COLLOQUIUM MATHEMATICUM

VOL. LXIII 1992 FASC. 2

A MINIMAX INEQUALITY WITH APPLICATIONS TO EXISTENCE OF EQUILIBRIUM POINT AND FIXED POINT THEOREMS

BY

XIE PING DING (CHENGDU) AND
KOK-KEONG TAN (HALIFAX, NOVA SCOTIA)

1. Introduction. Ky Fan's minimax inequality [8, Theorem 1] has become a versatile tool in nonlinear and convex analysis. In this paper, we shall first obtain a minimax inequality which generalizes those generalizations of Ky Fan's minimax inequality due to Allen [1], Yen [18], Tan [16], Bae–Kim–Tan [3] and Fan himself [9]. Several equivalent forms are then formulated and one of them, the maximal element version, is used to obtain a fixed point theorem which in turn is applied to obtain an existence theorem of an equilibrium point in a one-person game. Next, by applying the minimax inequality, we present some fixed point theorems for set-valued inward and outward mappings on a non-compact convex set in a topological vector space. These results generalize the corresponding results due to Browder [5], Jiang [11] and Shih–Tan [15] in several aspects.

2. Preliminaries. Let X be a non-empty set. We shall denote by 2^X the family of all non-empty subsets of X, by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X and by $\mathbb{R}$ the set of all real numbers. If A is a subset of a topological vector space E, we shall denote by $\operatorname{co}(A)$ the convex hull of A and by $\bar{A}$ the closure of A in E. Let X be a topological space and $A \subset X$; then $\mathrm{cl}_X A$ denotes the closure of A in X. A function $g : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is said to be *upper* (resp. *lower*) *semicontinuous* on A if for each $\lambda \in \mathbb{R}$, the set $\{x \in A : g(x) \geq \lambda\}$ (resp. $\{x \in A : g(x) \leq \lambda\}$) is closed in A. If Y is another topological space, a set-valued map $T : X \to 2^Y$ is said to be

(i) *upper* (resp. *lower*) *semicontinuous at* $x_0 \in X$ if for each open set G in Y with $T(x_0) \subset G$ (resp. with $T(x_0) \cap G \neq \emptyset$), there exists an open neighborhood U of x_0 in X such that $T(x) \subset G$ (resp. $T(x) \cap G \neq \emptyset$) for all $x \in U$;

This work was partially supported by NSERC of Canada under grant A-8096.

(ii) *upper* (resp. *lower*) *semicontinuous on* X if T is upper (resp. lower) semicontinuous at each point of X;

(iii) *continuous on* X if T is both lower and upper semicontinuous on X.

If X is a convex subset of a topological vector space, a map $P : X \to 2^X \cup \{\emptyset\}$ is said to be of *class* L_C if for each $x \in X$, $x \notin \mathrm{co}(P(x))$, and for each non-empty compact subset C of X and for each $y \in X$, $P^{-1}(y) \cap C$ is open in C.

The following Lemma 1 is Theorem 2.5.1 of Aubin [2, p. 67]:

LEMMA 1. *Let X and Y be topological spaces. Suppose $W : X \times Y \to \mathbb{R}$ is lower semicontinuous on $X \times Y$ and $G : X \to 2^Y$ is upper semicontinuous at $x_0 \in X$ such that $G(x_0)$ is compact. Then the function $U : X \to [-\infty, \infty)$ defined by*

$$U(x) = \inf_{y \in G(x)} W(x, y)$$

is lower semicontinuous at x_0.

The following Lemma 2 is Theorem 2.5.2 of Aubin [2, p. 69]:

LEMMA 2. *Let X and Y be topological spaces. Suppose $W : X \times Y \to \mathbb{R}$ is upper semicontinuous on $X \times Y$ and $G : X \to 2^Y$ is lower semicontinuous at $x_0 \in X$. Then the function $V : X \to [-\infty, \infty)$ defined by*

$$V(x) = \inf_{y \in G(x)} W(x, y)$$

is upper semicontinuous at x_0.

The proof of Lemma 1 of Fan [7] can be slightly modified to give a proof of the following

LEMMA 3. *Let X and Y be non-empty sets in a topological vector space E and let $F : X \to 2^Y$ be such that*

(i) *for each $x \in X$, $F(x)$ is closed in Y;*

(ii) *for each $A \in \mathcal{F}(X)$, $\mathrm{co}(A) \subset \bigcup_{x \in A} F(x)$;*

(iii) *there exists an $x_0 \in X$ such that $F(x_0)$ is compact.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

We shall remark here that even although Fan [7] implicitly assumed all topological vector spaces to satisfy the Hausdorff separation axiom, in proving Lemma 1 in [7], "Hausdorff" is never needed. We note that the above Lemma 3 differs from Lemma 1 of Fan [7] in the following ways: (a) E is not required to be Hausdorff and (b) Y need not be the whole space E.

3. A minimax inequality. We shall first prove the following very general minimax inequality:

THEOREM 1. *Let X be a non-empty convex subset of a topological vector space and let $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(i) *for each fixed $x \in X$, $f(x,y)$ is a lower semicontinuous function of y on each non-empty compact subset C of X;*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, $\min_{x\in A} f(x,y) \le 0$;*

(iii) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $f(x,y) > 0$.*

Then there exists $\widehat{y} \in K$ such that $f(x,\widehat{y}) \le 0$ for all $x \in X$.

Proof. For each $x \in X$, let

$$K(x) = \{y \in K : f(x,y) \le 0\}.$$

By (i), $K(x)$ is closed in K for each $x \in X$. We claim that the family $\{K(x) : x \in X\}$ has the finite intersection property. Indeed, let $\{x_1,\dots,x_n\}$ be any finite subset of X and let $D = \mathrm{co}(X_0 \cup \{x_1,\dots,x_n\})$; then D is a compact convex subset of X. First we note that by (ii), $f(x,x) \le 0$ for each $x \in X$. Define $F : D \to 2^D$ by $F(x) = \{y \in D : f(x,y) \le 0\}$. Then

(a) for each $x \in D$, $F(x)$ is closed in D by (i), and hence it is compact;

(b) for each $A \in \mathcal{F}(D)$, $\mathrm{co}(A) \subset \bigcup_{x\in A} F(x)$.

