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SOME CHARACTERIZATIONS OF BLOCH FUNCTIONS ON STRONGLY PSEUDOCONVEX DOMAINS

BY

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1. Introduction. Bloch functions have been studied deeply and systematically for a long time. S. Axler [1] gave some new characterizations of Bloch functions on the unit disk U. They are:

(i) $\|f\|_{B(U)} < \infty$;

(ii) $\sup\Big\{ \int\limits_{U(q,r)} |f(z) - f(q)|^p dA(z)/|U(q,r)|\Big\} < \infty$;

(iii) $\sup\Big\{ \int\limits_{U(q,r)} |f(z) - f_{U(q,r)}|^p dA(z)/|U(q,r)|\Big\} < \infty$;

(iv) $\sup\{\operatorname{dist}(\overline{f}|_{U(q,r)}, H^\infty(U(q,r))\} < \infty$;

(v) $\sup\{\operatorname{area}(f(U(q,r)))\} < \infty$;

(vi) $\sup\Big\{ \int\limits_{U(q,r)} |f'(z)|^2\, dA(z)\Big\} < \infty$.

Here $0 < r < 1 \le p$, $S_q(z) = (q-z)/(1-\overline{q}z)$, $U(q,r) = S_q(\{z : |z| < r\})$, $f_{U(q,r)} = |U(q,r)|^{-1} \int_{U(q,r)} f(z)\, dA(z)$, A is the usual area measure on $\mathbb{C}$, $|K|$ is the measure of a set $K \subset \mathbb{C}$ with respect to dA, f is an analytic function on U, and all the suprema are taken over $q \in U$.

Z. J. Hu proved that similar results hold on bounded symmetric domains (see the graduation papers of Hangzhou University). Axler's and Hu's proofs depend strongly on the homogeneity and Bergman kernels of bounded symmetric domains. Since strongly pseudoconvex domains are not necessarily homogeneous, we have to search for another method. We use mainly the boundary value estimate of the Kobayashi metric.

In Section 2, some preliminaries are given. In Section 3, we describe a kind of polydisk which will play a key role in proving the main results of this paper. In Section 4, we generalize some results of Axler. Furthermore,

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the condition (vi) is extended to all $p > 0$ and a weight $|\varrho(z)|^{p-n-1}$. In Section 5, we give another characterization of Bloch functions on some bounded domains in $\mathbb{C}^n$.

2. Preliminaries. First, we give some definitions:

DEFINITION 2.1 (cf. [5]). Let f be an analytic function on the unit disk U. Then f is called a *Bloch function* on U if

$$\|f\|_{B(U)} = \sup\{|f'(z)|(1-|z|^2) : z \in U\} < \infty .$$

DEFINITION 2.2 (cf. [6]). Let D be a bounded homogeneous domain in $\mathbb{C}^n$, let $H_z(x,\overline{x})$ be the Bergman metric on D and f a holomorphic function on D. Write

$$\begin{aligned} Q_f(z) &= \sup\{|\nabla f(z)\cdot x|/H_z(x,\overline{x})^{1/2} : x \in \mathbb{C}^n \setminus \{0\}\} < \infty , \\ \|f\|_{B(D)} &= \sup\{Q_f(z) : z \in D\} . \end{aligned}$$

If $\|f\|_{B(D)} < \infty$, then f is called a *Bloch function* on D. Here $\nabla f(q) = (\frac{\partial f}{\partial z_1}(q), \ldots, \frac{\partial f}{\partial z_n}(q)), \nabla f(z)\cdot x = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z)\cdot x_j$.

Before we introduce Bloch functions on strongly pseudoconvex domains, we define the Kobayashi metric.

DEFINITION 2.3 (cf. [3]). Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$. The *Kobayashi metric for D at z in the direction x* is defined as

$$F_{\mathrm{K}}^D(z,x) = \inf\{c : \text{ there is } f \in D(U) \text{ with } f(z) = 0,\ f'(z) = x/c,\ c > 0\} ,$$

where U is the unit disk and $D(U)$ is the set of all holomorphic mappings which map U into D.

DEFINITION 2.4 (cf. [2]). Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$, and let $F_{\mathrm{K}}^D(z,x)$ be the Kobayashi metric. A holomorphic function f on D is called a *Bloch function* if

$$\|f\|_{B(D)} = \sup\{|f_*(z)\cdot x|/F_{\mathrm{K}}^D(z,x) : z \in D \text{ and } x \in \mathbb{C}^n \setminus \{0\}\} < \infty .$$

Here $f_*(z)$ is the mapping from $T_z(D)$ to $T_{f(z)}(\mathbb{C})$ induced by f.

In $\|f\|_{B(D)}$, $F_{\mathrm{K}}^D(z,x)$, we shall omit D or K or both when no confusion can arise.

