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# ON THE NUMBER OF PAIRS OF PARTITIONS OF $n$ WITHOUT COMMON SUBSUMS

BY

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**1. Introduction.** To every (unrestricted or unequal) partition $\tau$ of $n$:

$$n = n_1 + \ldots + n_t$$

we assign the multiset $\mathcal{N} = \{n_1, \ldots, n_t\}$.

Two partitions $\tau_1, \tau_2$ of the same number $n$ are said to be *additively independent* if for $\mathcal{N}_1' \subset \mathcal{N}_1$, $\mathcal{N}_2' \subset \mathcal{N}_2$ the sum of the elements of $\mathcal{N}_1'$ can be equal to the sum of the elements of $\mathcal{N}_2'$ only in the two cases $\mathcal{N}_1' = \mathcal{N}_2' = \emptyset$ (so that both sums are 0) and $\mathcal{N}_1' = \mathcal{N}_1$, $\mathcal{N}_2' = \mathcal{N}_2$ (so that both sums are $n$).

We denote by $\pi(n)$ the set of unrestricted partitions of $n$, and by $\pi'(n)$ the set of unequal partitions of $n$, and as usual we set

$$p(n) = \operatorname{card} \pi(n), \qquad q(n) = \operatorname{card} \pi'(n) .$$

Let $G(n)$ and $H(n)$ denote the number of pairs of additively independent unrestricted or unequal partitions of $n$, respectively. We shall prove

THEOREM 1. *For all integers $k$ there are coefficients $\alpha_1, \ldots, \alpha_k$ such that*

$$G(n) = 2p(n)\left(1 + \frac{\alpha_1}{\sqrt{n}} + \frac{\alpha_2}{n} + \ldots + \frac{\alpha_k}{n^{k/2}} + O\left(\frac{1}{n^{(k+1)/2}}\right)\right) \tag{1.1}$$

*with*

$$\alpha_1 = \frac{\pi}{\sqrt{6}} = 1.28\ldots, \qquad \alpha_2 = \frac{17}{12}\pi^2 - 1 = 12.98\ldots$$

*The coefficients $\alpha_3, \ldots, \alpha_{17}$ have been computed by the computer algebra system MAPLE*:

$$\alpha_3 = \frac{1}{\sqrt{6}}\left(\frac{337}{36}\pi^3 - \frac{1019}{48}\pi + \frac{3}{2\pi}\right) = 91.46\ldots,$$
$$\alpha_4 = \frac{7889}{864}\pi^4 - \frac{12115}{288}\pi^2 + \frac{509}{24} + \frac{3}{4\pi^2} = 495.53\ldots,$$

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$$\begin{array}{lll}
\alpha_5 = 10450.82\,, & \alpha_6 = 43427.98\,, & \alpha_7 = -848498.0\,, \\
\alpha_8 = 7.67 \cdot 10^7\,, & \alpha_9 = -1.897 \cdot 10^9\,, & \alpha_{10} = 4.42 \cdot 10^{10}\,, \\
\alpha_{11} = -7.28 \cdot 10^{11}\,, & \alpha_{12} = 1.23 \cdot 10^{13}\,, & \alpha_{13} = -4.04 \cdot 10^{14}\,, \\
\alpha_{14} = 2.53 \cdot 10^{16}\,, & \alpha_{15} = -1.42 \cdot 10^{18}\,, & \alpha_{16} = 6.51 \cdot 10^{19}\,, \\
\alpha_{17} = -2.53 \cdot 10^{21}\,. & &
\end{array}$$

THEOREM 2. *There exists a real number $c$ such that*

$$H(n) = cq(n)(1 + O(\log^2 n/\sqrt{n}))\,. \tag{1.2}$$

*The value of $c$ satisfies* $13.83 \le c \le 14.29$.

To any set or multiset $\mathcal{A}$ of integers, we associate

$$S(\mathcal{A}) = \sum_{a \in \mathcal{A}} a \quad \text{and} \quad |\mathcal{A}| = \operatorname{card} \mathcal{A} = \sum_{a \in \mathcal{A}} 1\,.$$

If $\tau$ is an unrestricted or unequal partition of $n$, and $\mathcal{N}$ is the set of its parts, we associate to them $\mathcal{P}(\tau) = \mathcal{P}(\mathcal{N})$, the set of non-zero subsums:

$$\mathcal{P}(\tau) = \mathcal{P}(\mathcal{N}) = \{S(\mathcal{N}') : \mathcal{N}' \ne \emptyset\,,\ \mathcal{N}' \subset \mathcal{N}\}\,.$$

We shall say that the partition $\tau$ *represents* $a$ if $a \in \mathcal{P}(\tau)$. We observe the obvious symmetry:

$$0 < x < n \text{ and } x \in \mathcal{P}(\tau) \Rightarrow n - x \in \mathcal{P}(\tau)\,,$$

and it is convenient to introduce

$$\mathcal{P}^*(\tau) = \mathcal{P}^*(\mathcal{N}) = \{x \in \mathcal{P}(\tau) : x \le n/2\}\,.$$

It is easy to see that two partitions $\tau_1$ and $\tau_2$ are additively independent if and only if

$$\mathcal{P}^*(\tau_1) \cap \mathcal{P}^*(\tau_2) = \emptyset\,.$$

Let us denote the set of positive integers by $\mathbb{N} = \{1, 2, \ldots\}$ and for $N \in \mathbb{N}$ the set of positive integers up to $N$ by $\mathbb{N}_N = \{1, \ldots, N\}$.

We say that a partition $\tau$ of $n$ is *practical* if $\mathcal{P}(\tau) = \mathbb{N}_n$. It has been proved that almost all unrestricted partitions of $n$ are practical (cf. [10], or [5]), that is, the number $\widetilde{p}(n)$ of practical unrestricted partitions of $n$ satisfies

$$\widetilde{p}(n) \sim p(n)\,.$$

This is no longer true for unequal partitions, because $1 \notin \mathcal{P}(\tau)$ for about half of the partitions.

Let us calculate $G(7)$. We have $p(7) = 15$, $\widetilde{p}(7) = 8$.

| $\sigma_i$ | $\mathcal{P}^*(\sigma_i)$ | $\{j : \mathcal{P}^*(\sigma_i) \cap \mathcal{P}^*(\sigma_j) = \emptyset\}$ | number of $j$'s |
|---|---|---|---|
| $\sigma_1 = 7$ | $\emptyset$ | 1,...,15 | 15 |
| $\sigma_2 = 6+1$ | 1 | 1,3,5,9 | 4 |
| $\sigma_3 = 5+2$ | 2 | 1,2,5,8 | 4 |
| $\sigma_4 = 5+1+1$ | 1,2 | 1,5 | 2 |
| $\sigma_5 = 4+3$ | 3 | 1,2,3,4 | 4 |
| $\sigma_6 = 4+2+1$ | 1,2,3 | 1 | 1 |
| $\sigma_7 = 4+1+1+1$ | 1,2,3 | 1 | 1 |
| $\sigma_8 = 3+3+1$ | 1,3 | 1,3 | 2 |
| $\sigma_9 = 3+2+2$ | 2,3 | 1,2 | 2 |
| $\sigma_{10} = 3+2+1+1$ | 1,2,3 | 1 | 1 |
| $\sigma_{11} = 3+1+1+1+1$ | 1,2,3 | 1 | 1 |
| $\sigma_{12} = 2+2+2+1$ | 1,2,3 | 1 | 1 |
| $\sigma_{13} = 2+2+1+1+1$ | 1,2,3 | 1 | 1 |
| $\sigma_{14} = 2+1+1+1+1+1$ | 1,2,3 | 1 | 1 |
| $\sigma_{15} = 1+1+1+1+1+1+1$ | 1,2,3 | 1 | 1 |
| | | Total: | 41 |

We see that $G(7) = 41$. It is clear that $G(n)$ is always an odd number.