Indeed, if (b) were false, then there would exist $A \in \mathcal{F}(D)$ and $y \in \mathrm{co}(A)$ such that $y \notin \bigcup_{x\in A} F(x)$. It follows that $f(x,y) > 0$ for all $x \in A$, which contradicts (ii).

By Lemma 3, $\bigcap_{x\in D} F(x) \ne \emptyset$; that is, there exists $\overline{y} \in D$ such that $f(x,\overline{y}) \le 0$ for all $x \in D$. By (iii), we must have $\overline{y} \in K$, so that $\overline{y} \in \bigcap_{i=1}^n K(x_i)$. This proves that $\{K(x) : x \in X\}$ has the finite intersection property. By the compactness of K, $\bigcap_{x\in X} K(x) \ne \emptyset$. Take any $\widehat{y} \in \bigcap_{x\in X} K(x)$; then $\widehat{y} \in K$ and $f(x,\widehat{y}) \le 0$ for all $x \in X$. ■

As an immediate consequence of Theorem 1, we have the following minimax inequality, which is essentially Theorem 1 of Bae–Kim–Tan [3], which in turn generalizes minimax inequalities due to Tan [16, Theorem 1] and Fan [9, Theorem 6] (and hence also [8, Theorem 1]).

THEOREM 2. *Let X be a non-empty convex subset of a topological vector space and let $f, g : X \times X \to \mathbb{R} \cup \{-\infty, \infty\}$ be such that*

(a) *$f(x,y) \le g(x,y)$ for all $x,y \in X$ and $g(x,x) \le 0$ for all $x \in X$;*

(b) *for each fixed $x \in X$, $f(x,y)$ is a lower semicontinuous function of y on each non-empty compact subset C of X;*

(c) *for each $y \in X$, the set $\{x \in X : g(x,y) > 0\}$ is convex;*

(d) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$, *there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $f(x,y) > 0$.

Then there exists $\widehat{y} \in K$ *such that* $f(x,\widehat{y}) \leq 0$ *for all* $x \in X$.

Proof. By Theorem 1, it is sufficient to show that (a) and (c) imply the condition (ii) of Theorem 1. Suppose not. Then there exist $A \in \mathcal{F}(X)$ and $y \in \mathrm{co}(A)$ such that $\min_{x\in A} f(x,y) > 0$; but then by (a), $\min_{x\in A} g(x,y) > 0$; it follows that $A \subset \{x \in X : g(x,y) > 0\}$. By (c), $y \in \mathrm{co}(A) \subset \{x \in X : g(x,y) > 0\}$, so that $g(y,y) > 0$, which contradicts (a). ■

The following result, which is equivalent to Theorem 2.11 of Zhou–Chen [19], is also an immediate consequence of Theorem 1.

Corollary 1. *Let* X *be a non-empty compact convex subset of a topological vector space and let* $f : X \times X \to \mathbb{R} \cup \{-\infty, \infty\}$ *be such that for each* $x \in X$, $f(x,y)$ *is a lower semicontinuous function of* y *on* X. *Then for each* $t \in \mathbb{R}$, *one of the following properties holds*:

(1) *there exists* $\widehat{y} \in X$ *such that* $f(x,\widehat{y}) \leq t$ *for all* $x \in X$;
(2) *there exist* $A \in \mathcal{F}(X)$ *and* $y \in \mathrm{co}(A)$ *such that* $\min_{x\in A} f(x,y) > t$.

Proof. Let $F(x,y) = f(x,y) - t$ for all $x,y \in X$; then for each $x \in X$, $F(x,y)$ is a lower semicontinuous function of y on X. Take $X_0 = K = X$. Then the condition (iii) in Theorem 1 is satisfied trivially. If for each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, $\min_{x\in A} F(x,y) \leq 0$, then by Theorem 1, there exists $\widehat{y} \in X$ such that $F(x,\widehat{y}) \leq 0$ for all $x \in X$. It follows that $f(x,\widehat{y}) \leq t$ for all $x \in X$, and (1) holds. On the other hand, if there exist $A \in \mathcal{F}(X)$ and $y \in \mathrm{co}(A)$ such that $\min_{x\in A} F(x,y) > 0$, then $\min_{x\in A} f(x,y) > t$, so that (2) holds. ■

The following result is essentially Theorem 1 of Yen [18].

Corollary 2. *Let* X *be a non-empty compact convex subset of a topological vector space and let* $f,g : X \times X \to \mathbb{R} \cup \{-\infty, \infty\}$ *be such that*

(i) $f(x,y) \leq g(x,y)$ *for all* $x,y \in X$;
(ii) *for each* $x \in X$, $f(x,y)$ *is a lower semicontinuous function of* y *on* X;
(iii) *for each* $y \in X$, $g(x,y)$ *is a quasi-concave function of* x *on* X; *i.e. for each* $t \in \mathbb{R}$, *the set* $\{x \in X : g(x,y) > t\}$ *is convex.*

Then the minimax inequality

$$\min_{y\in X} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} g(x,x)$$

holds.

Proof. It suffices to assume that $t = \sup_{x\in X} g(x,x) < \infty$. We shall show that case (2) of Corollary 1 cannot occur. Indeed, if there exist $A \in \mathcal{F}(X)$ and $y \in \mathrm{co}(A)$ such that $\min_{x\in A} f(x,y) > t$, then by (i), we must have $\min_{x\in A} g(x,y) > t$. It follows from (iii) that $g(y,y) > t$, contradicting $t = \sup_{x\in X} g(x,x)$. Hence the conclusion follows from Corollary 1. ∎

We observe that for $t = \sup_{x\in X} g(x,x) < \infty$, the above result also follows from Theorem 2 by replacing f and g by $f-t$ and $g-t$ respectively and by taking $X_0 = K = X$.