If $\gamma : [0,1] \to D$ is a C^1 curve, the *Kobayashi length* of γ is $l_{\mathrm{K}}(\gamma) = \int_0^1 F_{\mathrm{K}}(\gamma(t),\gamma'(t))\,dt$. If both z_1 and z_2 are in D, then the *Kobayashi distance* between them is

$$d_{\mathrm{K}}(z_1,z_2) = \inf\{l_{\mathrm{K}}(\gamma) : \gamma(0) = z_1,\ \gamma(1) = z_2\} .$$

The *Kobayashi ball* is $B_{\mathrm{K}}(q,r) = \{z \in D : d_{\mathrm{K}}(q,z) < r\}$, where $q \in D$ and $r > 0$.

The Kobayashi metric has the following properties (cf. [3]):

(a) $F_{\mathrm{K}}(z, cx) = |c| F_{\mathrm{K}}(z, x)$ for all $z \in D$ and $x \in \mathbb{C}^n$.

(b) Let $D \subset \mathbb{C}^n$, $D' \subset \mathbb{C}^m$ be strongly pseudoconvex domains and let $g : D \to D'$ be a holomorphic mapping. Then

$$F_{\mathrm{K}}^{D'}(g(z), g_*(x)) \le F_{\mathrm{K}}^{D}(z, x) \quad \text{for all } z \in D \text{ and } x \in \mathbb{C}^n$$

where $g_*(x)$ is the mapping from $T_x(D)$ to $T_{g(x)}(\mathbb{C}^m)$ induced by g.

(c) If $\varrho(z)$ is a defining function for a strongly pseudoconvex domain D in $\mathbb{C}^n$, and if z is sufficiently near ∂D, from [6] we have

$$F_{\mathrm{K}}^{D}(z, x) \approx |x_{\mathrm{N}}||\varrho(z)|^{-1} + |x_{\mathrm{T}}||\varrho(z)|^{-1/2} \quad \text{for all } z \in D\,,$$

where $x = x_{\mathrm{N}} + x_{\mathrm{T}}$ is the decomposition of x into the complex normal and complex tangential pieces at z.

If $dw(z)$ denotes the Euclidean volume element of $\mathbb{C}^n$, then the Kobayashi volume element $dD(z)$ is equivalent to $|\varrho(z)|^{-n-1}\, dw(z)$ (see p. 59 in [4]). Thus when $r > 0$ is fixed, the Kobayashi volume of $B_{\mathrm{K}}(q, r)$ is comparable to that of some polydisk. In order to describe the polydisk explicitly, assume that the complex normal direction at q is the z_1 direction, the vector $v = (1, 0, \ldots, 0)$ is the complex outward normal direction at q, and $z_2, z_3, \ldots, z_n$ are the complex tangential directions at q. Denote by $d(q)$ the distance between q and the boundary of D. Then $B_{\mathrm{K}}(q, r)$ is comparable to the following polydisk:

$$\begin{aligned} P_{q,r} = \{z \in D : |z_1 - q_1'| < c_1 d(q),\ |z_2 - q_2| < c_2 d(q)^{1/2}, \ldots \\ \ldots, |z_n - q_n| < c_2 d(q)^{1/2}\}\,; \end{aligned}$$

here c_1, c_2 depend on r (but not on q—in particular, not on $d(q)$) and $q_1' = q_1 + c_3 v$ with c_3 depending on r and $d(q)$.

Finally, we give three equivalent conditions for Bloch functions on strongly pseudoconvex domains (see [2]):

PROPOSITION 2.5 (cf. [2]). *Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain, and let $\varrho(z)$ be a defining function for D. If f is a holomorphic function on D, then the following statements are equivalent*:

(i) $\|f\|_{B(D)} < \infty$.

(ii) *The radii of schlicht disks in the range of f are bounded above.*

(iii) $\sup\{|\nabla_\nu f(z)| \cdot |\varrho(z)| : z \in D\} < \infty$.

Here ∇_ν is the normal derivative.

3. Some lemmas. In this section, D always denotes a strongly pseudoconvex domain in $\mathbb{C}^n$.

LEMMA 3.1. *For any $q' \in \partial D$, if $v(q')$ is the unit complex outward normal vector at q', then there exists a unique ball $B(q')$ such that*

(i) $B(q')$ *is tangent to* ∂D *from the inside and* q' *is one of the tangency points.*

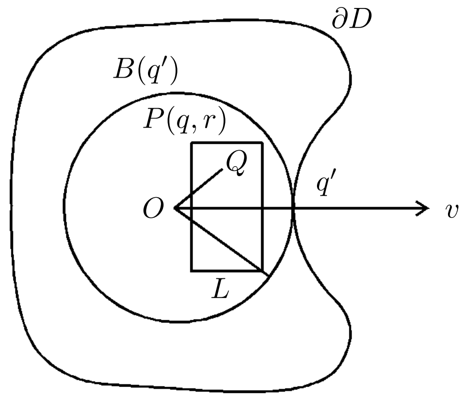
(ii) $B(q')$ *does not intersect the boundary of* D.