From the above table it can be guessed that the main contribution to $G(n)$ is given by the pairs $(\tau', \tau'')$ such that either $\tau'$ or $\tau''$ is the partition with only one part. The number of such couples is $2p(n) - 1$, and this explains the first term of (1.1).

A table of $G(n)$ for $n \leq 70$ is given at the end of the paper. As the other tables appearing in this paper, it has been calculated by M. Deléglise, and we are pleased to thank him very much. The method used to compute $G(n)$ is a double back-tracking. It is not clear from this table that $G(n)/p(n)$ tends to 2, but this phenomenon can be explained by the large size of the coefficients $\alpha_i$ in (1.1).

One of the main tools in the proof of both Theorems 1 and 2 is Lemma 3 below. It is a result in additive number theory which has already been used in [8] and [4] (cf. also [12] and [13]).

Let $P$ be a non-empty subset of $\mathbb{N}$, and $a = \max P$. We define $\mathcal{R}(n, P)$ as the set of unrestricted partitions $\tau$ of $n$ such that $\mathcal{P}(\tau) \cap P = \emptyset$ and we set $R(n, P) = |\mathcal{R}(n, P)|$. This quantity has been extensively studied by J. Dixmier, and we shall use the following results:

There exist three integers $d(P), \varphi(P)$ and $u(P)$, and a polynomial

$$f(P; X) = \sum_{i=0}^{d(P)} a_i X^i \in \mathbb{Z}[X] \tag{1.3}$$

such that ([1], 4.2)

$$R(n,P)=\sum_{i=0}^{d(P)} a_i p(n-i). \tag{1.4}$$

For $P$ fixed and $n\to\infty$ we have (cf. [1], 4.8)

$$R(n,P)\sim p(n)(\pi/\sqrt{6n})^{\varphi(P)}u(P)\,, \tag{1.5}$$

$$[a/2]+1\le\varphi(P)\le a \tag{1.6}$$

where $[x]$ denotes the integral part of $x$.

In [1], 4.10, an algorithm to calculate $f(P;X)$ is given, and at the end of [1] a table of $f(P;X)$ is given for all $P\subset\mathbb{N}_5$. Some more information about $f(P;X)$ can be found in [2].

We denote $R(n,\{a\})$ by $R(n,a)$. For $a$ fixed and $n\to\infty$ we have (cf. [1], 4.22)

$$R(n,a)\sim p(n)(\pi/\sqrt{6n})^{\varphi(a)}u(a)\,, \tag{1.7}$$

$$\varphi(a)=[a/2]+1\,. \tag{1.8}$$

Various estimates for $u(a)$ can be found in [1] and [3], but we shall not use them here.

Clearly, as $a=\max P\in P$, we have

$$R(n,P)\le R(n,a). \tag{1.9}$$

The number of unequal partitions of $n$ which do not represent any element of $P$ will be denoted by $Q(n,P)$, and we shall write $Q(n,a)$ instead of $Q(n,\{a\})$. Obviously, as in (1.9) we have

$$Q(n,P)\le Q(n,a)\quad \text{for } a=\max P\,. \tag{1.10}$$

We shall need the following results, valid if $a$ goes to infinity with $n$: there exists $\varepsilon_0>0$ such that uniformly for $1\le a\le\varepsilon_0\sqrt{n}$ we have (cf. [7], Theorem 1)

$$\log\frac{R(n,a)}{p(n)}\le\varphi(a)\log\frac{\pi a}{\sqrt{6n}}+O(1/\sqrt{n})\,, \tag{1.11}$$

$$Q(n,a)\le q(n)\exp\left(-a\log\frac{2}{\sqrt3}+\pi\frac{a^2}{8\sqrt{3n}}+O(1)\right). \tag{1.12}$$

For $a=a(n)$ such that $\sqrt{n}\log n\le a\le n-\sqrt{n}\log n$ we have uniformly (this is a consequence of Theorems 1 and 2 of [8])

$$R(n,a)=p([n/2])^{1+o(1)}\,, \tag{1.13}$$

$$Q(n,a)=q([n/2])^{1+o(1)}\,. \tag{1.14}$$

For all $\varepsilon > 0$ there exists $\delta < 1$ such that for all $n \geq 1$ and $a$ with $\varepsilon\sqrt{n} \leq a \leq n - \varepsilon\sqrt{n}$ we have (cf. [4], 2.1)

$$R(n,a) \leq p(n)^{\delta}\,, \tag{1.15}$$

$$Q(n,a) \leq q(n)^{\delta}\,. \tag{1.16}$$

In fact, (1.16) is not proved in [4]; but, starting with (1.12), (1.16) can be proved in the same way as (1.15).

From the famous result of Hardy and Ramanujan (cf. [11]) we know that

$$p(n) \sim \frac{1}{4\sqrt{3}\,n}\exp(C\sqrt{n}) \tag{1.17}$$

where $C = \pi\sqrt{2/3} = 2.56\ldots$ (throughout the paper). As explained in [5], it is also possible to deduce from the result of Hardy and Ramanujan an asymptotic expansion for $p(n-\mu)/p(n)$, where $\mu$ is a fixed integer:

$$\frac{p(n-\mu)}{p(n)} = 1 + \sum_{i=1}^{k} \beta_i\, n^{-i/2} + O(n^{-(k+1)/2}). \tag{1.18}$$

The first values of $\beta_i$ are

$$\begin{aligned} &\beta_1 = -\frac{C\mu}{2}\,, \qquad \beta_2 = \mu + \frac{C^2\mu^2}{8}\,,\\ &\beta_3 = -\frac{C^3\mu^3}{48} - \frac{5}{8}C\mu^2 - \frac{1}{96}\frac{(48+C^2)\mu}{C}\,. \end{aligned} \tag{1.19}$$

From Hardy and Ramanujan we also know that

$$q(n) \sim \frac{1}{4(3n^3)^{1/4}}\exp(\pi\sqrt{n/3}). \tag{1.20}$$

It is easy to deduce from (1.20) that for $h = h(n) = o(n^{3/4})$ we have (cf. [9], Lemma 3)

$$q(n+h) \sim q(n)\exp\left(\frac{\pi h}{2\sqrt{3n}}\right). \tag{1.21}$$

We finally introduce $\rho(n,m)$, which is the number of partitions of $n$ into unequal parts $\geq m$. From ([9], Theorem 1) we know that for all $n \geq 1$ and $1 \leq m \leq n$,

$$\frac{1}{2^{m-1}}q(n) \leq \rho(n,m) \leq \frac{1}{2^{m-1}}q\left(n + \frac{m(m-1)}{2}\right). \tag{1.22}$$

Perhaps, in Theorem 2 it is possible to get an asymptotic expansion of $H(n)$ in the powers of $1/\sqrt{n}$. In a letter to J.-L. Nicolas, M. Szalay explains how to obtain such an asymptotic expansion for $\rho(n,2)$ by an analytic method. If this result can be extended to $\rho(n,m)$ for $m = O(\log^2 n)$, then our proof yields an asymptotic expansion for $H(n)$.

As in [8], we have not tried to choose optimally the parameters in the proofs of Theorems 1 and 2.

## 2. The lower bound

PROPOSITION 1. *For $h_0 < n/2$ we have*

$$G(n) \geq 2 \sum_{h=0}^{h_0} \sum_{\sigma \in \pi(h)} R(n, \mathcal{P}(\sigma)) - E(h_0) \tag{2.1}$$

*with*

$$E(h_0) \leq ((1+h_0)p(h_0))^2 \,, \tag{2.2}$$

$$E(h_0) \leq G(2h_0) \,. \tag{2.3}$$

*Similarly, for unequal partitions we have*

$$H(n) \geq 2 \sum_{h=0}^{h_0} \sum_{\sigma \in \pi'(h)} Q(n, \mathcal{P}(\sigma)) - E'(h_0) \tag{2.4}$$

*with*

$$E'(h_0) \leq ((1+h_0)q(h_0))^2 \,, \tag{2.5}$$

$$E'(h_0) \leq H(2h_0+1) \,. \tag{2.6}$$

P r o o f. We shall prove (2.1)–(2.3). The proof of (2.4)–(2.6) is the same, just considering unequal partitions instead of unrestricted ones.