Next we remark that while Theorem 2 (also Theorem 1 of Tan [13]) is a generalization of Fan's minimax inequality [7, Theorem 1] *from a single function on a compact set to a pair of functions on a non-compact set*, Theorem 1 is a generalization of Theorem 1 of Tan [13] (and hence also of Theorem 1 of Yen [15]) *from a pair of functions to a single function*. We should point out that a function $f : X \times X \to \mathbb{R}$ satisfying the condition (ii) in Theorem 1 is said to be 0-*diagonally quasi-concave in* y in [16]. For other related but not comparable results, we refer to Deguire–Granas [6, Theorem 1], Granas–Liu [10, Theorem 5.1] and Shih–Tan [12, Theorem 1].

4. Equivalent forms. Following Ky Fan's idea in [8], we shall now give various equivalent formulations of Theorem 1:

THEOREM 1′ (First Geometric Form). *Let X be a non-empty convex subset of a topological vector space and let $N \subset X \times X$ be such that*

(i) *for each fixed $x \in X$ and for each non-empty compact subset C of X, the set $\{y \in C : (x,y) \in N\}$ is open in C;*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, there exists $x \in A$ such that $(x,y) \notin N$;*

(iii) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $(x,y) \in N$.*

Then there exists a point $\widehat{y} \in K$ such that $\{x \in X : (x,\widehat{y}) \in N\} = \emptyset$.

THEOREM 1″ (Second Geometric Form). *Let X be a non-empty convex subset of a topological vector space and let $M \subset X \times X$ be such that*

(i) *for each fixed $x \in X$ and for each non-empty compact subset C of X, the set $\{y \in C : (x,y) \in M\}$ is closed in C;*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, there exists $x \in A$ such that $(x,y) \in M$;*

(iii) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $(x,y) \notin M$.*

Then there exists a point $\widehat{y} \in K$ such that $X \times \{\widehat{y}\} \subset M$.

THEOREM $1'''$ (Maximal Element Version). *Let X be non-empty convex subset of a topological vector space and let $G : X \to 2^X \cup \{\emptyset\}$ be a set-valued map such that*

(i) *for $x \in X$ and for each non-empty compact subset C of X, $G^{-1}(x) \cap C$ is open in C (where $G^{-1}(x) = \{y \in X : x \in G(y)\}$);*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, there exists $x \in A$ such that $x \notin G(y)$;*

(iii) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $x \in G(y)$.*

Then there exists $\widehat{y} \in K$ such that $G(\widehat{y}) = \emptyset$.

Sketch of proofs. Theorem 1$\Rightarrow$Theorem $1'$: Let $f : X \times X \to \mathbb{R}$ be the characteristic function on N. ∎

Theorem $1'$ $\Rightarrow$Theorem 1: Define $N = \{(x, y) \in X \times X : f(x, y) > 0\}$. ∎

Theorem $1'$ $\Rightarrow$Theorem $1''$: Let $N = X \times X \setminus M$. ∎

Theorem $1''$ $\Rightarrow$Theorem $1'$: Let $M = X \times X \setminus N$. ∎

Theorem $1''$ $\Rightarrow$Theorem $1'''$: Let $M = \{(x, y) \in X \times X : x \notin G(y)\}$. ∎

Theorem $1'''$ $\Rightarrow$Theorem $1''$: Define $G : X \to 2^X \cup \{\emptyset\}$ by $G(y) = \{x \in X : (x, y) \notin M\}$ for all $y \in X$. ∎

Theorem $1'$ (respectively, Theorem $1''$) generalizes Theorem 3 (respectively, Theorem 4) of Shih–Tan [13].

As an immediate consequence of Theorem $1'''$, the maximal element version of our minimax inequality, we have the following result:

THEOREM 3. *Let X be a non-empty convex subset of a topological vector space and let $G : X \to 2^X$ be a set-valued map such that*

(i) *for each $y \in X$ and for each non-empty compact subset C of X, $G^{-1}(y) \cap C$ is open in C;*

(ii) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $x \in G(y)$.*

Then there exists $\widehat{y} \in X$ such that $\widehat{y} \in \mathrm{co}(G(\widehat{y}))$.

Proof. Since $G(y) \neq \emptyset$ for each $y \in X$, by Theorem $1'''$, there exist $A \in \mathcal{F}(X)$ and $\widehat{y} \in \mathrm{co}(A)$ such that $x \in G(\widehat{y})$ for all $x \in A$. Thus $A \subset G(\widehat{y})$, so that $\widehat{y} \in \mathrm{co}(A) \subset \mathrm{co}(G(\widehat{y}))$. ∎

The following result is an immediate consequence of Theorem 3:

THEOREM $3'$. *Let X be a non-empty convex subset of a topological vector space and let $G : X \to 2^X$ be a set-valued map such that*

(i) *for each* $x \in X$ *and for each non-empty compact subset* C *of* X, $G^{-1}(x) \cap C$ *is open in* C;

(ii) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$, *there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $x \in G(y)$;

(iii) *for each* $y \in X$, $G(y)$ *is convex.*

Then there exists $\widehat{y} \in X$ *such that* $\widehat{y} \in G(\widehat{y})$.

Theorem 3′ implies the following:

THEOREM 3″. *Let* X *be a non-empty convex subset of a topological vector space and* $G : X \to 2^X$ *be a set-valued map such that*

(i) *for each* $x \in X$ *and for each non-empty compact subset* C *of* X, $G^{-1}(x) \cap C$ *is open in* C;

(ii) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$, *there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $x \in \mathrm{co}(G(y))$.