(iii) $B(q')$ *is the largest ball which satisfies* (i) *and* (ii).

The proof is trivial.

LEMMA 3.2. *There exists a positive constant* c_0 *depending on* D *and a neighbourhood* G *of* ∂D *which depends on* c_0 *and* D *such that for any* $q \in G$, *there exists a ball* $B(q')$ (*as in Lemma* 3.1) *such that* $P(q,r) = \{z \in D : |z_1 - q_1| < c_0 d(q),\ |z_j - q_j| < c_0 d(q)^{1/2},\ 2 \le j \le n\} \subset B(q')$.

Proof. For $r(q')$ the radius of $B(q')$, let r' be the infimum of $r(q')$ as q' runs through ∂D. Since ∂D is compact and $r(q')$ is continuous with respect to q' on ∂D, r' is positive.



$$O = q' - r(q')v(q')$$
$$\overline{OQ} \le \overline{OL}$$

Take

$$c_0 = r'/(n + r'),$$
$$G = \{q \in D : d(q) < \min\{r', (n-1)c_0^2/(1-c_0)^2\}\}.$$

For any $q \in G$, there exists a unique ball $B(q')$ such that q is in $\{q' - tr(q')v(q') : 0 \le t \le 1\}$ ($v(q')$ is the unit complex outward normal at q'). The distance $\mathrm{dist}(q' - r(q')v(q'), Q)$ between $q' - r(q')v(q')$ and any point $Q \in P(q,r)$ is less than or equal to $[(r(q') - (1-c_0)d(q))^2 + (n-1)c_0^2 d(q)]^{1/2}$, thus

$$\begin{aligned} r(q') - [(r(q') - &(1-c_0)d(q))^2 + (n-1)c_0^2 d(q)]^{1/2} \\ &= \frac{[2(1-c_0)r(q') - (n-1)c_0^2 - (1-c_0)^2 d(q)]d(q)}{r(q') + ((r(q') - (1-c_0)d(q))^2 + (n-1)c_0^2 d(q))^{1/2}} \\ &> 2(1-c_0)r(q') - (n-1)c_0^2 - (1-c_0)^2 d(q) \\ &> 2(1-c_0)r' - 2(n-1)c_0 \ge 0. \end{aligned}$$

This means that $\mathrm{dist}(q' - r(q')v(q'), Q) < r(q')$, and thus $P(q,r) \subset B(q')$.

LEMMA 3.3. *There exist positive constants c and N, which depend on r, q and D, such that*

$$Nd(q) \le d(z) \quad \text{for all } q \in D \text{ and } z \in P(q,r)\,.$$

Proof. Without loss of generality, we only have to prove this lemma in some neighbourhood of ∂D. Take $q \in D$ such that $P(q,r)$ is the polydisk of Lemma 3.2. We may assume that $c \le \min\{1/2, c_0\}$, where c_0 satisfies the conclusion of Lemma 3.2. For $r(q')$ and $B(q')$ as in the proof of Lemma 3.2, $q \in \{q' - tr(q')v(q') : 0 \le t \le 1\}$. For any $z \in P(q,r)$,

$$d(z) = \operatorname{dist}(z, \partial D) \ge r(q') - [(r(q') - (1-c_0)\,d(q))^2 + c_0^2(n-1)\,d(q)]^{1/2}\,.$$

When $n = 1$,

$$d(z) \ge (1-c_0)\,d(q) > d(q)/2\,.$$

When $n > 1$,

$$d(z) \ge \frac{[2(1-c_0)r(q') - (n-1)c_0^2 - (1-c_0)^2\,d(q)]\,d(q)}{r(q') + [r(q') - (1-c_0)\,d(q)]^2 + (n-1)c_0^2\,d(q))^{1/2}}\,.$$

Choose c and $d(q)$ such that

$$c^2 = \min\{c_0^2, 1/4, r(q')/[4(n-1)]\}\,,$$
$$d(q) < \min\{r(q')/4, (n-1)c_0^2/(1-c_0)^2\}\,.$$

It is not difficult to obtain $d(z) \ge d(q)/6$. Thus we may choose

$$\begin{aligned} c &= \inf\{\min\{c_0^2, r(q')/(4n), 1/4\} : q' \in \partial D\}\,,\\ R &= \inf\{\min\{r(q')/4, (n-1)c_0^2/(1-c_0)^2\} : q' \in \partial D\}\,,\\ G &= \{z \in D : d(z) < R\}\,. \end{aligned}$$

Because of the compactness of ∂D and the continuity of c and $r(q')$ with respect to q', it is easy to show that the assertion of the lemma holds.

