi_n\) as an operator from \(L^p\) to \(L^q\). Then \(\varphi \in M(p,q)\) if and only if \(\prod \|\varphi_n\|_{p,q} < \infty\) and in this case \(\|\varphi\|_{p,q} \leq \prod \|\varphi_n\|_{p,q}\).

**Proof of Theorem 2.1 (ctd.).** Sufficiency for one-sided Riesz products. Let \(p = 2s\) (so \(s > 1\)) and set \(\varepsilon_n = |t_n|^2 - 2/p\). Let \(s_0 = 1\) and let

\[ s_k = \frac{s(s-1)\ldots(s-k+1)}{k!} \quad \text{if} \ k \neq 0. \]

Thus \(s_k = \binom{s}{k}\) if \(s\) is an integer (where \(\binom{s}{k} = 0\) if \(k > s\)). One can easily check that \(0 \leq s_k \leq s^k/k!\) if \(k \leq [s] + 1\) and \(|s_k| \leq s^{[s]+1}/(k(k-1))\) if \(k > [s]+1\).

Certainly the assumption that \(\sum \varepsilon_n^2 < \infty\) implies that \(\varepsilon_n \to 0\) so we may choose \(N\) so that for all \(n > N\) we have \(|t_n| < 1\), \(s_k(1/s + \varepsilon_n)^k < 3/4\) if \(k = 2, 3, \ldots, [s]\), and \(\varepsilon_n < \varepsilon = \varepsilon(s)\) where \(0 < \varepsilon \leq 1 - 1/s\) will be specified later.

It is easy to see that if \(\varphi_n = 1 + t_n e^{ix}\) then

\[ \|\varphi_n\|_{2,p} = \sup_b \frac{\|1 + bt_n e^{ix}\|_p}{\|1 + be^{ix}\|_2}. \]

**Claim.** For \(|r| \leq 1\) and any complex number \(b\) with \(|b| > 1\), \(|1 + bre^{ix}| \leq |b + re^{ix}|\).

To prove this observe that

\[ |b + re^{ix}|^2 - |1 + bre^{ix}|^2 = |b|^2 - 1 + |r|^2 - |br|^2. \]

The latter expression is a decreasing function of \(|r|^2\), whose value at \(|r|^2 = 1\) is zero. This proves the claim.

From this inequality we see that if \(|b| > 1\) and \(|t_n| \leq 1\) then

\[ \|1 + bt_n e^{ix}\|_p \leq \|b + t_n e^{ix}\|_p = |b| \|1 + b^{-1}t_n e^{ix}\|_p. \]

As \(\|1 + be^{ix}\|_2 = |b| \|1 + b^{-1}e^{ix}\|_2\) and \(|b^{-1}| < 1\) it follows that in computing \(\|\varphi_n\|_{2,p}\), for \(n > N\), we need only take the supremum over \(b \in \mathbb{C}\) with \(|b| \leq 1\). By taking limits we may further reduce to

\[ \|\varphi_n\|_{2,p} = \sup_{|b| < 1} \frac{\|1 + bt_n e^{ix}\|_p}{\|1 + be^{ix}\|_2}. \]
Thus we now assume $|b| < 1$ and $n \geq N$. Compute the Taylor series expansion for

$$(1 + bte^{ix})^s = \sum_{k=0}^{\infty} s_k (bt)^k e^{ikx}$$

(of course the sum terminates at $k = s$ if $s$ is an integer). Since $|bte^{ix}| \leq |b| < 1$ this series converges uniformly so

$$
\left\| 1 + bte^{ix} \right\|_p^p = \left\| (1 + bte^{ix})^s \right\|_2^2 = \sum_{k=0}^{\infty} s_k^2 |bt|^2k,
$$

and this series converges absolutely. Also,

$$
\left\| 1 + be^{ix} \right\|_2^2 = (1 + |b|^2)^s = \sum_{k=0}^{\infty} s_k^2 |b|^2k,
$$

and this series converges absolutely as well.