Then there exists $\widehat{y} \in X$ *such that* $\widehat{y} \in \mathrm{co}(G(\widehat{y}))$.

Proof. By Theorem 3′, it remains to show that the map $\mathrm{co}\, G : X \to 2^X$ defined by $(\mathrm{co}\, G)(x) = \mathrm{co}(G(x))$ has the property: for each $x \in X$ and for each non-empty compact subset C of X, $(\mathrm{co}\, G)^{-1}(x) \cap C$ is open in C. Indeed, if $y \in (\mathrm{co}\, G)^{-1}(x) \cap C$, then $y \in C$ and $x \in \mathrm{co}(G(y))$; let $y_1, \ldots, y_n \in G(y)$ and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $x = \sum_{i=1}^n \lambda_i y_i$. For each $i = 1, \ldots, n$, $G^{-1}(y_i) \cap C$ is open in C and $y \in G^{-1}(y_i) \cap C$; let $U = \bigcap_{i=1}^n G^{-1}(y_i) \cap C$. Then U is an open neighbourhood of y in C. If $z \in U$, then $z \in C$ and $y_i \in G(z)$ for each $i = 1, \ldots, n$, so that $x = \sum_{i=1}^n \lambda_i y_i \in \mathrm{co}(G(z))$ and hence $z \in (\mathrm{co}\, G)^{-1}(x) \cap C$, for all $z \in U$. Therefore $(\mathrm{co}\, G)^{-1}(x) \cap C$ is open in C. ∎

The above proof that $(\mathrm{co}\, G)^{-1}(x) \cap C$ is open in C is a modification of the corresponding proof of Lemma 5.1 of Yannelis–Prabhakar [17]. As the condition (ii) of Theorem 3 implies the condition (ii) of Theorem 3″, Theorem 3 follows from Theorem 3″. Therefore Theorems 3, 3′ and 3″ are all equivalent. Theorem 3′ generalizes Theorem 1 of Browder [4].

5. Application to the existence of an equilibrium point. A quadruple (X, A, B, P) is a *one-person game* or a *one-agent abstract economy* if X is a non-empty convex subset of a topological vector space, $A, B : X \to 2^X \cup \{\emptyset\}$ are constraint correspondences and $P : X \to 2^X \cup \{\emptyset\}$ is a preference correspondence. An *equilibrium point* for (X, A, B, P) is a point $\widehat{x} \in X$ such that $\widehat{x} \in \mathrm{cl}_X B(\widehat{x})$ and $A(\widehat{x}) \cap P(\widehat{x}) = \emptyset$.

As an application of Theorem 3″, we have the following existence theorem of an equilibrium point for a one-person game:

THEOREM 4. *Let* (X,A,B,P) *be a one-person game such that*

(i) P *is of class* L_C;

(ii) *for each* $x\in X$, $A(x)$ *is non-empty and* $\mathrm{co}(A(x))\subset B(x)$;

(iii) *for each* $y\in X$, $A^{-1}(y)\cap C$ *is open in each non-empty compact subset* C *of* X;

(iv) *the map* $\mathrm{cl}\,B: X\to 2^X$ *defined by* $(\mathrm{cl}\,B)(x)=\mathrm{cl}_X\,B(x)$ *is upper semicontinuous*;

(v) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y\in X\setminus K$,

$$\mathrm{co}(X_0\cup\{y\})\cap\mathrm{co}(A(y)\cap P(y))\neq\emptyset\,.$$

Then (X,A,B,P) *has an equilibrium point* $\widehat{x}\in K$.

Proof. Suppose that for each $x\in X$, we have either $x\notin \mathrm{cl}\,B(x)$ or $A(x)\cap P(x)\neq\emptyset$. Define $G: X\to 2^X$ by

$$G(x)=\begin{cases} A(x)\cap P(x) & \text{if } x\in \mathrm{cl}_X B(x),\\ A(x) & \text{if } x\notin \mathrm{cl}_X B(x).\end{cases}$$

Let $y\in X$; for each non-empty compact subset C of X, we shall prove that $G^{-1}(y)\cap C$ is open in C. Let

$$U_1=\{x\in C: y\in A(x)\cap P(x)\}\,,$$
$$U_2=\{x\in C: y\in A(x) \text{ and } x\notin \mathrm{cl}_X B(x)\}\,.$$

Then $U_1=C\cap A^{-1}(y)\cap P^{-1}(y)$ is open in C by (ii) and P being of class L_C. Note that

$$\begin{aligned} U_2&=\{x\in C: y\in A(x)\}\cap\{x\in C: x\notin \mathrm{cl}_X B(x)\}\\ &=(C\cap A^{-1}(y))\cap[C\cap(X\setminus\{x\in X: x\in \mathrm{cl}_X B(x)\}]\,. \end{aligned}$$

By (ii), $C\cap A^{-1}(y)$ is open in C. By the upper semicontinuity of $\mathrm{cl}\,B$, the set $\{x\in X: x\in \mathrm{cl}_X B(x)\}$ is closed in X, so that $C\cap(X\setminus\{x\in X: x\in \mathrm{cl}_X B(x)\})$ is open in C; it follows that U_2 is also open in C. It is clear that $G^{-1}(y)\cap C=\{x\in C: y\in G(x)\}\subset U_1\cup U_2$. Conversely, if $x\in U_1$, then $x\in C$ and $y\in A(x)\cap P(x)$. We consider two cases:

(i) if $x\notin \mathrm{cl}_X B(x)$, then $y\in A(x)\cap P(x)\subset A(x)=G(x)$;

(ii) if $x\in \mathrm{cl}_X B(x)$, then $y\in A(x)\cap P(x)=G(x)$.