LEMMA 3.4. *There exist two positive constants c and M, which depend only on D (and not on r, q and z), such that*

$$d(z) \le Md(q) \quad \text{for all } q \in D \text{ and } z \in P(q,r)\,.$$

Proof. To prove this lemma, we have to improve the ball $B(q')$ of Lemma 3.1 as follows:

(i) $B(q')$ is tangent from the inside to ∂D at q'.

(ii) $B(q')$ and the ball symmetric to it with respect to $T(q')$ do not intersect D. Here $T(q')$ is the complex tangent plane to D at q'.

(iii) $B(q')$ is the largest ball which satisfies (i) and (ii).

For any $q' \in \partial D$, we can always find such a ball $B(q')$. From the proof of Lemma 3.2, there exist some positive constant c and some neighbourhood G of ∂D such that for any $q \in G$ and any fixed positive r, there exists a corresponding $B(q')$ which satisfies $P(q,r) \subset B(q')$ and the above conditions (i)–(iii).

The set of the intersection points of $\{z + tv(q') : z \in P(q,r) \text{ and } t \in \mathbb{C}\}$ and ∂D near q' is included in $\{sQ + (1-s)Q' : 0 \le s \le 1,\ Q \in P(q,r),\ Q'$ is symmetric to Q with respect to $T(q')\}$. We obtain

$$d(z) \le 2(1+c)d(q) \quad \text{for all } z \in P(q,r)\,.$$

The remainder is similar to the proof of Lemma 3.3.

LEMMA 3.5. *There exists a positive number c, which is independent of q but may depend on r (fixed $r > 0$), such that $P(q,r)$ is included in the Kobayashi ball $B_{\mathrm{K}}(q,r)$ for q sufficiently near ∂D.*

Proof. Because $|\varrho(z)|$ is equivalent to $d(z)$ from Section 2, we obtain

$$F_{\mathrm{K}}(z,x) \approx |x_{\mathrm{N}}|/d(z) + |x_{\mathrm{T}}|/d(z)^{1/2}\,.$$

Here x_{N} and x_{T} are the components of $x \in \mathbb{C}^n \setminus \{0\}$ at z.

Assume the constant c' to be the constant c as in Lemma 3.3. There exists a positive constant M' depending on D such that

$$\begin{aligned} d_{\mathrm{K}}(q,z) &\le \int_0^1 F_{\mathrm{K}}(q + t(z-q), z-q)\,dt \\ &\le M' \int_0^1 |(z-q)_{\mathrm{N}}|\,dt/d(q+t(z-q)) \\ &\quad + M' \int_0^1 |(z-q)_{\mathrm{T}}|\,dt/d(q+t(z-q))^{1/2}\,. \end{aligned}$$

By Lemma 3.3,

$$d_{\mathrm{K}}(q,z) \le M' \int_0^1 |(z-q)_{\mathrm{N}}|\,dt/[Nd(q)] + M' \int_0^1 |(z-q)_{\mathrm{T}}| dt/[Nd(q)]^{1/2}\,.$$

From the proof of Lemma 3.4 and the definition of $P(q,r)$, we have

$$|(z-q)_{\mathrm{N}}| \le c'd(q) \quad \text{and} \quad |(z-q)_{\mathrm{T}}| \le c'(n-1)\,d(q)^{1/2}\,.$$

Thus $d_{\mathrm{K}}(q,z) < r$ by taking $c = \min\{c', rN/(nM')\}$, i.e. $P(q,r) \subset B_{\mathrm{K}}(q,r)$.

Remark. The point q may be any point in D, provided that a suitable decomposition of x is given whenever q is away from ∂D.

LEMMA 3.6. *Let $P(q,r)$ be as in Lemma* 3.5. *Then* $|P(q,r)| \approx |B_{\mathrm{K}}(q,r)|$.

The proof is trivial.

4. The main theorem and its proof

THEOREM 4.1. *Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ and let $\varrho(z)$ be a defining function for D. For $0 < r, p < \infty$, the following conditions are equivalent*:

(i) $\|f\|_{B(D)} < \infty$;

(ii) $\sup\Big\{\Big[\int\limits_{B_{\mathrm{K}}(q,r)} |f(z) - f(q)|^p\, dw(z)/|B_{\mathrm{K}}(q,r)|\Big]^{1/p}\Big\} < \infty$;

(iii) $\sup\Big\{\Big[\int\limits_{B_{\mathrm{K}}(q,r)} |f(z) - f_{B_{\mathrm{K}}(q,r)}|^p\, dw(z)/|B_{\mathrm{K}}(q,r)|\Big]^{1/p}\Big\} < \infty$;

(iv) $\sup\{\mathrm{dist}[\bar{f}|_{B_{\mathrm{K}}(q,r)}, H^\infty(B_{\mathrm{K}}(q,r))]\} < \infty$;

(v) $\sup\{\mathrm{area}[f(B_{\mathrm{K}}(q,r))]\} < \infty$;

(vi) $\sup\Big\{\int\limits_{B_{\mathrm{K}}(q,r)} |\nabla f(z)|^p |\varrho(z)|^{p-n-1}\, dw(z)\Big\} < \infty$.