We must estimate

$$
\frac{\left\| 1 + bte^{ix} \right\|_p^p}{\left\| 1 + be^{ix} \right\|_2^2} = \frac{\sum_{k=0}^{\infty} s_k^2 |bt|^2k}{\sum_{k=0}^{\infty} s_k |b|^2k} = 1 + s^2 |b|^2 \varepsilon_n + |b|^4 \sum_{k=2}^{\infty} s_k |b|^{2(k-2)} (s_k (1/s + \varepsilon_n)^k - 1) \quad \frac{1}{(1 + |b|^2)^s}
$$

We break the infinite sum into two terms:

(i) $$
\sum_{k=2}^{[s]} s_k |b|^{2(k-2)} (s_k (1/s + \varepsilon_n)^k - 1)
$$

(If $[s] = 1$ this term is not present.)

(ii) $$
\sum_{k=[s]+1}^{\infty} s_k |b|^{2(k-2)} (s_k (1/s + \varepsilon_n)^k - 1)
$$

(If $s$ is an integer this term is not present.)

In (i) the choice of $n \geq N$ ensures that $s_k (1/s + \varepsilon_n)^k - 1 < -1/4$, and as $s_k > 0$ for $k = 2, \ldots, [s]$ the first sum is at most $-s_2/4$ if $[s] \neq 1$.

Sum (ii) we further break down as

$$
\sum_{k=[s]+1}^{\infty} s_k |b|^{2(k-2)} (s_k s^{-1} - 1) + \sum_{k=[s]+1}^{\infty} s_k^2 |b|^{2(k-2)} ((1/s + \varepsilon_n)^k - s^{-k}).
$$

By the mean-value theorem and the assumption that $\varepsilon_n \leq \varepsilon \leq 1 - 1/s$,

$$
(1/s + \varepsilon_n)^k - s^{-k} \leq \varepsilon_n k(1/s + \varepsilon_n)^{k-1} \leq \varepsilon k.
$$
Thus for some constant $C_1(s)$,
\[
\sum_{k=\lfloor s \rfloor + 1}^{\infty} s_k^2 |b|^{2(k-2)}((1/s + \varepsilon_n)^k - s^{-k}) \leq \sum_{k=\lfloor s \rfloor + 1}^{\infty} \left( \frac{s[k+1]}{k(k-1)} \right)^2 |b|^{2(k-2)} \varepsilon_k \\
\leq |b|^{2(\lfloor s \rfloor - 1)} \leq C_1(s).
\]

Clearly \( \{s_k(s_k^{-k} - 1)\}_{k=\lfloor s \rfloor + 1}^{\infty} \) is an alternating sequence tending to zero, with first term negative. We claim that it is a (strictly) decreasing sequence (in absolute value). To prove this we first remark that as \( s_{k+1}/s_k = (s - k)/(k + 1) \) it suffices to show that for \( k \geq \lfloor s \rfloor + 1 \),
\[
\frac{s_k}{s^2} \left( k + 1 + \frac{(k - s)^2}{(k + 1)s} \right) < s + 1.
\]
Since \(|s_k s^{-k}| \leq 1/(k(k - 1)) \) and \( k^2 + 1 + s + k^2/s \leq 2k(k + 1) \),
\[
\frac{s_k}{s^2} \left( k + 1 + \frac{(k - s)^2}{(k + 1)s} \right) \leq \frac{1}{k(k - 1)} \left( \frac{2k(k + 1)}{k + 1} \right) = \frac{2}{k - 1} < s + 1,
\]
as desired. Hence the first sum in (ii) is at most the sum of its first two terms, which is at most \(|b|^{2(\lfloor s \rfloor - 1)}C_2(s)\) where \( C_2(s) < 0 \). If \( \varepsilon > 0 \) is chosen so that \( \varepsilon C_1(s) < |C_2(s)|/2 \) then sum (ii) is negative, and more specifically, if \( \lfloor s \rfloor = 1 \) then (ii) is at most \( C_2(s)/2 \).