To each $h$ with $0 \leq h \leq h_0$ and each partition $\sigma \in \pi(h)$ we associate a partition of $n$, say $\tau \in \pi(n)$, by adding to $\sigma$ a large element $n-h > n/2$. So, $\mathcal{P}^*(\tau) = \mathcal{P}(\sigma)$. Clearly, there are $R(n, \mathcal{P}(\sigma))$ partitions of $n$ which are additively independent of $\tau$.

Now, we may consider all the pairs $(\tau', \tau'')$ in $\pi(n)$ with either $\tau' = \tau$ and $\tau'' \in \mathcal{R}(n, \mathcal{P}(\sigma))$, or $\tau'' = \tau$ and $\tau' \in \mathcal{R}(n, \mathcal{P}(\sigma))$. The number of such pairs is the first term on the right hand side of (2.1).

But some pairs are counted twice: for instance, the pair $\tau' = n$, $\tau'' = 1 + (n-1)$, if $h_0 \geq 1$. Denote the number of such pairs by $E(h_0)$. To be counted twice, the pair $(\tau', \tau'')$ must be of the following form: there exist $h'$ and $h''$ with $0 \leq h' \leq h_0$, $0 \leq h'' \leq h_0$ and $\sigma' \in \pi(h')$, $\sigma'' \in \pi(h'')$ such that $\tau' = \sigma' + (n-h')$, $\tau'' = \sigma'' + (n-h'')$ and

$$\mathcal{P}(\sigma') \cap \mathcal{P}(\sigma'') = \emptyset. \tag{2.7}$$

If we neglect (2.7), the number of such exceptions is

$$\Big(\sum_{h=0}^{h_0} p(h)\Big)^2 \leq ((h_0+1)p(h_0))^2 \,,$$

which proves (2.2).

To prove (2.3), we associate with $\sigma'$ and $\sigma''$ two partitions $\xi'$ and $\xi''$ of $2h_0$, by adding a large element $2h_0 - h'$ or $2h_0 - h''$. We have

$$\mathcal{P}^*(\xi') = \mathcal{P}(\sigma') \tag{2.8}$$

and similarly, $\mathcal{P}^*(\xi'') = \mathcal{P}(\sigma'')$. (2.8) can be easily seen when $h' < h_0$, and it also holds when $h' = h_0$, because in this case $h_0 = h' \in \mathcal{P}(\sigma')$.

From (2.7) and (2.8) we see that $\xi'$ and $\xi''$ are two additively independent partitions of $2h_0$. Observing that to distinct $\sigma'$ correspond distinct $\xi'$ completes the proof of (2.3).

To prove (2.6), it suffices to observe that $\xi'$ and $\xi''$ belong to $\pi'(2h_0+1)$, because if $h' = h_0$ and $\sigma' = h_0$, then $\xi'$ must have unequal parts.

## 3. The upper bound

LEMMA 1. *Let $s(m)$ denote the smallest integer which does not divide $m$. If $m$ is not a divisor of* 12, *we have*

$$s(m) \le \frac{7}{\log 60} \log m \le 1.71 \log m\,. \tag{3.1}$$

Proof. This is an improvement of Lemma 1 of [8]. After Chebyshev, we define

$$\psi(x) = \sum_{p^m \le x} \log p\,,$$

and it follows from Chebyshev's results that $\psi(x)/x \ge (\log 60)/7$ for $x \ge 5$. Thus for $s(m) \ge 6$ we have

$$\log m \ge \psi(s(m)-1) = \psi(s(m)-c) \ge \frac{\log 60}{7}(s(m)-c)$$

for any $c$ such that $0 < c < 1$. Letting $c$ tend to 0 we obtain

$$s(m) \ge 6 \Rightarrow s(m) \le \frac{7}{\log 60} \log m. \tag{3.2}$$

It remains to prove the lemma for $s(m) \le 5$. But in this case (3.1) holds for $\log m \ge \frac{5}{7}\log 60$, i.e., for $m \ge 19$. Calculating $s(m)$ for $2 \le m \le 18$ completes the proof.

LEMMA 2. *Let $\tau$ be a partition of $n$. Consider the following property*:

$$\tau \text{ has at least } \frac{\sqrt{n}}{100} \text{ distinct parts not exceeding } 100\sqrt{n}. \tag{3.3}$$

(i) *The number of unrestricted partitions of $n$ which do not satisfy* (3.3) *is* $O(p(n)^{1/4})$.

(ii) *The number of unequal partitions of $n$ which do not satisfy* (3.3) *is* $O(q(n)^{1/4})$.

P r o o f. We shall use Lemma 4 of [8], which claims that if $Z(n,t,m)$ denotes the number of unrestricted partitions of $n$ such that at most $t$ distinct parts not exceeding $m$ may occur, and $t < m/2$, then

$$Z(n,t,m) \le 6tn^2 \binom{m}{t} p(n,t)p(n,n/m).$$

Now, if $m = [100\sqrt{n}]$ and $t = [\sqrt{n}/100]$, then

$$\binom{m}{t} \le \frac{m^t}{t!} \le \frac{m^t}{t^t e^{-t}} = \left(\frac{me}{t}\right)^t \le \exp\left(\frac{\sqrt{n}}{100}\log(10000e)\right) \le \exp(0.11\sqrt{n}).$$

By Lemma 2 of [8], $p(n, \sqrt{n}/100) \le \exp(0.12\sqrt{n})$, and therefore, for $n$ large enough,

$$Z(n, \sqrt{n}/100,\ 100\sqrt{n}) \le \exp(0.4\sqrt{n})\,,$$

which together with (1.17) proves (i).

To prove (ii), we observe that a partition of $n$ which does not satisfy (3.3) has at most $\sqrt{n}/100$ parts smaller than $100\sqrt{n}$, and also at most $\sqrt{n}/100$ parts bigger than $100\sqrt{n}$. Such a partition has at most $2\sqrt{n}/100$ parts, and their number is bounded by $p(n, \sqrt{n}/50) \le \exp(0.21\sqrt{n})$ by Lemma 2 of [8], which, in view of (1.20), completes the proof of (ii).

LEMMA 3. *Let* $N \in \mathbb{N}$, $N \ge 2500$, *and* $m \in \mathbb{N}$ *satisfy*

(3.4) $$7Ns(m) \le m \le 10^3 N^2 s(m)^{-2}\,,$$

*and suppose that*

(3.5) $$\mathcal{A} \subset \mathbb{N}_N\,,$$

(3.6) $$|\mathcal{A}| \ge 10^4 N/s(m)\,.$$

*Then* $m \in \mathcal{P}(\mathcal{A})$.

P r o o f. This is Lemma 10 of [8].

LEMMA 4. *Let* $m$ *and* $n$ *be two positive integers*, $n \ge 2$, *such that*

(3.7) $$1500\sqrt{n}\log n \le m \le 2.10^6 n(\log n)^{-2}\,,$$

(3.8) $$s(m) > 10^8\,,$$

*and suppose that*

(3.9) $$\mathcal{A} \subset \mathbb{N}_{[100\sqrt{n}]}\,,$$

(3.10) $$|\mathcal{A}| \ge \sqrt{n}/100\,.$$

*Then* $m \in \mathcal{P}(\mathcal{A})$.

P r o o f. We set $N = [100\sqrt{n}\,]$, so we have $99\sqrt{n} \le N \le 100\sqrt{n}$. We apply Lemma 3: (3.5) comes from (3.9), and (3.6) follows from (3.8) and (3.10). Moreover, from (3.2) and (3.8) we have $\log m > \frac{1}{2}\cdot 10^8$, and (3.7) implies $n \ge m$.