Here $B_{\mathrm{K}}(q,r)$ is a Kobayashi metric ball,

$$f_{B_{\mathrm{K}}(q,r)} = \int\limits_{B_{\mathrm{K}}(q,r)} f(z)\, dw(z)/|B_{\mathrm{K}}(q,r)|$$

and the suprema are taken over $q \in D$.

P r o o f. Since a strongly pseudoconvex domain has C^2 boundary, there exists a neighbourhood G in D such that for any $g \in G$ there exists a unique $q' \in \partial D$ which satisfies $d(q) = |q - q'|$.

From the conditions of Theorem 4.1 we can see that it is sufficient to prove this theorem for some G. Thus we shall not make any distinction between $q \in D$ and $q \in G$ in the following proof. It is worth pointing out that the constants c may be different at every occurrence. Unless otherwise stated, they will not depend on q in $B_{\mathrm{K}}(q,r)$ but may depend on r.

For any $q \in G$, let v_1 be the unit complex outward normal vector at q and let $v_2, v_3, \ldots, v_n$ be orthonormal complex tangential vectors at q. Write $z = q + \sum_{j=1}^n z_j v_j$ for all $z \in D$. We choose a positive constant $c = c(r)$ such that

$$P(q,r) = \{z \in D : |z_1| < cd(q),\ |z_j| < cd(q)^{1/2},\ 2 \le j \le n\}$$

satisfies the conditions of Lemmas 3.3 and 3.6. This can be done by the proof of the above lemmas.

Now we prove (iii)⇒(i). Without loss of generality, assume $P(q,r) \subset G$.

By Cauchy's integral formula, for any $z \in P(q,r)$ we have

$$f(z) = (2\pi)^{-n} \int\limits_{T^n} f(cd(q)x_1, cd(q)^{1/2}x_2, \ldots, cd(q)^{1/2}x_n)$$
$$\times (1 - z_1\overline{x}_1/[cd(q)])^{-1} \prod_{j=2}^{n} (1 - z_j\overline{x}_j/[cd(q)^{1/2}])^{-1}\, d\theta(x_1)\, d\theta(x_2) \ldots d\theta(x_n)\,.$$

Calculating directly and taking $z = q$, we get

$$\begin{aligned}
\frac{\partial f}{\partial z_1}(q) &= ((2\pi)^{-n}/[cd(q)]) \int\limits_{T^n} \overline{x}_1 f(cd(q)x_1, cd(q)^{1/2}x_2, \ldots, cd(q)^{1/2}x_n) \\
&\qquad \times d\theta(x_1)\, d\theta(x_2) \ldots d\theta(x_n) \\
&= cd(q)^{-1} \int\limits_0^{2\pi} \Big[d(q)^{-2} \int\limits_0^{cd(q)} r\, dr\Big] \int\limits_0^{2\pi} \Big[d(q)^{-1} \int\limits_0^{cd(q)^{1/2}} r\, dr\Big] \ldots \\
&\qquad \times \int\limits_0^{2\pi} \Big[d(q)^{-1} \int\limits_0^{cd(q)^{1/2}} r\, dr\Big] \\
&\quad \times f(cd(q)x_1, cd(q)^{1/2}x_2, \ldots, cd(q)^{1/2}x_n)\, d\theta(x_1) d\theta(x_2) \ldots d\theta(x_n) \\
&= cd(q)^{-n-2} \int\limits_{|y_1| \le cd(q)} \int\limits_{|y_2| \le cd(q)^{1/2}} \ldots \int\limits_{|y_n| \le cd(q)^{1/2}} f(y_1, y_2, \ldots, y_n) \\
&\qquad \times dA(y_1)\, dA(y_2) \ldots dA(y_n)\,.
\end{aligned}$$

So Fubini's Theorem gives

$$\left|\frac{\partial f}{\partial z_1}(q)\right| \le cd(q)^{-n-2} \int\limits_{P(q,r)} |f(z)|\, dw(z)\,.$$

Similarly we have

$$\left|\frac{\partial f}{\partial z_j}(q)\right| \le cd(q)^{-n-1} d(q)^{-1/2} \int\limits_{P(q,r)} |f(z)|\, dw(z), \qquad 2 \le j \le n\,.$$

Thus

$$|\nabla f(q)| \le cd(q)^{-n-2} \int\limits_{P(q,r)} |f(z)|\, dw(z)\,.$$

By Lemmas 3.5 and 3.6,

$$|\nabla f(q)| d(q) \le c \int\limits_{B_{\mathrm{K}}(q,r)} |f(z)|\, dw(z)/|B_{\mathrm{K}}(q,r)|\,.$$