Combining (i) and (ii) we get
\[
\sum_{k=2}^{\infty} s_k |b|^{2(k-2)} (s_k(1/s + \varepsilon_n)^k - 1) \leq C_3(s) \equiv \begin{cases} -s_2/4 & \text{if } \lfloor s \rfloor \neq 1, \\ C_2(s)/2 & \text{if } \lfloor s \rfloor = 1. \end{cases}
\]

Thus for \(|b| < 1 \) and \( n \geq N \),
\[
\frac{||1 + b\varepsilon_n e^{ix}||_p^p}{||1 + b e^{ix}||_2^p} \leq 1 + \frac{s^2 |b|^2 \varepsilon_n + |b|^4 C_3(s)}{(1 + |b|^2)^s}.
\]

If \(|b|^2 \leq 2s^2 \varepsilon_n/|C_3(s)|\) then clearly
\[
\frac{||1 + b\varepsilon_n e^{ix}||_p^p}{||1 + b e^{ix}||_2^p} \leq 1 + \varepsilon_n^2 C_4(s)
\]
for \( C_4(s) = 2s^2/|C_3(s)| \), while if \( 2s^2 \varepsilon_n/|C_3(s)| \leq |b|^2 < 1 \),
\[
\frac{||1 + b\varepsilon_n e^{ix}||_p^p}{||1 + b e^{ix}||_2^p} \leq 1 + \frac{s^2 |b|^2 (\varepsilon_n - 2\varepsilon_n)}{(1 + |b|^2)^s} \leq 1.
\]

Thus \( ||\varphi_n||_{2,p} \leq 1 + \varepsilon_n^2 C_4(s) \) whenever \( n \geq N \). As \( ||\varphi_n||_{2,p} < \infty \) for all \( n \), \( \prod ||\varphi_n||_{2,p} < \infty \) when \( \sum \varepsilon_n^2 < \infty \). By Lemma 2.2, \( \varphi \in M(2,p) \).

**Sufficiency for Riesz products.** The proof is similar to that for one-sided Riesz products so only the main ideas will be sketched.
Let \( \phi_n = 1 + r_n e^{ix} + r_n e^{-ix} \). We need to bound \( \| \phi_n \|_{2,p} \). Since \( \mu \in M(2,p) \) if and only if \( \prod (1 + |r_n| (e^{ix_n} + e^{-ix_n})) \in M(2,p) \), without loss of generality we may assume \( r_n \geq 0 \). Since this operator maps real-valued functions to real-valued functions, Bonami [2, p. 377] has shown that

\[
\| \phi_n \|_{2,p} = \sup_{b \in \mathbb{R}} \| 1 + b r_n \cos x \|_{2,p}.
\]

For \( 0 \leq r \leq 1 \) and \( |b| > 1 \)

\[
|1 + b r \cos x| \leq |b| |1 + rb^{-1} \cos x|.
\]

This simple inequality shows that whenever \( r_n \leq 1 \) then in computing \( \| \phi_n \|_{2,p} \) we may restrict ourselves to \( |b| \leq 1 \). Choose \( N \) so that \( r_n < 1 \) for \( n \geq N \) and let \( p = 2s \).

The power series expansion of \( (1 + x)^{2s} \) converges uniformly on \([-\alpha, \alpha]\) for any \( \alpha < 1 \), thus for \( n \geq N \) and \( |b| \leq 1 \)

\[
\|1 + b r_n \cos x\|_p = \sum_{k=0}^{\infty} \int_0^{2\pi} (2s)_k (b r_n)^k \cos^k x \, dx
\]

\[
= 1 + \sum_{k=1}^{\infty} (2s)_{2k} (b r_n)^{2k} \frac{(2k-1)(2k-3)\ldots1}{2k(2k-2)\ldots2}.
\]

and the latter series converges absolutely. (Of course, this is a finite sum if \( 2s \) is an integer.) It follows that

\[
\frac{\|1 + b r_n \cos x\|_p}{\|1 + b \cos x\|_2^p}
\]

\[
= 1 + \sum_{k=1}^{\infty} s_k b^{2k} \left[ \frac{1}{2s-1} + \varepsilon_n \right]^k \frac{(2s-1)(2s-3)\ldots(2s-2k+1)}{k!} - 1 \right] \frac{1}{(1 + b^2/2)^s}.
\]