Hence $x\in G^{-1}(y)\cap C$. If $x\in U_2$, then $x\in C$ and $y\in A(x)$ and $x\notin \mathrm{cl}_X B(x)$, so that $y\in G(x)$ and $x\in G^{-1}(y)\cap C$. Therefore $G^{-1}(y)\cap C=U_1\cup U_2$ is open in C.

By (iv) and the definition of G, for each $y\in X\setminus K$, there exists $x\in \mathrm{co}(X_0\cup\{y\})$ such that $x\in \mathrm{co}\,G(y)$.

By Theorem 3$''$ there exists $\widehat{y}\in X$ such that $\widehat{y}\in \mathrm{co}(G(\widehat{y}))$. If $\widehat{y}\in \mathrm{cl}_X B(\widehat{y})$, then $\widehat{y}\in \mathrm{co}(A(\widehat{y})\cap P(\widehat{y}))\subset \mathrm{co}(P(\widehat{y}))$, which contradicts the assumption that P is of class L_C. If $\widehat{y} \notin \mathrm{cl}_X B(\widehat{y})$, then $\widehat{y} \in \mathrm{co}(A(\widehat{y})) \subset B(\widehat{y})$, which is impossible. Therefore there must exist $\widehat{x} \in X$ such that $\widehat{x} \in \mathrm{cl}_X B(\widehat{x})$ and $A(\widehat{x}) \cap P(\widehat{x}) = \emptyset$; that is, $\widehat{x}$ is an equilibrium point for (X, A, B, P). By (v), $\widehat{x}$ is necessarily in K. ■

For the existence of equilibrium points for an abstract economy with an infinite set of agents, we refer to Yannelis–Prabhakar [17, Theorem 6.1].

6. Fixed point theorems. In this section, we shall establish several fixed point theorems for set-valued inward and outward mappings in topological vector spaces (which need not be Hausdorff).

THEOREM 5. *Let X be a non-empty convex subset of a topological vector space E, and let $G : X \to 2^E$ be continuous on each non-empty compact subset C of X and such that for each $x \in X$, $G(x)$ is compact and convex. Let $p : X \times E \to \mathbb{R}$ be such that*

(a) *p is continuous on $C \times E$ for each non-empty compact subset C of X;*
(b) *for each $x \in X$, $p(x, \cdot)$ is a convex function on E.*

Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that

(i) *for each $y \in K$ with $y \notin G(y)$, there exist $x \in \overline{y + \bigcup_{\lambda > 0} \lambda(X - y)}$ and $v \in G(y)$ such that*

$$p(y, x - v) < \inf_{u \in G(y)} p(y, y - u)\,;$$

(ii) *for each $y \in X \backslash K$ with $y \notin G(y)$, there exist $x \in \overline{y + \bigcup_{\lambda > 0} \lambda(X_0 - y)}$ and $v \in G(y)$ such that*

$$p(y, x - v) < \inf_{u \in G(y)} p(y, y - u)\,.$$

Then G has a fixed point in X.

Proof. Assume that G has no fixed point in X. Define the function $f : X \times X \to \mathbb{R}$ by

$$f(x, y) = \inf_{u \in G(y)} p(y, y - u) - \inf_{v \in G(y)} p(y, x - v)\,.$$

For each fixed $x \in X$, by the continuity of p and G, it follows from Lemmas 1 and 2 that $f(x, y)$ is a lower semicontinuous function of y on each non-empty compact subset C of X.

The condition (ii) of Theorem 1 holds. Indeed, if it does not hold, then there exist $A = \{x_1, \ldots, x_n\} \in \mathcal{F}(X)$ and $\overline{y} = \sum_{i=1}^n \lambda_i x_i \in \mathrm{co}(A)$ with $\lambda_i > 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i = 1$ such that

$$f(x, \overline{y}) = \inf_{u \in G(\overline{y})} p(\overline{y}, \overline{y} - u) - \inf_{v \in G(\overline{y})} p(\overline{y}, x - v) > 0 \quad \text{for all } x \in A\,.$$

Hence we have

$$\inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u) > \inf_{v\in G(\bar y)} p(\overline{y},x_i-v) \quad \text{for all } x_i\in A\,. \tag{6.1}$$

Since $G(\overline{y})$ is compact and convex and p is continuous, for each $x_i \in A$ there exists $v_i \in G(\overline{y})$ such that

$$\inf_{v\in G(\bar y)} p(\overline{y},x_i-v) = p(\overline{y},x_i-v_i) \quad \text{and} \quad \overline{v}=\sum_{i=1}^n \lambda_i v_i \in G(\overline{y})\,.$$

From the convexity of the function $p(x,\cdot)$ and (6.1) it follows that

$$\begin{aligned}\inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u) &\le p(\overline{y},\overline{y}-\overline{v}) = p\Big(\overline{y},\sum_{i=1}^n \lambda_i(x_i-v_i)\Big)\\ &\le \sum_{i=1}^n \lambda_i p(\overline{y},x_i-v_i) = \sum_{i=1}^n \lambda_i \inf_{v\in G(\bar y)} p(\overline{y},x_i-v)\\ &< \inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u)\,,\end{aligned}$$

which is a contradiction. Hence the condition (ii) of Theorem 1 holds.