Further, by Lemma 1,

$$\frac{10^3N^2}{s(m)^2} \geq \frac{10^3(9801)n}{(2\log m)^2} \geq 2\cdot 10^6\frac{n}{(\log n)^2} \geq m$$

and

$$7Ns(m) \leq 700\sqrt{n}(2\log m) < 1500\sqrt{n}\log n \leq m\,,$$

so that (3.4) holds, and Lemma 3 yields $m \in \mathcal{P}(\mathcal{A})$.

LEMMA 5. *If $d \in \mathbb{N}$ and $n_1,\ldots,n_d$ are integers, then there is a sum of the form $n_{i_1}+\ldots+n_{i_t}$ $(1 \leq i_1 < \ldots < i_t \leq d)$ such that $d$ divides $n_{i_1}+\ldots+n_{i_t}$.*

Proof. Apply the pigeon-hole principle to the $d+1$ sums $\sum_{i=0}^{k} n_i$, for $0 \leq k \leq d$, modulo $d$.

LEMMA 6. *Let $D = \exp(\psi(10^8))$ denote the least common multiple of the numbers $1,2,\ldots,10^8$. Assume that $\tau_1$ and $\tau_2$ are two additively independent partitions of $n \geq 3$, and $\tau_1$ satisfies* (3.3). *To $\tau_1$ and $\tau_2$ assign the multisets $\mathcal{N}_1$ and $\mathcal{N}_2$ of their parts. Then $\mathcal{N}_2$ can be written as*

$$\mathcal{N}_2 = \mathcal{N}' \cup \mathcal{N}'' \tag{3.11}$$

*where*

$$S(\mathcal{N}') < 1500\sqrt{n}\log n \tag{3.12}$$

*(or $\mathcal{N}' = \emptyset$) and*

$$|\mathcal{N}''| < D(\log n)^2. \tag{3.13}$$

Proof. Let $u$ denote the greatest integer such that

$$D|u\,, \tag{3.14}$$

$$u \leq 2\cdot 10^6 n(\log n)^{-2}\,, \tag{3.15}$$

and there is a subset $\mathcal{N}^* \subset \mathcal{N}_2$ with

$$S(\mathcal{N}^*) = u; \tag{3.16}$$

if such an integer does not exist, then we write $u = 0$. We are going to show that $\mathcal{N}' = \mathcal{N}^*$ (if $u = 0$, then $\mathcal{N}' = \emptyset$) and $\mathcal{N}'' = \mathcal{N}_2\backslash\mathcal{N}'$ satisfy (3.11)–(3.13).

Here and everywhere the occurring subsets may also contain multiple elements; e.g., the multiplicity of $m$ in $\mathcal{N}\backslash\mathcal{N}'$ is the multiplicity in $\mathcal{N}$ minus the multiplicity in $\mathcal{N}'$.

By (3.16), $\mathcal{N}_2$ represents $u$. This implies that

$$S(\mathcal{N}') < 1500\sqrt{n}\log n \tag{3.17}$$

since otherwise, by Lemma 4 (with $u$ in place of $m$, and $\mathcal{A}$ being the subset of $\mathcal{N}_1$ consisting of elements smaller than $100\sqrt{n}$), it would follow from (3.14) and (3.15) that also $\tau_1$ represents $u$, and this is impossible since $\tau_1$ and $\tau_2$ are additively independent.

(3.11) holds trivially, while (3.12) holds by (3.17). To show that also (3.13) holds, assume indirectly that

$$|\mathcal{N}''| = |\mathcal{N}_2 \backslash \mathcal{N}'| \ge D(\log n)^2 . \tag{3.18}$$

Write $\mathcal{N}'' = \{n''_1, \dots, n''_v\}$ where $n''_1 \le \dots \le n''_v$ and by (3.18), $v = |\mathcal{N}''| \ge D(\log n)^2$. Thus we have

$$\begin{aligned}(n''_1 + \dots + n''_D)[\log n]^2 &\le \sum_{j=0}^{[\log n]^2-1} \sum_{i=1}^{D} n''_{jD+i} \\ &\le n''_1 + \dots + n''_v = S(\mathcal{N}'') \le S(\mathcal{N}_2) = n .\end{aligned} \tag{3.19}$$

By Lemma 5, there is a (non-empty) subset $\mathcal{N}_3 \subset \{n''_1, \dots, n''_D\}$ such that

$$D | S(\mathcal{N}_3). \tag{3.20}$$

Then writing $u' = u + S(\mathcal{N}_3)$, by (3.14) and (3.20) we have

$$D | (u + S(\mathcal{N}_3)) = u' . \tag{3.21}$$

Furthermore, it follows from (3.17) and (3.19) that for $n \ge 3$,

$$\begin{aligned} u < u' = u + S(\mathcal{N}_3) = S(\mathcal{N}') + S(\mathcal{N}_3) \le S(\mathcal{N}') + n''_1 + \dots + n''_D \\ \le 1500\sqrt{n}\,\log n + n[\log n]^{-2} < 2 \cdot 10^6 n(\log n)^{-2} .\end{aligned} \tag{3.22}$$

(3.21) and (3.22) contradict the maximum property of $u$, which proves (3.13) and completes the proof of the lemma.

LEMMA 7. *For all $\varepsilon > 0$ the number of pairs $(\tau_1, \tau_2)$ of partitions of $n$ such that $\tau_1$ satisfies* (3.3), *$\tau_1$ and $\tau_2$ are additively independent and the greatest part of $\tau_2$ is less than $n - 3000\sqrt{n}\,\log n$ is*

(i) *$O_\varepsilon(p(n)^{1/\sqrt{2}+\varepsilon})$ for unrestricted partitions*,
(ii) *$O_\varepsilon(q(n)^{1/\sqrt{2}+\varepsilon})$ for unequal partitions.*

Proof. We shall prove (i); for unequal partitions, everything goes in the same way. To the couple $(\tau_1, \tau_2)$ we assign the sets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}', \mathcal{N}''$ satisfying the conditions in Lemma 6. Write $\mathcal{N}'' = \{n''_1, \dots, n''_v\}$ where $n''_1 \le \dots \le n''_v$, and define $x$ by

$$\begin{aligned} S(\mathcal{N}') + (n''_1 + \dots + n''_{x-1}) < 1500\sqrt{n}\log n \\ \le S(\mathcal{N}') + (n''_1 + \dots + n''_x)\end{aligned} \tag{3.23}$$

(and $x = 1$ for $1500\sqrt{n}\,\log n \le S(\mathcal{N}') + n''_1$). Let

$$a = S(\mathcal{N}') + n''_1 + \dots + n''_x. \tag{3.24}$$

By (3.23) and (3.24) we have

$$\begin{aligned}(3.25)\qquad 1500\sqrt{n}\,\log n < a &= S(\mathcal{N}') + (n''_1 + \ldots + n''_{x-1}) + n''_x \\ &< 1500\sqrt{n}\,\log n + (n - 3000\sqrt{n}\,\log n) = n - 1500\sqrt{n}\,\log n\,.\end{aligned}$$

By (3.24) $\tau_2$ represents $a$. Since $\tau_1$ and $\tau_2$ are additively independent, $\tau_1$ does not represent $a$, so that, for fixed $a$, it can be chosen in at most $R(n,a)$ ways.

Writing $k = S(\mathcal{N}')$, by (3.12) we have

$$(3.26)\qquad k < 1500\sqrt{n}\,\log n$$

and clearly $\mathcal{N}'$ can be chosen in at most $p(k)$ ways. Finally, all the elements of $\mathcal{N}''$ can be chosen in at most $n$ ways, and by (3.13) their number is at most $D(\log n)^2$, so that $\mathcal{N}''$ can be chosen in at most

$$n^{D(\log n)^2} = \exp(D(\log n)^3)$$

ways.