Let

\[
a_k(s) \equiv a_k \equiv \frac{1}{(2s-1)^k} \frac{(2s-1)\ldots(2s-2k+1)}{k!}.
\]

When \( s \geq 3/2 \) then \( (2s-1) \geq 2 \) and with this observation it is not hard to show that \( |a_k| \leq 1/k \). (It is helpful to consider the cases \( 2s \) an even or odd integer separately.) Also, \( \{ a_k (a_k - 1/2)^k \}^\infty_{k=1} \) is an alternating sequence which is decreasing (in absolute value) to zero and with first term negative. Thus arguments similar to those used for the one-sided Riesz products show that \( \| \phi_n \|_{2,p} \leq (1 + C(s) \varepsilon_n^2)^{1/p} \) for \( n \geq N \).

When \( 1 < s < 3/2 \), the factors \( (2s)_{2k} \) are negative for \( k \geq 2 \). Thus

\[
\|1 + b r_n \cos x\|_p \leq 1 + (2s)_2 (b r_n)^2 / 2.
\]
Hence
\[\frac{\|1 + br_n \cos x\|_p^n}{\|1 + b \cos x\|_p^n} \leq 1 + \frac{1}{b^2} \sum_{k=2}^{\infty} s_k(b^2/2)^k.\]

Since \( \{s_k/b^2\} \) is an alternating sequence which is decreasing (in absolute value) to zero and with first term positive, the same sort of arguments as before again prove that \( \|\varphi_n\|_{2, p} \leq (1 + C(s)b_n^2)^{1/p} \) for \( n \geq N. \)

Since \( \varphi_n \in M(2, p) \) for all \( n \) we can conclude (in either case) that \( \varphi \in M(2, p) \) when \( \sum \varepsilon_n^2 < \infty. \)

An obvious corollary to this theorem is

**Corollary 2.3.** The one-sided Riesz product \( \varphi = \prod (1 + re^{ix}) \) belongs to \( M(2, p) \) if and only if \( |r| \leq \sqrt{2/p}. \)

The next corollary is in the same spirit as [2, p. 387].

**Corollary 2.4.** Let \( 1 < p \leq 2 < q < \infty \) and
\[|r_n|^2 = \frac{p-1}{q-1} + \varepsilon_n\]
where \( \varepsilon_n \geq 0. \) Then the Riesz product \( \mu \) on \( T^\infty \) given by \( \mu = \prod (1 + 2r_n \cos x_j) \) belongs to \( M(p, q) \) if and only if \( \sum \varepsilon_n^2 < \infty. \)

**Proof.** First we prove sufficiency. Let \( t_n = r_n/\sqrt{p-1}, \) \( \nu_1 = \prod (1 + 2\sqrt{p-1} \cos x_j) \) and \( \nu_2 = \prod (1 + 2t_n \cos x_j). \) Clearly \( \mu \) is the composition of the multipliers \( \nu_1 \) and \( \nu_2. \) Since \( |t_n|^2 = r_n^2/(p-1) \geq 1/(q-1) \) and
\[\sum (|t_n|^2 - \frac{1}{q-1})^2 = \frac{1}{(p-1)^2} \sum (|r_n|^2 - \frac{p-1}{q-1})^2 < \infty,\]
by the theorem \( \nu_2 \in M(2, q). \) If \( 1/p + 1/p' = 1 \) then \( p-1 = 1/(p' - 1), \) so \( \nu_1 \in M(2, p') = M(p, 2). \) Therefore \( \mu \in M(p, q). \)

Necessity follows from Proposition 1.4.