Since $|f(z)|$ is a plurisubharmornic function,

$$|f(q)| \le \Big[\int_{P(q,r)} |f(z)|^p \, dw(z)/|P(q,r)|\Big]^{1/p}$$
$$\le \Big[\int_{B_{\mathrm{K}}(q,r)} |f(z)|^p \, dw(z)/|B_{\mathrm{K}}(q,r)|\Big]^{1/p},$$

that is,

$$|\nabla f(q)| d(q) \le c \sup \Big\{ \Big[\int_{B_{\mathrm{K}}(q,r)} |f(z)|^p \, dw(z)/|B_{\mathrm{K}}(q,r)|\Big]^{1/p} \Big\}.$$

We replace $f(z)$ with $f(z) - f_{B_{\mathrm{K}}(q,r)}$ to get

$$|\nabla f(q)| d(q) \le c \sup \Big\{ \Big[\int_{B_{\mathrm{K}}(q,r)} |f(z) - f_{B_{\mathrm{K}}(q,r)}|^p \, dw(z)/|B_{\mathrm{K}}(q,r)|\Big]^{1/p} \Big\}.$$

Since $|\varrho(z)| \approx d(z)$, we have

$$|\nabla f(q) \cdot v_1||\varrho(q)| \le c|\nabla f(q)|\, d(q)$$
$$\le c \sup \Big\{ \Big[\int_{B_{\mathrm{K}}(q,r)} |f(z) - f_{B_{\mathrm{K}}(q,r)}|^p \, dw(z)/|B_{\mathrm{K}}(q,r)|\Big]^{1/p} \Big\}.$$

According to the assumption and Proposition 2.5, the quantity in (i) is bounded by a multiple of the quantity in (iii).

Let $\gamma : [0,1] \to D$ be a C^1 curve and $\gamma(0) = z,\ \gamma(1) = q$. Then

$$\int_{B_{\mathrm{K}}(q,r)} |f(z) - f(q)|^p \, dw(z)/|B_{\mathrm{K}}(q,r)|$$
$$\le \int_{B_{\mathrm{K}}(q,r)} \Big[\int_0^1 |\nabla f(\gamma(t)) \cdot \gamma'(t)| \, dt \Big] \, dw(z)/|B_{\mathrm{K}}(q,r)|$$
$$\le \|f\|^p_{B(D)} \int_{B_{\mathrm{K}}(q,r)} \Big[\int_0^1 F_{\mathrm{K}}(\gamma(t), \gamma'(t)) \, dt \Big] \, dw(\gamma(0))/|B_{\mathrm{K}}(q,r)|.$$

The infimum of the right-hand side over all C^1 curves γ as above is $\le \|f\|^p_B r^p$. This shows that the quantity in (ii) is bounded by a multiple of that in (i).

It is trivial that (ii)$\Rightarrow$(iii).

The proof that (v)$\Rightarrow$(iv) is similar to that in [1].

Suppose g is any function in $H^\infty(B_{\mathrm{K}}(q,r))$ and $f(z)$ is a holomorphic function on D. For any $z \in P(q,r)$, we can write $z = q + \sum_{j=1}^{n} z_j v_j$. Then

$$\int\limits_{\partial P(q,r)} g(z) z_j \, dz_1 \, dz_2 \dots dz_n/(z_1 z_2 \dots z_n) = 0, \qquad 1 \le j \le n\,,$$

where $\partial P(q,r)$ is the feature boundary of $P(q,r)$.

By simple calculation we obtain

$$\begin{aligned}&\int\limits_{\partial P(q,r)} f(z)\overline{z}_1 \, dz_1 \, dz_2 \dots dz_n/(z_1 z_2 \dots z_n)\\ &\qquad\qquad = c \int\limits_{|z_1|=cd(q)} f(z_1 v_1 + q)\overline{z}_1 \, dz_1/z_1 = cd(q)^2 \frac{\partial f}{\partial z_1}(q)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int\limits_{\partial P(q,r)} f(z)\overline{z}_j \, dz_1 \, dz_2 \dots dz_n/(z_1 z_2 \dots z_n)\\ &\quad = c \int\limits_{|z_j|=cd(q)^{1/2}} f(z_j v_j + q)\overline{z}_j \, dz_j/z_j = cd(q)\frac{\partial f}{\partial z_j}(q), \qquad 2 \le j \le n\,.\end{aligned}$$