Collecting the results above, and summing over the $a$'s in (3.25) and $k$'s in (3.26), we find that the number of pairs considered in this lemma is at most

$$\Big(\sum_{1500\sqrt{n}\,\log n < a < n - 1500\sqrt{n}\,\log n} R(n,a)\Big)\Big(\sum_{k=1}^{1500\sqrt{n}\,\log n} p(k)\Big)\exp(D(\log n)^3)\,,$$

and by (1.13) and (1.17) this quantity is $O(p(n)^{1/\sqrt{2}+\varepsilon})$ for all $\varepsilon > 0$.

To prove (ii) we just have to replace $p(k)$ by $q(k)$, $R(n,a)$ by $Q(n,a)$ and to use (1.14) and (1.20) instead of (1.13) and (1.17).

PROPOSITION 2. *For all $\varepsilon > 0$ we have*

$$\text{(i)}\qquad G(n) = 2\sum_{h=0}^{3000\sqrt{n}\,\log n}\ \sum_{\sigma\in\pi(h)} R(n,\mathcal{P}(\sigma)) + O_\varepsilon(p(n)^{1/\sqrt{2}+\varepsilon})\,,$$

$$\text{(ii)}\qquad H(n) = 2\sum_{h=0}^{3000\sqrt{n}\,\log n}\ \sum_{\sigma\in\pi'(h)} Q(n,\mathcal{P}(\sigma)) + O_\varepsilon(q(n)^{1/\sqrt{2}+\varepsilon})\,.$$

Proof. First we observe that the lower bounds for $G(n)$ and $H(n)$ follow from Proposition 1, and more precisely from (2.1), (2.2) and (1.17), and respectively from (2.4), (2.5) and (1.20), with $h_0 = [3000\sqrt{n}\,\log n]$.

For the upper bounds, we notice that if $(\tau_1,\tau_2)$ is a pair of additively independent partitions of $n$, at least one of the following 5 cases must occur:

Case 1: Neither $\tau_1$ nor $\tau_2$ satisfies (3.3). From Lemma 2, the number of such pairs is $O(p(n)^{1/2})$ for unrestricted partitions, and $O(q(n)^{1/2})$ for unequal partitions.

Case 2: $\tau_1$ satisfies (3.3), and the greatest element of $\tau_2$ is less than $n-3000\sqrt{n}\,\log n$. By Lemma 7, the number of such pairs is $O(p(n)^{1/\sqrt{2}+\varepsilon})$ resp. $O(q(n)^{1/\sqrt{2}+\varepsilon})$.

Case 3: $\tau_2$ satisfies (3.3) and the greatest element of $\tau_1$ is less than $n-3000\sqrt{n}\,\log n$. The number of such pairs is the same as for Case 2.

Case 4: The greatest element $t$ of $\tau_2$ is greater than or equal to $n-3000\sqrt{n}\,\log n$, so that

$$h \stackrel{\text{def}}{=} n-t \le 3000\sqrt{n}\,\log n\,.$$

For a fixed $h$, $\tau_2$ is the union of a large part $t=n-h$ and of a partition $\sigma\in\pi(h)$. So $\tau_1$ can be chosen in at most $R(n,\mathcal{P}(\sigma))$ ways. The number of such pairs is at most

$$\sum_{h=0}^{3000\sqrt{n}\,\log n}\ \sum_{\sigma\in\pi(h)} R(n,\mathcal{P}(\sigma))\,.$$

Case 5: The greatest element of $\tau_1$ is greater than or equal to $n-3000\sqrt{n}\,\log n$. The number of pairs is the same as for Case 4.

For Cases 4 or 5 we have only considered unrestricted partitions. For unequal partitions, replace $\pi(h)$ by $\pi'(h)$ and $R(n,\mathcal{P}(\sigma))$ by $Q(n,\mathcal{P}(\sigma))$.

**4. Proof of Theorem 1.** To obtain an asymptotic expansion of order $k$, we shall split the summation over $h$ appearing in Proposition 2(i) into four subsums:

- $S_1$ where $h$ runs between 0 and $h_0 \stackrel{\text{def}}{=} 2k-1$,
- $S_2$ where $h$ runs between $h_0+1=2k$ and $h_0+4$,
- $S_3$ where $h$ runs between $h_0+5$ and $\varepsilon\sqrt{n}$ with $\varepsilon$ smaller than $\varepsilon_0$ considered in (1.11),
- $S_4$ where $h$ runs between $\varepsilon\sqrt{n}$ and $3000\sqrt{n}\,\log n$.

We have to prove that $S_2+S_3+S_4=O(p(n)n^{-(k+1)/2})$.

First, it follows from (1.9) that

$$\sum_{\sigma\in\pi(h)} R(n,\mathcal{P}(\sigma)) \le p(h)R(n,h).$$

To estimate $S_4$, we use (1.15). So, for $n$ large enough,

$$S_4=\sum_{h=[\varepsilon\sqrt{n}]+1}^{3000\sqrt{n}\,\log n} p(h)R(n,h) \le n\,p([3000\sqrt{n}\,\log n])p(n)^{\delta}$$

and $S_4 = O(p(n)n^{-(k+1)/2})$ for all $k$, by (1.17). For $S_3$, we use (1.11):

$$R(n,h) = O\biggl(p(n)\exp\left(\frac{h}{2}\log\frac{\pi h}{\sqrt{6n}}\right)\biggr).$$

Now, from (1.17) we have $p(h) = O(\exp(C\sqrt{h}))$ so that

$$p(h)R(n,h) = O(p(n)\exp g(h,n))$$

where

$$g(h,n) = C\sqrt{h} + \frac{h}{2}\log\frac{\pi h}{\sqrt{6n}}\,.$$

We notice that for $\varepsilon$ small enough and $n$ fixed, $g(h,n)$ is decreasing in $h$ for $h \le \varepsilon\sqrt{n}$, therefore

$$S_3 \le \sum_{h=h_0+5}^{\varepsilon\sqrt{n}} p(h)R(n,h) = O(\varepsilon\sqrt{n}\,p(n)\exp\, g(h_0+5,n))$$

and so $S_3 = O(p(n)n^{-(k+1)/2})$. The same estimate for $S_2$ follows easily from (1.7).

It remains to obtain an asymptotic expansion of $S_1$, of order $k$. By using the polynomials introduced in (1.3) and setting

$$F(X) = \sum_{h=0}^{h_0}\sum_{\sigma\in\pi(h)} f(\mathcal{P}(\sigma);X) = \sum_{i=0}^{d'} b_i X^i\,,$$

we obtain by (1.4)

$$S_1 = \sum_{h=0}^{h_0}\sum_{\sigma\in\pi(h)} R(n,\mathcal{P}(\sigma)) = \sum_{i=0}^{d'} b_i p(n-i)$$

and by (1.18), each $p(n-i)/p(n)$ can be expanded in the powers of $1/\sqrt{n}$.