**Example 2.5.** Let \( 1 < p \leq 2 < q < \infty. \) The multiplier on \( T^\infty \) given by
\[\varphi = \prod (1 + 2\sqrt{a_n} \cos x_n) \text{ where } a_n = \frac{p-1}{q-1} + \frac{1}{\sqrt{n}}\]
does not belong to \( M(p, q) \) but does belong to \( M(s, t) \) for all \( 1 < s \leq 2 < t < \infty \) satisfying \( (p-1)/(q-1) < (s-1)/(t-1). \)

**Proof.** By the previous corollary \( \varphi \notin M(p, q). \) Suppose \( (s-1)/(t-1) > (p-1)/(q-1). \) Let \( \varphi_1 = \prod (1 + 2\sqrt{s-1} \cos x_n) \) and \( \varphi_2 = \prod (1 + 2\sqrt{a_n}/(s-1) \cos x_n). \) Clearly \( \varphi_1 \in M(s, 2) \) and as \( a_n/(s-1) < 1/(t-1) \) for \( n \) sufficiently large, \( \varphi_2 \in M(2, t). \) Since \( \varphi \) is the composition of \( \varphi_1 \) and \( \varphi_2, \) we see that \( \varphi \in M(s, t). \)
Just as in \cite[pp. 392–393]{2} the following is another consequence of Theorem 2.1:

**Corollary 2.6.** Let \( p > 2 \) and let \( \{n_i\} \) be a lacunary sequence of positive integers satisfying \( n_{i+1}/n_i \geq 3 \). Then \( \varphi = \prod (1 + r e^{i n_j}) \in M(2,p) \) if \( |r| \leq \sqrt{1/2p} \), and if in addition \( \sum n_i/n_{i+1} < \infty \), then \( \varphi \in M(2,p) \) if and only if \( |r| \leq \sqrt{2/p} \).

We will omit the proofs as they are similar to the corresponding results in \cite{2}.

Let \( \epsilon_n \) be the character on \( D^n \) given by \( \epsilon_n((x_j)) = x_n \). Similar arguments to those used in Theorem 2.1 enable one to prove

**Proposition 2.7.** Let \( 1 < p \leq 2 < q < \infty \) and let \( |r_n|^2 = (p-1)/(q-1) + \epsilon_n \geq 0 \). Then the Riesz product \( \mu = \prod (1 + r_n \epsilon_n(x)) \) on \( D^n \) belongs to \( M(p,q) \) if and only if \( \sum \epsilon_n^2 < \infty \).

We leave the details to the reader.

### 3. Computation of \( A(p) \) constants.

Let \( p > 2 \). A subset \( E \) of \( \Gamma \) is called a \( A(p) \) set if there is a constant \( C_p \) such that \( ||f||_p \leq C_p ||f||_2 \) for all \( f \in \{g \in L^2 : \text{supp} \hat{g} \subseteq E\} \). The least such constant \( C_p \) is called the \( A(p) \) constant of \( E \) and is denoted by \( A(E,p) \). For standard results on \( A(p) \) sets see \cite{10} or \cite{15}.

Let \( \{\chi_i\} \subseteq \Gamma \) be a dissociate set. Sets of the form

\[
\left\{ \prod \chi_i^{e_i} : \sum |e_i| \leq n, \ e_i = 0, \pm 1 \text{ (or } e_i = 0,1) \right\}
\]

are well known examples of \( A(p) \) sets for all \( 2 < p < \infty \), but are not Sidon sets. Using mainly combinatorial methods Bonami found estimates for the \( A(p) \) constants of such sets \cite[Ch. 2]{2}. She then used her estimates in the proof of her result for \( (L^2,L^p) \) one-sided Riesz products. Here we take the opposite approach and use the earlier results of this paper to improve upon Bonami’s estimates of \( A(p) \) constants (when they are not already optimal).

The connection between the two subjects is due to the following theorem which almost characterizes \( (L^2,L^p) \) multipliers.