Thus

$$\begin{aligned}|\nabla f(q)| &\le \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(q)\\ &\le cd(q)^{-2}\Big|\int\limits_{\partial P(q,r)} z_1(\bar f - g)\, dz_1 \, dz_2 \dots dz_n/(z_1 z_2 \dots z_n)\Big|\\ &\quad + cd(q)^{-1}\Big|\int\limits_{\partial P(q,r)} \sum_{j=2}^{n} z_j(\bar f - g)\, dz_1 \, dz_2 \dots dz_n/(z_1 z_2 \dots z_n)\Big|\\ &\le c\{d(q)^{-1} + d(q)^{-1/2}\}\|\bar f - g\|_{H^\infty(B_{\mathrm{K}}(q,r))}\\ &\le cd(q)^{-1}\|\bar f - g\|_{H^\infty(B_{\mathrm{K}}(q,r))}\,.\end{aligned}$$

Hence

$$|\nabla f(q)\cdot v_1||\varrho(q)| \le c|\nabla f(q)|\, d(q) \le c\|\bar f - g\|_{H^\infty(B_{\mathrm{K}}(q,r))}\,.$$

By Proposition 2.5 the quantity in (i) is bounded by a multiple of that in (iv).

For any z in $B_{\mathrm{K}}(q,r)$, and $\gamma : [0,1] \to D$ a curve C^1 with $\gamma(0) = z$, $\gamma(1) = q$, we have

$$|f(z) - f(g)| \le \int_0^1 |\nabla f(\gamma(t),\gamma'(t))\cdot\gamma'(t)|\, dt \le \|f\|_B \int_0^1 F(\gamma(t),\gamma'(t))\, dt\,.$$

Taking the infimum of the right-hand side over all γ, we obtain $|f(z) - f(q)| \le r\|f\|_B$, that is,

$$\sup\{\mathrm{area}[f(B_{\mathrm{K}}(q,r))]\} \le \pi r^2 \|f\|_B^2 .$$

Hence the quantity in (v) is bounded by a multiple of that in (i).

Assume (i). By Proposition 2.5,

$$\sup\{|\nabla f(z) \cdot v_1||\varrho(z)|\} < \infty .$$

From Theorem 2.1 of [2], we have

$$\sup\{|\nabla f(z) \cdot v_j||\varrho(z)|^{1/2}\} < \infty , \qquad 2 \le j \le n .$$

Since $|\varrho(z)|$ is continuous and there exists a unit vector $x \in \mathbb{C}^n$ such that $|\nabla f(z)| = |\nabla f(z) \cdot x|$, we get

$$\begin{aligned}\sup\{|\nabla f(z)||\varrho(z)|\} &= \sup\{|\nabla f(z) \cdot x||\varrho(z)|\} \\ &\le \sup\{|\nabla f(z) \cdot v_1||\varrho(z)|\} \\ &\quad + \sup\{|\varrho(z)|^{1/2}\} \sum_{j=2}^{n} \sup\{|\nabla f(z) \cdot v_j||\varrho(z)|^{1/2}\} .\end{aligned}$$

Now $\int_{B_{\mathrm{K}}(q,r)} |\varrho(z)|^{-n-1} dw(z) = c(r) < \infty$ yields

$$\begin{aligned}&\int_{B_{\mathrm{K}}(q,r)} |\nabla f(z)|^p |\varrho(z)|^{p-n-1}\, dw(z) \\ &\qquad \le \sup\{[|\nabla f(z)||\varrho(z)|]^p\} \int_{B_{\mathrm{K}}(q,r)} |\varrho(z)|^{-n-1}\, dw(z) < \infty ,\end{aligned}$$

that is,

$$\sup\Big\{ \int_{B_{\mathrm{K}}(q,r)} |\nabla f(z)|^p |\varrho(z)|^{p-n-1}\, dw(z) \Big\} < \infty .$$

Hence the quantity in (vi) is bounded by that in (i).

Since $|\nabla f(q)|$ is a plurisubharmonic function,

$$|\nabla f(q)|^{p/2} \le \int_{P(q,r)} |\nabla f(z)|^{p/2}\, dw(z)/|P(q,r)| .$$

Applying Hölder's inequality to the right-hand side, we obtain

$$\begin{aligned}|\nabla f(q)|^{p/2} &\le \Big\{ \int_{P(q,r)} |\nabla f(z)|^p |\varrho(z)|^{p-n-1}\, dw(z)/|P(q,r)| \Big\}^{1/2} \\ &\quad \times \Big\{ \int_{P(q,r)} |\varrho(z)|^{n+1-p}\, dw(z)/|P(q,r)| \Big\}^{1/2}.\end{aligned}$$

By Lemmas 3.3 and 3.4,

$$|\nabla f(q)| \le \Big\{cd(q)^{n+1-p} \int\limits_{P(q,r)} |\nabla f(z)|^p |\varrho(z)|^{p-n-1}\, dw(z)/|P(q,r)|\Big\}^{1/p}$$
$$\le cd(q)^{-1+(n+1)/p} |P(q,r)|^{-1/p} \Big\{ \int\limits_{B_{\mathrm{K}}(q,r)} |\nabla f(z)|^p |\varrho(z)|^{p-n-1} dw(z)\Big\}.$$

This completes the proof of Theorem 4.1.