For instance, we choose $k=2$, $h_0=3$. From Dixmier's table (cf. [1]) we have

| $h$ | $\sigma$ | $\mathcal{P}(\sigma)$ | $f(\mathcal{P}(\sigma);X)$ |
|---|---|---|---|
| 0 | | $\emptyset$ | $1$ |
| 1 | 1 | $\{1\}$ | $1-X$ |
| 2 | 2 | $\{2\}$ | $1-2X^2+X^4$ |
| 2 | $1+1$ | $\{1,2\}$ | $1-X-X^2+X^3$ |
| 3 | 3 | $\{3\}$ | $1-3X^3+X^5+2X^6-X^8$ |
| 3 | $2+1$ | $\{1,2,3\}$ | $1-X-X^2+X^4+X^5-X^6$ |
| 3 | $1+1+1$ | $\{1,2,3\}$ | $1-X-X^2+X^4+X^5-X^6$ |

$$F(X) = 7-4X-5X^2-2X^3+3X^4+3X^5-X^8 = \sum_{\mu=0}^{8} b_\mu X^\mu\,.$$

Then we have $S_1 = \sum_{\mu=0}^{8} b_\mu\, p(n-\mu)$, and by (1.18) and (1.19)

$$S_1 = p(n)\biggl(s_0 - \frac{C}{2\sqrt{n}}s_1 + \biggl(s_1 + \frac{C^2}{8}s_2\biggr)\frac{1}{n} + O\biggl(\frac{1}{n^{3/2}}\biggr)\biggr)$$

with

$$s_0 = \sum_{\mu=0}^{8} b_\mu = 1\,, \quad s_1 = \sum_{\mu=0}^{8} \mu b_\mu = -1\,, \quad s_2 = \sum_{\mu=0}^{8} \mu^2 b_\mu = 17\,.$$

**5. Proof of Theorem 2.** Here we split the summation over $h$ appearing in Proposition 2(ii) in the following way:

$S_1$ with $0 \le h \le 5\log n$,
$S_2$ with $5\log n < h \le 10\log n$,
$S_3$ with $10\log n < h \le \varepsilon\sqrt{n}$ where $\varepsilon$ is small enough, and
$S_4$ with $\varepsilon\sqrt{n} < h \le 3000\sqrt{n}\,\log n$.

First, by (1.10), we notice that

$$\sum_{\sigma\in\pi'(h)} Q(n, \mathcal{P}(\sigma)) \le q(h)Q(n,h).$$

Then we obtain an upper bound for $S_4$ and $S_3$ by using (1.14) and (1.12) in the same way as in Section 4:

$$S_3 + S_4 = O(q(n)/\sqrt{n}). \tag{5.1}$$

Now, for $5\log n < h \le 10\log n$, (1.12) yields

$$Q(n,h) = O(q(n)\exp(-h\log(2/\sqrt{3}))$$

and (1.20) yields

$$q(h) = O(\exp(0.03h))\,, \tag{5.2}$$

so that $q(h)Q(n,h) = O(q(n)\exp(-h/10))$ and

$$S_2 = O(q(n)\log n/\sqrt{n}). \tag{5.3}$$

To deal with $S_1$, we have to introduce a new definition.

For $P \subset \mathbb{N}$ and $h \ge 1$ let $\mathcal{W}(h,P)$ denote the set of subsets $\mathcal{A} \subset \{1,\dots,h\}$ such that $\mathcal{P}(\mathcal{A}) \cap P = \emptyset$. Then

$$Q(n,P) = \sum_{\mathcal{A}\subset\mathcal{W}(h,P),\, S(\mathcal{A})\le n} \rho(n - S(\mathcal{A}), h+1) \tag{5.4}$$

provided that $\max P \le h$. To prove this, we just have to distinguish for a partition of $n$ the parts smaller than $h$ and those larger than $h+1$.

With (5.4) we have

$$S_1 = q(n) + \sum_{h=1}^{5\log n} \sum_{\sigma\in\pi'(h)} \sum_{\mathcal{A}\subset\mathcal{W}(h,\mathcal{P}(\sigma))} \rho(n - S(\mathcal{A}), h+1). \tag{5.5}$$

Further, remembering that $\rho(n,m)$ is non-decreasing in $n$, observing that in (5.5) we have

$$S(\mathcal{A}) \le h(h+1)/2 = O(\log^2 n) \tag{5.6}$$

and using (1.22), we get

$$\rho(n - S(\mathcal{A}), h+1) \le \rho(n, h+1) \le \frac{1}{2^h} q\left(n + \frac{(h+1)(h+2)}{2}\right) \tag{5.7}$$

$$\begin{aligned}\rho(n - S(\mathcal{A}), h+1) &\ge \rho\left(n - \frac{h(h+1)}{2}, h+1\right)\\ &\ge \frac{1}{2^h} q\left(n - \frac{h(h+1)}{2}\right).\end{aligned} \tag{5.8}$$

Now, (5.6)–(5.8) and (1.21) give

$$\rho(n - S(\mathcal{A}), h+1) = \frac{1}{2^h} q(n)(1 + O(\log^2 n/\sqrt{n})). \tag{5.9}$$

We set $W(h,P) = |\mathcal{W}(h,P)|$ and we observe, as in the proof of Theorem 5 of [8], that if $h = \max P$, then

$$W(h,P) \le 3^{h/2}. \tag{5.10}$$

Indeed, for all $i$ with $1 \le i < h/2$ there are 3 possibilities: $i \in \mathcal{A},\ h-i \notin \mathcal{A}$ ; $i \notin \mathcal{A},\ h-i \in \mathcal{A}$ ; and $i \notin \mathcal{A},\ (h-i) \notin \mathcal{A}$.

Further, we define

$$z(h) = \sum_{\sigma\in\pi'(h)} W(h, \mathcal{P}(\sigma)) \le q(h)3^{h/2} \tag{5.11}$$

and $z(0) = 1$. We denote by $c/2$ the sum of the convergent series

$$c/2 = \sum_{h=0}^{\infty} z(h)2^{-h}. \tag{5.12}$$

From (5.2) and (5.11) we get

$$\sum_{h>5\log n} z(h)2^{-h} = O\Big(\sum_{h>5\log n} \exp(-h/10)\Big) = O(1/\sqrt{n})\,. \tag{5.13}$$

Finally, (5.5) and (5.9) give

$$S_1 = q(n)\Big(\sum_{h=0}^{5\log n} z(h)2^{-h}\Big)(1 + O(\log^2 n/\sqrt{n}))\,. \tag{5.14}$$

Combining this with (5.12) and (5.13) yields

$$S_1 = \frac{c}{2} q(n)(1 + O(\log^2 n/\sqrt{n})), \tag{5.15}$$

which together with Proposition 2(ii), (5.1) and (5.3) completes the proof of (1.2).

**6. Calculation of $c$.** The real number $c$ is defined by (5.12). Unfortunately, $z(h)$ is not easy to calculate, and M. Deléglise has calculated it for $h \le 40$ after a long running time of the computer.

The upper bound (5.11) is rather poor: for $h = 40$ we have $z(h)3^{-h/2} = 0.266$ and $q(h) = 1113$. So, we need an improved upper bound for $z(h)$.

LEMMA 8. *Let $\sigma \in \pi'(h)$ and $j = \operatorname{card} \mathcal{P}(\sigma)$. We have*

$$W(h, \mathcal{P}(\sigma)) \le \tfrac{2}{3} \cdot 3^{(h-j+1)/2}. \tag{6.1}$$

Proof. First recall that if $x \in \mathcal{P}(\sigma)$, then $h - x \in \mathcal{P}(\sigma)$. We shall consider 3 cases:

Case 1: $h$ is odd. By the above remark, $j$ must be odd. Write $\mathcal{P}(\sigma) = \{x_1, \ldots, x_j\}$ with $x_1 < \ldots < x_{(j-1)/2} \le (h-1)/2$ and $x_i = h - x_{j-i}$ for $i > (j-1)/2$.