We claim that the condition (iii) of Theorem 1 holds. Indeed, if it were false, then there would exist $\overline{y} \in X\setminus K$ such that $f(x,\overline{y}) \le 0$ for all $x \in \mathrm{co}(X_0\cup\{\overline{y}\})$. Hence we have

$$\inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u) \le \inf_{v\in G(\bar y)} p(\overline{y},x-v) \quad \text{for all } x\in \mathrm{co}(X_0\cup\{\overline{y}\})\,.$$

Note that $\mathrm{co}(X_0\cup\{\overline{y}\}) = \overline{y}+\bigcup_{0\le\lambda\le1}\lambda(X_0-\overline{y})$, so we have

$$\inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u) \le p(\overline{y},x-v) \tag{6.2}$$

$$\text{for all } v\in G(\overline{y}) \text{ and } x\in \overline{y}+\bigcup_{0\le\lambda<1}\lambda(X_0-\overline{y})\,.$$

Since $\overline{y}\notin G(\overline{y})$, by (ii) and the continuity of $p(x,\cdot)$ there exist $x_0\in X_0$, $\lambda>0$ and $\overline{v}\in G(\overline{y})$ such that $x=\overline{y}+\lambda(x_0-\overline{y})$ and

$$p(\overline{y},x-\overline{v}) < \inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u)\,. \tag{6.3}$$

By (6.2), we must have $\lambda>1$ so that

$$x_0=\frac{\lambda-1}{\lambda}\overline{y}+\frac{1}{\lambda}x\,.$$

By the continuity of $p(\overline{y},\cdot)$ and the compactness of $G(\overline{y})$, there exists $u_0\in G(\overline{y})$ such that $p(\overline{y},\overline{y}-u_0)=\inf_{u\in G(\bar y)} p(\overline{y},\overline{y}-u)$. Since $G(\overline{y})$ is convex,

$$w=\frac{\lambda-1}{\lambda}u_0+\frac{1}{\lambda}\overline{v}\in G(\overline{y})\,.$$

Again from the convexity of $p(\overline{y},\cdot)$ it follows that

$$p(\overline{y},x_0-w)=p\bigg(\overline{y},\frac{\lambda-1}{\lambda}(\overline{y}-u_0)+\frac{1}{\lambda}(x-\overline{v})\bigg)$$
$$\le\frac{\lambda-1}{\lambda}p(\overline{y},\overline{y}-u_0)+\frac{1}{\lambda}p(\overline{y},x-\overline{v})<\inf_{u\in G(\overline{y})}p(\overline{y},\overline{y}-u)\,,$$

which contradicts (6.2). Thus the condition (iii) of Theorem 1 also holds.

By Theorem 1, there exists $\widehat{y}\in K$ such that $f(x,\widehat{y})\le 0$ for all $x\in X$. It follows that

$$\inf_{u\in G(\hat{y})}p(\widehat{y},\widehat{y}-u)\le p(\widehat{y},x-v)\quad\text{ for all } x\in X \text{ and } v\in G(\widehat{y})\,. \tag{6.4}$$

Since $\widehat{y}\in K$ and $\widehat{y}\notin G(\widehat{y})$, by (i) and continuity of $p(\widehat{y},\cdot)$, there exist $\widehat{x}\in X$, $\lambda>0$ and $\widehat{v}\in G(\widehat{y})$ such that $x=\widehat{y}+\lambda(\widehat{x}-\widehat{y})$ and

$$p(\widehat{y},x-\widehat{v})<\inf_{u\in G(\hat{y})}p(\widehat{y},\widehat{y}-u)\,. \tag{6.5}$$

If $\lambda\le 1$, then $x\in X$ so that (6.5) contradicts (6.4). If $\lambda>1$, using a similar argument to the above proof, we also obtain a contradiction. Therefore G must have a fixed point in X. ∎

Theorem 5 generalizes Theorem 3.3 of Jiang [11] to the non-compact setting and Theorem 10 of Shih–Tan [15], which in turn generalizes Theorem 1 of Browder [5].

THEOREM 5′. *Let X be a non-empty convex subset of a topological vector space E, and let $G:X\to 2^E$ be continuous on each non-empty compact subset C of X and such that for each $x\in X$, $G(x)$ is compact and convex. Let $p:X\times E\to\mathbb{R}$ be such that*

(a) *p is continuous on $C\times E$ for each non-empty compact subset C of X;*
(b) *for each $x\in X$, $p(x,\cdot)$ is a convex function on E.*

Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that

(i) *for each $y\in K$ with $y\notin G(y)$, there exist $x\in\overline{y+\bigcup_{\lambda<0}\lambda(X-y)}$ and $v\in G(y)$ such that*

$$p(y,x-v)<\inf_{u\in G(y)}p(y,y-u);$$

(ii) *for each $y\in X\setminus K$ with $y\notin G(y)$, there exist $x\in\overline{y+\bigcup_{\lambda<0}(X_0-y)}$ and $v\in G(y)$ such that*

$$p(y,x-v)<\inf_{u\in G(y)}p(y,y-u)\,.$$

Then G has a fixed point in X.

Proof. Define the maps $F : X \to 2^E$ and $q : X \times E \to \mathbb{R}$ by $F(x) = 2x - G(x)$ and $q(x,y) = p(x,-y)$. It is easy to check that F and q satisfy the hypotheses of Theorem 5. By Theorem 5, F has a fixed point in X, so that G has a fixed point in X. ∎

Theorem 5′ generalizes Theorem 2 of Browder [5] to a set-valued map on a non-compact set in a topological vector space which is not necessarily locally convex (as is required in [5]) and Corollary 3.4 of Jiang [11] to the non-compact setting.

COROLLARY 3. *Let X be a non-empty convex subset of a normed space E, and let $G : X \to 2^E$ be continuous on each non-empty compact subset C of X and such that for each $x \in X$, $G(x)$ is compact convex. Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that*

(i) *for each $y \in K$, $G(y) \cap \overline{(y + \bigcup_{\lambda>0} \lambda(X-y))} \neq \emptyset$ (respectively, $G(y) \cap \overline{(y + \bigcup_{\lambda<0} \lambda(X-y))} \neq \emptyset$);*

(ii) *for each $y \in X \setminus K$, $G(y) \cap \overline{(y + \bigcap_{\lambda>0} \lambda(X_0-y))} \neq \emptyset$ (respectively, $G(y) \cap \overline{(y + \bigcup_{\lambda<0} \lambda(X_0-y))} \neq \emptyset$).*

Then G has a fixed point in X.