If we use the weighted volume element

$$dQ_s(z) = |\varrho(z)|^{s-n-1}\, dw(z)\,,$$

with integer $s > n$, then the following result is obvious.

THEOREM 4.2. *Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ with defining function $\varrho(z)$ and $0 < r, p < \infty$. Then the following conditions are equivalent*:

(a) $\|f\|_B < \infty\,;$

(b) $\sup\Big\{\Big[\int\limits_{B_{\mathrm{K}}(q,r)} |f(z) - f(q)|^p \, dQ_s(z)/Q_s(B_{\mathrm{K}}(q,r))\Big]^{1/p}\Big\} < \infty\,;$

(c) $\sup\Big\{\Big[\int\limits_{B_{\mathrm{K}}(q,r)} |f(z) - f_{B_{\mathrm{K}}(q,r)}|^p \, dQ_s(z)/Q_s(B_{\mathrm{K}}(q,r))\Big]^{1/p}\Big\} < \infty\,;$

(d) $\sup\{\mathrm{dist}[\bar{f}|_{B_{\mathrm{K}}(q,r)}\,, H^\infty(B_{\mathrm{K}}(q,r))]\} < \infty\,;$

(e) $\sup\{\mathrm{area}[f(B_K(q,r))]\} < \infty\,;$

(f) $\sup\Big\{\int\limits_{B_{\mathrm{K}}(q,r)} |\nabla f(z)|^p |\varrho(z)|^p \, dQ_s(z)/Q_s(B_{\mathrm{K}}(q,r))\Big\} < \infty\,.$

Here $f_{B_{\mathrm{K}}(q,r)} = \int_{B_{\mathrm{K}}(q,r)} f(z)\, dQ_s(z)/Q_s(B_{\mathrm{K}}(q,r))$, and the suprema are taken over $q \in D$.

5. A characterization of the value distribution of Bloch functions. R. M. Timoney [5] gave a characterization of the value distribution of Bloch functions on the unit disk in $\mathbb{C}$. Here we give a useful characterization on some bounded domains in $\mathbb{C}^n$ and a corollary. The proof of the characterization is omitted. By the properties of the Kobayashi metric in Section 2 and the result in [5], the proof is not difficult.

THEOREM 5.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and let f be a holomorphic function on D. Suppose there exist two holomorphic mappings $h : D \to U$ and $g : U \to D$, where U is the unit disk in $\mathbb{C}$,* *satisfying* $h \circ g = \mathrm{id}$. *Let* $E \subset \mathbb{C}$. *Then the radii of all disks in* $\mathbb{C} \setminus E$ *are bounded above if and only if*

$$\sup\{|\nabla f(z) \cdot x| / H_z(x, \overline{x})^{1/2} : x \in \mathbb{C}^n \setminus \{0\},\ z \in f^{-1}(E)\} < \infty$$

implies that f *is a Bloch function on* D.

THEOREM 5.2. *Let* D *be a strongly pseudoconvex domain in* $\mathbb{C}^n$. *Let* $E \subset \mathbb{C}$, *and let* f *be a holomorphic function on* D. *If there exist two holomorphic mappings* h *and* g *as in Theorem* 1, *then the radii of all disks in* $\mathbb{C} \setminus E$ *are bounded above if and only if*

$$\sup\{|f_*(z) \cdot x| / F_{\mathrm{K}}(z, x) : x \in \mathbb{C}^n \setminus \{0\},\ z \in f^{-1}(E)\} < \infty$$

implies that f *is a Bloch function on* D.

COROLLARY 5.3. *Let* D *be a bounded symmetric domain in* $\mathbb{C}^n$, *let* $E \subset \mathbb{C}$, *and let* f *be a holomorphic function on* D. *Then the radii of all disks in* $\mathbb{C} \setminus E$ *are bounded above if and only if*

$$\sup\{|\nabla f(z) \cdot x| / H_z(x, \overline{x})^{1/2} : x \in \mathbb{C}^n \setminus \{0\},\ z \in f^{-1}(E)\} < \infty$$

implies that f *is a Bloch function on* D.

Proof. Without loss of generality, we may assume that D is a circular domain. Let $P : \mathbb{C}^n \to \mathbb{C}$, $P(z_1, \ldots, z_n) = z_1$. Then $P(D)$ is a disk. Let r denote its radius and take $h = P/r$ and $U = h(D)$. Thus U is a unit disk. For any z_1 on the boundary of U, take $x_1 = P(x)$, $g(z) = zx\overline{x}_1/|x_1|$ (for any $z \in U$ and a fixed x in the intersection of $h^{-1}(z)$ and the boundary of D). So $|x_1| = r$, and both h and g are holomorphic mappings such that $h \circ g = \mathrm{id}$. By Theorem 5.1, the proof is complete.

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