How to choose $\mathcal{A} \in \mathcal{W}(h, \mathcal{P}(\sigma))$? For all $i$ with $1 \le i \le j$, $x_i \in \mathcal{A}$ is impossible. For the $\frac{1}{2}(h-1) - \frac{1}{2}(j-1) = \frac{1}{2}(h-j)$ $x$'s up to $\frac{1}{2}(h-1)$ and $\neq x_i$ we have at most 3 possibilities: $x \in \mathcal{A}$ and $h - x \notin \mathcal{A}$; $x \in \mathcal{A}$ and $h - x \in \mathcal{A}$; $x \notin \mathcal{A}$ and $h - x \notin \mathcal{A}$; and thus,

$$W(h, \mathcal{P}(\sigma)) \le 3^{(h-j)/2}. \tag{6.2}$$

Case 2: $h$ even, $j$ odd. Necessarily we have $h/2 \notin \mathcal{P}(\sigma)$. Now, we have $\frac{1}{2}(h-2) - \frac{1}{2}(j-1) = \frac{1}{2}(h-j-1)$ $x$'s for which there are 3 possibilities, and for $x = h/2$ we have two possibilities: $x \in \mathcal{A}$ or $x \notin \mathcal{A}$, and thus

$$W(h, \mathcal{P}(\sigma)) \le 2 \cdot 3^{(h-j-1)/2}. \tag{6.3}$$

Case 3: $h$ even, $j$ even. We have $h/2 \in \mathcal{P}(\sigma)$ and $h/2 \notin \mathcal{A}$. The number of free $x$'s is $\frac{1}{2}(h-2) - \frac{1}{2}(j-2) = \frac{1}{2}(h-j)$, and thus

$$W(h, \mathcal{P}(\sigma)) \le 3^{(h-j)/2}. \tag{6.4}$$

It remains to observe that (6.2)–(6.4) imply (6.1).

Let us introduce now for $1 \le j \le h$,

$$\rho(j) = \operatorname{card}\{\sigma \in \pi'(h) : \operatorname{card} \mathcal{P}(\sigma) = j\}. \tag{6.5}$$

It is easy to see that $\rho(1) = 1$, $\rho(2) = 0$, $\rho(3) = [(h-1)/2]$, the number of partitions with 2 parts. We define

$$\widehat{s}(h) = \sum_{\sigma\in\pi'(h)} 3^{(1-\operatorname{card}\mathcal{P}(\sigma))/2} = \sum_{j=1}^{h} \rho(j)3^{-(j-1)/2}. \tag{6.6}$$

It follows from (5.11), (6.6), and Lemma 8 that

$$\frac{z(h)}{2^h} \le \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^h \widehat{s}(h)\,. \tag{6.7}$$

LEMMA 9. *For all real positive $x$, we have*

$$n \le 3^{1/4}3^{x/2}x^2 \Rightarrow \widehat{s}(n) \le \frac{0.4}{\sqrt{n}}\, 3^{x^2/4}e^{2x}\,, \tag{6.8}$$

*and*

$$\widehat{s}(n) \le \frac{0.4}{\sqrt{n}} \exp\left(\frac{\log n(\log n+4)}{\log 3}\right). \tag{6.9}$$

Proof. Consider a partition of $n$ into $m$ distinct parts. We clearly have $m \le \sqrt{2n}$, and for such a partition $\sigma$, $\operatorname{card}\mathcal{P}(\sigma) \ge m(m+1)/2$. Indeed, if the parts are $a_1 < \ldots < a_m$, the subsums $a_1, \ldots, a_m, a_m + a_1, \ldots, a_m + a_{m-1}, a_m + a_{m-1} + a_1, \ldots, a_m + a_{m-1} + a_{m-2}, \ldots, a_m + a_{m-1} + \ldots + a_1 = n$ are all distinct.

Now, it is known that the number of partitions of $n$ into $m$ distinct parts is at most

$$\frac{1}{m!}\binom{n-1}{m-1} = \frac{1}{m!}\,\frac{m}{n}\binom{n}{m}.$$

Therefore, from (6.6) we have

$$\widehat{s}(n) \le \sum_{m=1}^{\sqrt{2n}} \frac{1}{m!}\,\frac{m}{n}\binom{n}{m}3^{1/2-m(m+1)/4}.$$

By Stirling's formula, $m! \ge m^m e^{-m}\sqrt{2\pi m}$, and thus,

$$\widehat{s}(n) \le \frac{\sqrt{3}}{2\pi n} \sum_{m=1}^{\sqrt{2n}} \left(\frac{ne^2}{m^2 3^{(m+1)/4}}\right)^m. \tag{6.10}$$

Now, we set

$$y = x\left(\log(ne^2) - \frac{x+1}{4}\log 3 - 2\log x\right),$$
$$y' = \log n - \frac{\log 3}{4} - \frac{x}{2}\log 3 - 2\log x\,.$$

The derivative $y'$ vanishes at $x_0 = x_0(n)$. It is easy to see that

$$x_0 < \frac{2\log n}{\log 3}\,, \tag{6.11}$$

$$x_0(n) \text{ is increasing in } n\,, \tag{6.12}$$

$$n = 3^{1/4} 3^{x_0/2} x_0^2\,. \tag{6.13}$$

Further, for $x = x_0$ we have $y = y_0 = x_0(2 + \frac{1}{4}x_0 \log 3)$, so that (6.10) becomes

$$\widehat{s}(n) \le \frac{\sqrt{6}}{2\pi\sqrt{n}} 3^{x_0^2/4} e^{2x_0} \le \frac{0.4}{\sqrt{n}}\, 3^{x_0^2/4} e^{2x_0}\,. \tag{6.14}$$

Then (6.8) follows from (6.12)–(6.14), while (6.9) follows from (6.11) and (6.14).

Now, we write

$$\begin{aligned} c/2 = \sum_{h=0}^{\infty} z(h) 2^{-h} &= \sum_{0}^{40} + \sum_{41}^{100} + \sum_{101}^{137} + \sum_{138}^{512} + \sum_{513}^{\infty} \\ &= T_1 + T_2 + T_3 + T_4 + T_5\,. \end{aligned}$$

From the table at the end of the paper we have $T_1 = 6.9168\ldots$ The upper bound $T_2 \le 0.212$ has been obtained by using (6.7), and by computing the exact value of $\widehat{s}(h)$ from (6.6) (cf. table at the end of the paper). For $T_3$, we use (6.8) with $x = 3.7$, which yields

$$n \le 137 \Rightarrow \widehat{s}(n) \le 28106/\sqrt{n}\,,$$

so that by (6.7)

$$T_3 \le \frac{2}{3}\,\frac{28106}{10} \sum_{h \ge 101} (\sqrt{3}/2)^h \le 0.007\,.$$

Using (6.8) with $x = 5$ yields

$$n \le 512 \Rightarrow \widehat{s}(n) \le 8.5 \cdot 10^6/\sqrt{n}$$

and allows us to get $T_4 \le 0.009$.

For $T_5$, we use (6.9), and we observe that for $n \ge 513$,

$$\frac{\log n(\log n + 4)}{\log 3} \le 0.114n\,.$$

Therefore, by (6.7), we have

$$T_5 \le \sum_{h \ge 513} \frac{2}{3}\,\frac{0.4}{\sqrt{513}} \exp\left(\left(0.114 - \log\frac{2}{\sqrt{3}}\right)h\right) \le 4 \cdot 1\,10^{-9}\,.$$

In conclusion, $6.916 \le c/2 \le 7.145$, which completes the proof of Theorem 2.