Proof. Since E is a normed space, by setting $p(x,y) = ||y||$ for all $(x,y) \in X \times E$, it follows from Theorem 5 (respectively, Theorem 5′) that the conclusion holds. ∎

Corollary 3 generalizes Corollary 2 (resp. Corollary 2′) of Browder [5] and Corollary 1 of Shih–Tan [15].

THEOREM 6. *Let X be a non-empty convex subset of a topological vector space E, and let $G : X \to 2^E$ be upper semicontinuous on each non-empty compact subset C of X and such that for each $x \in X$, $G(x)$ is compact. Let $p : X \times E \to \mathbb{R}$ be continuous on $C \times D$ for any non-empty compact subsets C and D of X and E, respectively, such that for each $x \in X$, $p(x,\cdot)$ is a convex function on E. Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that*

(i) *for each $y \in K$ with $y \notin G(y)$, there exists $x \in y + \bigcup_{\lambda>0} \lambda(X-y)$ such that $p(y, x-u) < p(y, y-u)$ for all $u \in G(y)$;*

(ii) *for each $y \in X \setminus K$ with $y \notin G(y)$, there exists $x \in y + \bigcup_{\lambda>0} \lambda(X_0-y)$ such that $p(y, x-u) < p(y, y-u)$ for all $u \in G(y)$.*

Then G has a fixed point in X.

Proof. Assume that G has no fixed point in X. Define the function $f : X \times X \to \mathbb{R}$ by

$$f(x,y) = \inf_{u \in G(y)} [p(y, y-u) - p(y, x-u)] .$$

For each non-empty compact subset C of X, by the assumption on G, $G(C)$ is compact in E. By the continuity assumption on p, for each fixed $x \in X$ the function $W(y,u) = p(y, y-u) - p(y, x-u)$ is continuous on $C \times G(C)$ so that from Lemma 1 it follows that for each fixed $x \in X$, $f(x,y)$ is a lower semicontinuous function of y on each non-empty compact subset C of X.

The condition (ii) of Theorem 1 is satisfied: Indeed, otherwise there would exist $A = \{x_1, \ldots, x_n\} \in \mathcal{F}(X)$ and $\overline{y} = \sum_{i=1}^n \lambda_i x_i \in \mathrm{co}(A)$ with $\lambda_i > 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i = 1$ such that $\min_{x \in A} f(x, \overline{y}) > 0$, so that

$$p(\overline{y}, \overline{y} - u) > p(\overline{y}, x-u) \quad \text{for all } x \in A \text{ and } u \in G(\overline{y}) . \tag{6.6}$$

Since $p(\overline{y}, \cdot)$ is a convex function, we have, for each $u \in G(\overline{y})$,

$$\begin{aligned} p(\overline{y}, \overline{y}-u) &= p\Big(\overline{y}, \sum_{i=1}^n \lambda_i x_i - u\Big) = p\Big(\overline{y}, \sum_{i=1}^n \lambda_i (x_i - u)\Big) \\ &\le \sum_{i=1}^n \lambda_i p(\overline{y}, x_i - u) < p(\overline{y}, \overline{y} - u) \quad \text{by (6.6)}, \end{aligned}$$

which is a contradiction. Hence the condition (ii) of Theorem 1 holds.

The condition (iii) of Theorem 1 is also satisfied: Suppose that there exists $\overline{y} \in X \setminus K$ such that

$$f(x, \overline{y}) \le 0 \quad \text{for all } x \in \mathrm{co}(X_0 \cup \{\overline{y}\}) . \tag{6.7}$$

Since $\overline{y} \in X \setminus K$, by (ii) there exists $\overline{x} \in \overline{y} + \bigcup_{\lambda > 0} \lambda(X_0 - \overline{y})$, say $\overline{x} = \overline{y} + \lambda(x_0 - \overline{y})$ for some $\lambda > 0$ and $x_0 \in X_0$, such that

$$p(\overline{y}, \overline{x} - u) < p(\overline{y}, \overline{y} - u) \quad \text{for all } u \in G(\overline{y}) . \tag{6.8}$$

Case 1: If $0 < \lambda \le 1$, then $\overline{x} = \lambda x_0 + (1-\lambda)\overline{y} \in \mathrm{co}(X_0 \cup \{\overline{y}\})$, so that by (6.7),

$$0 \ge f(\overline{x}, \overline{y}) = \inf_{u \in G(\overline{y})} [p(\overline{y}, \overline{y} - u) - p(\overline{y}, \overline{x} - u)] = p(\overline{y}, \overline{y} - \overline{u}) - p(\overline{y}, \overline{x} - \overline{u})$$

for some $\overline{u} \in G(\overline{y})$ since $G(\overline{y})$ is compact; this contradicts (6.8).

Case 2: If $\lambda > 1$ then

$$x_0 = \frac{1}{\lambda}\overline{x} + \frac{\lambda - 1}{\lambda}\overline{y}$$

is a convex combination of $\overline{x}$ and $\overline{y}$; as $p(\overline{y}, \cdot)$ is convex, we have, for each $u \in G(\overline{y})$,

$$\begin{aligned} p(\overline{y}, x_0 - u) &= p\bigg(\overline{y}, \frac{1}{\lambda}(\overline{x} - u) + \frac{\lambda - 1}{\lambda}(\overline{y} - u)\bigg) \\ &\le \frac{1}{\lambda} p(\overline{y}, \overline{x} - u) + \frac{\lambda - 1}{\lambda} p(\overline{y}, \overline{y} - u) \end{aligned} \tag{6.9}$$

$$< \frac{1}{\lambda}p(\overline{y},\overline{y}-u)+\frac{\lambda-1}{\lambda}p(\overline{y},\overline{y}-u) \quad \text{by}(6.8)$$
$$= p(\overline{y},\overline{y}-u)\,.$$

By (6.7), since $x_0 \in X_0 \subset \mathrm{co}(X_0 \cup \{\overline{y}\})$,

$$0 \geq f(x_0,\overline{y}) = \inf_{u\in G(\bar{y})}[p(\overline{y},\overline{y}-u)-p(\overline{y},x_0-u)] = p(\overline{y},\overline{y}-u_0)-p(\overline{y},x_0-u_0)$$

for some $u_0 \in G(\overline{y})$ as $G(\overline{y})$ is compact; this contradicts (6.9). Hence the condition (iii) of Theorem 1 holds.