**Theorem 3.1** \cite{8}. Let \( \varphi \) be a bounded function on \( \Gamma \) and for each \( \varphi > 0 \) let \( E(\varphi) = \{ \chi : |\varphi(\chi)| \geq \varepsilon \} \). If \( \varphi \in M(2,p) \) for some \( p > 2 \), then for each \( \varepsilon > 0, E(\varepsilon) \) is a \( A(p) \) set and \( A(\varepsilon, p) \leq ||\varphi||_2 e^{-1} \). If \( E(\varepsilon) \) is a \( A(p) \) set for every \( \varepsilon > 0 \) and \( A(\varepsilon, p) = O(e^{-1}) \), then \( \varphi \in M(2,r) \) for all \( r < p \).

Before applying this theorem it is convenient to establish some notation.

**Notation.** Let

\[
T_k = \left\{ (n_i) : \sum n_i = 0, \pm 1, \sum |n_i| = k \right\},
\]
The proof of Theorem 2.1 shows that

\[ \Lambda \]

Given \( E \) is satisfying from Theorem 3.1 and \( r > q \) false for direction of Theorem 3.1 tells us

\[ \phi \]

Let \( \phi \). As

\[ q < p \]

\( \phi \) and \( \phi \) are exactly dominated by \((G, F)\) for all \( k \in \mathbb{N} \), and for every \( 2 < q < p \), \( \lim \sup k F(k, p)/G(k, q) = \infty \).

**Proposition 3.2.** Let \( p > 2 \). Then both \( \Lambda(T_k, p) \) and \( \Lambda(I_k, p) \) are exactly dominated by \((p - 1)^{k/2}\), and \( \Lambda(T_k^+, p) \) is exactly dominated by \((p/2)^{k/2}\).

**Proof.** Let \( \varphi = \prod (1 + \sqrt[2]{2/p}\epsilon i x) \) be a one-sided Riesz product on \( T^\infty \). Then \( \varphi \) is an \((L^2, L^p)\) multiplier and

\[ E((2/p)^{k/2}, \varphi) = \left\{ (n_i) \in \sum \mathbb{Z} : |\varphi((n_i))| \geq (2/p)^{k/2} \right\} = \bigcup_{j=1}^{k} T_j^+ \]

The proof of Theorem 2.1 shows that \( ||\varphi||_{2,p} = 1 \), thus Theorem 3.1 gives \( \Lambda(T_k^+, p) \leq (p/2)^{k/2} \). Suppose \( \lim \sup k \Lambda(T_k^+, p) \leq C(q/2)^{k/2} \) for some \( 2 < q < p \). As \( T_k^+ \) is a \( \Lambda(p) \) set for every \( k \) there is a constant \( C_1 \) such that

\[ \Lambda\left( \bigcup_{j=1}^{k} T_j^+, p \right) \leq k \lim \sup_{1 \leq j \leq k} \Lambda(T_j^+, p) \leq C_1(q/2)^{k/2} \]

Let \( \varphi_1 = \prod (1 + \sqrt[2]{2/q}\epsilon i x) \). Since \( E((2/q)^{k/2}, \varphi_1) \leq \bigcup_{j=1}^{k} T_k^+ \), the converse direction of Theorem 3.1 tells us \( \varphi_1 \in M(2, r) \) for every \( r < p \). But this is false for \( r > q \).

The estimates of the \( \Lambda(p) \) constants for the sets \( T_k \) and \( I_k \) follow similarly from Theorem 3.1 and \( [2, \text{p. 376, 385}] \).

**Proposition 3.3.** Let \( E = \{ n_i \} \) be a lacunary set of positive integers satisfying \( n_{i+1}/n_i \geq 3 \) for all \( i \).

(a) \( \Lambda(E_k^+, p) \leq (2p)^{k/2} \) and \( \Lambda(E_k, p) \leq (4(p - 1))^{k/2} \) for all \( k \in \mathbb{N} \).

(b) If \( \sum n_i/n_{i+1} < \infty \), then for some constant \( C \), \( \Lambda(E_k^+, p) \) and \( \Lambda(E_k, p) \) are exactly dominated by \((C(p/2)^{k/2} \) respectively.

**Proof.** The proof is similar using Corollary 2.6 and \([2, \text{pp. 392–393}]\). We remark that in (a) the \((L^2, L^p)\) operator norm of the appropriate multiplier can be shown to be 1.
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