By Theorem 1, there exists $\widehat{y} \in K$ such that

$$f(x,\widehat{y}) = \inf_{u\in G(\hat{y})}[p(\widehat{y},\widehat{y}-u)-p(\widehat{y},x-u)] \leq 0 \quad \text{for all } x \in X\,.$$

It follows that for each $x \in X$, there exists $u_x \in G(\widehat{y})$ such that

$$p(\widehat{y},\widehat{y}-u_x) \leq p(\widehat{y},x-u_x)\,. \tag{6.10}$$

Since $\widehat{y} \in K$, by (i) there exists $\widehat{x} \in \widehat{y}+\bigcup_{\lambda>0}\lambda(X-\widehat{y})$, say $\widehat{x}=\widehat{y}+\lambda(\overline{x}-\widehat{y})$ for some $\lambda>0$ and $\overline{x}\in X$, such that

$$p(\widehat{y},\widehat{x}-u) < p(\widehat{y},\widehat{y}-u) \quad \text{for all } u \in G(\widehat{y})\,. \tag{6.11}$$

If $\lambda \leq 1$, then $\widehat{x} \in X$, so that (6.11) contradicts (6.10). If $\lambda > 1$, then

$$\overline{x} = \frac{1}{\lambda}\widehat{x}+\frac{\lambda-1}{\lambda}\widehat{y}$$

and for each $u \in G(\widehat{y})$.

$$p(\widehat{y},\overline{x}-u) = p\bigg(\widehat{y},\frac{1}{\lambda}(\widehat{x}-u)+\frac{\lambda-1}{\lambda}(\widehat{y}-u)\bigg)$$
$$\leq \frac{1}{\lambda}p(\widehat{y},\widehat{x}-u)+\frac{\lambda-1}{\lambda}p(\widehat{y},\widehat{y}-u)$$
$$< p(\widehat{y},\widehat{y}-u) \quad \text{by (6.11)}\,,$$

which again contradicts (6.10). Therefore G must have a fixed point in X. ∎

Theorem 6 also generalizes Theorem 10 of Shih–Tan [15] and Theorem 1 of Browder [5]. Similar to Theorem 5′, Theorem 6 remains valid if in both conditions (i) and (ii), "$\lambda > 0$" is replaced by "$\lambda < 0$".

REFERENCES

[1] G. Allen, *Variational inequalities, complementarity problems, and duality theorems*, J. Math. Anal. Appl. 58 (1977), 1–10.

[2] J. P. Aubin, *Mathematical Methods of Games and Economic Theory*, revised ed., Stud. Math. Appl. 7, North-Holland, 1982.

[3] J. S. Bae, W. K. Kim and K. K. Tan, *Another generalization of Fan's minimax inequality and its applications*, submitted.

[4] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. 177 (1968), 283–301.
[5] —, *On a sharpened form of the Schauder fixed point theorem*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 4749–4751.
[6] P. Deguire et A. Granas, *Sur une certaine alternative non-linéaire en analyse convexe*, Studia Math. 83 (1986), 127–138.
[7] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. 142 (1961), 305–310.
[8] —, *A minimax inequality and applications*, in: Inequalities III, O. Shisha (ed.), Acad. Press, 1972, 103–113.
[9] —, *Some properties of convex sets related to fixed point theorems*, Math. Ann. 266 (1984), 519–537.
[10] A. Granas and F. C. Liu, *Coincidences for set-valued maps and minimax inequalities*, J. Math. Pures Appl. 65 (1986), 119–148.
[11] J. Jiang, *Fixed point theorems for multi-valued mappings in locally convex spaces*, Acta Math. Sinica 25 (1982), 365–373.
[12] M. H. Shih and K. K. Tan, *A further generalization of Ky Fan's minimax inequality and its applications*, Studia Math. 77 (1984), 279–287.
[13] —, —, *The Ky Fan minimax principle, sets with convex sections, and variational inequalities*, in: Differential Geometry, Calculus of Variations and Their Applications, G. M. Rassias & T. M. Rassias (eds.), Lecture Notes in Pure Appl. Math. 100, Dekker, 1985, 471–481.
[14] —, —, *Covering theorems of convex sets related to fixed point theorems*, in: Nonlinear and Convex Analysis, B.L. Lin and S. Simons (eds.), Dekker, 1987, 235-244.
[15] —, —, *A geometric property of convex sets with applications to minimax type inequalities and fixed point theorems*, J. Austral. Math. Soc. Ser. A 45 (1988), 169–183.
[16] K. K. Tan, *Comparison theorems on minimax inequalities, variational inequalities, and fixed point theorems*, J. London Math. Soc. 23 (1983), 555–562.
[17] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological vector spaces*, Math. Economics 12 (1983), 233–245.
[18] C. L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific J. Math. 97 (1981), 477–481.
[19] J. X. Zhou and G. Chen, *Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities*, J. Math. Anal. Appl. 132 (1988), 213–225.

DEPARTMENT OF MATHEMATICS
SICHUAN NORMAL UNIVERSITY
CHENGDU, SICHUAN
CHINA

DEPARTMENT OF MATHEMATICS,
STATISTICS AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA, B3H 3J5

Reçu par la Rédaction le 7.2.1990;
en version modifiée le 4.3.1991