TABLE OF $G(n)$

| $n$ | $p(n)$ | $G(n)$ | $G(n)/p(n)$ | $n$ | $p(n)$ | $G(n)$ | $G(n)/p(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1.00 | 36 | 17977 | 151371 | 8.42 |
| 2 | 2 | 3 | 1.50 | 37 | 21637 | 171675 | 7.93 |
| 3 | 3 | 5 | 1.67 | 38 | 26015 | 214973 | 8.26 |
| 4 | 5 | 11 | 2.20 | 39 | 31185 | 251205 | 8.06 |
| 5 | 7 | 15 | 2.14 | 40 | 37338 | 313449 | 8.39 |
| 6 | 11 | 33 | 3.00 | 41 | 44583 | 356495 | 8.00 |
| 7 | 15 | 41 | 2.73 | 42 | 53174 | 447651 | 8.42 |
| 8 | 22 | 77 | 3.50 | 43 | 63261 | 506113 | 8.00 |
| 9 | 30 | 105 | 3.50 | 44 | 75175 | 625341 | 8.32 |
| 10 | 42 | 173 | 4.12 | 45 | 89134 | 721223 | 8.09 |
| 11 | 56 | 215 | 3.84 | 46 | 105558 | 868565 | 8.23 |
| 12 | 77 | 381 | 4.95 | 47 | 124754 | 989791 | 7.93 |
| 13 | 101 | 449 | 4.45 | 48 | 147273 | 1222899 | 8.30 |
| 14 | 135 | 699 | 5.18 | 49 | 173525 | 1372535 | 7.91 |
| 15 | 176 | 911 | 5.18 | 50 | 204226 | 1657831 | 8.12 |
| 16 | 231 | 1335 | 5.78 | 51 | 239943 | 1890863 | 7.88 |
| 17 | 297 | 1611 | 5.42 | 52 | 281589 | 2264913 | 8.04 |
| 18 | 385 | 2433 | 6.32 | 53 | 329931 | 2550905 | 7.73 |
| 19 | 490 | 2867 | 5.85 | 54 | 386155 | 3079125 | 7.97 |
| 20 | 627 | 4179 | 6.67 | 55 | 451276 | 3457885 | 7.66 |
| 21 | 792 | 5113 | 6.46 | 56 | 526823 | 4132983 | 7.85 |
| 22 | 1002 | 6903 | 6.89 | 57 | 614154 | 4662771 | 7.59 |
| 23 | 1255 | 8251 | 6.57 | 58 | 715220 | 5488969 | 7.67 |
| 24 | 1575 | 11769 | 7.47 | 59 | 831820 | 6172705 | 7.42 |
| 25 | 1958 | 13661 | 6.98 | 60 | 966467 | 7397379 | 7.65 |
| 26 | 2436 | 18177 | 7.46 | 61 | 1121505 | 8200197 | 7.31 |
| 27 | 3010 | 22011 | 7.31 | 62 | 1300156 | 9643057 | 7.42 |
| 28 | 3718 | 28997 | 7.80 | 63 | 1505499 | 10894619 | 7.24 |
| 29 | 4565 | 33711 | 7.38 | 64 | 1741630 | 12737677 | 7.31 |
| 30 | 5604 | 45251 | 8.07 | 65 | 2012558 | 14233625 | 7.07 |
| 31 | 6842 | 51891 | 7.58 | 66 | 2323520 | 16720939 | 7.20 |
| 32 | 8349 | 67697 | 8.11 | 67 | 2679689 | 18567877 | 6.93 |
| 33 | 10143 | 79499 | 7.84 | 68 | 3087735 | 21685005 | 7.02 |
| 34 | 12310 | 100123 | 8.13 | 69 | 3554345 | 24264927 | 6.83 |
| 35 | 14883 | 117307 | 7.88 | 70 | 4087968 | 28143245 | 6.88 |

## TABLE OF $H(n)$

$r_n := H(n)/q(n)$

| $n$ | $q(n)$ | $H(n)$ | $r_n$ | $r_{n+1} - r_{n-1}$ | $n$ | $q(n)$ | $H(n)$ | $r_n$ | $r_{n+1} - r_{n-1}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1.00 | | 41 | 1260 | 54541 | 43.29 | 3.71 |
| 2 | 1 | 1 | 1.00 | 0.50 | 42 | 1426 | 64497 | 45.23 | 3.81 |
| 3 | 2 | 3 | 1.50 | 0.50 | 43 | 1610 | 75823 | 47.10 | 3.82 |
| 4 | 2 | 3 | 1.50 | 0.83 | 44 | 1816 | 89067 | 49.05 | 4.01 |
| 5 | 3 | 7 | 2.33 | 0.75 | 45 | 2048 | 104665 | 51.11 | 4.13 |
| 6 | 4 | 9 | 2.25 | 0.67 | 46 | 2304 | 122527 | 53.18 | 3.98 |
| 7 | 5 | 15 | 3.00 | 0.92 | 47 | 2590 | 142677 | 55.09 | 4.03 |
| 8 | 6 | 19 | 3.17 | 1.12 | 48 | 2910 | 166471 | 57.21 | 4.09 |
| 9 | 8 | 33 | 4.12 | 1.13 | 49 | 3264 | 193149 | 59.18 | 4.36 |
| 10 | 10 | 43 | 4.30 | 1.13 | 50 | 3658 | 225237 | 61.57 | 4.30 |
| 11 | 12 | 63 | 5.25 | 1.10 | 51 | 4097 | 260071 | 63.48 | 4.17 |
| 12 | 15 | 81 | 5.40 | 1.25 | 52 | 4582 | 301201 | 65.74 | 4.23 |
| 13 | 18 | 117 | 6.50 | 1.46 | 53 | 5120 | 346697 | 67.71 | 4.44 |
| 14 | 22 | 151 | 6.86 | 1.54 | 54 | 5718 | 401399 | 70.20 | 4.40 |
| 15 | 27 | 217 | 8.04 | 1.67 | 55 | 6378 | 459917 | 72.11 | 4.23 |
| 16 | 32 | 273 | 8.53 | 1.72 | 56 | 7108 | 529029 | 74.43 | 4.31 |
| 17 | 38 | 371 | 9.76 | 1.93 | 57 | 7917 | 604999 | 76.42 | 4.49 |
| 18 | 46 | 481 | 10.46 | 1.85 | 58 | 8808 | 695093 | 78.92 | 4.39 |
| 19 | 54 | 627 | 11.61 | 1.90 | 59 | 9792 | 791261 | 80.81 | 4.48 |
| 20 | 64 | 791 | 12.36 | 1.82 | 60 | 10880 | 906317 | 83.30 | 4.40 |
| 21 | 76 | 1021 | 13.43 | 2.06 | 61 | 12076 | 1028939 | 85.21 | 4.45 |
| 22 | 89 | 1283 | 14.42 | 2.31 | 62 | 13394 | 1175301 | 87.75 | 4.41 |
| 23 | 104 | 1637 | 15.74 | 2.23 | 63 | 14848 | 1330657 | 89.62 | 4.40 |
| 24 | 122 | 2031 | 16.65 | 2.48 | 64 | 16444 | 1515269 | 92.15 | 4.42 |
| 25 | 142 | 2587 | 18.22 | 2.70 | 65 | 18200 | 1711531 | 94.04 | 4.47 |
| 26 | 165 | 3193 | 19.35 | 2.64 | 66 | 20132 | 1945243 | 96.62 | 4.45 |
| 27 | 192 | 4005 | 20.86 | 2.64 | 67 | 22250 | 2191343 | 98.49 | 4.40 |
| 28 | 222 | 4881 | 21.99 | 2.75 | 68 | 24576 | 2482699 | 101.02 | 4.43 |
| 29 | 256 | 6043 | 23.61 | 3.13 | 69 | 27130 | 2792127 | 102.92 | 4.40 |
| 30 | 296 | 7437 | 25.12 | 2.98 | 70 | 29927 | 3157955 | 105.52 | 4.44 |
| 31 | 340 | 9041 | 26.59 | 2.94 | 71 | 32992 | 3541887 | 107.36 | 4.41 |
| 32 | 390 | 10943 | 28.06 | 3.02 | 72 | 36352 | 3996105 | 109.93 | 4.43 |
| 33 | 448 | 13265 | 29.61 | 3.23 | 73 | 40026 | 4474421 | 111.79 | 4.46 |
| 34 | 512 | 16021 | 31.29 | 3.23 | 74 | 44046 | 5038449 | 114.39 | 4.49 |
| 35 | 585 | 19213 | 32.84 | 3.19 | 75 | 48446 | 5633187 | 116.28 | 4.38 |
| 36 | 668 | 23035 | 34.48 | 3.33 | 76 | 53250 | 6324539 | 118.77 | 4.29 |
| 37 | 760 | 27487 | 36.17 | 3.50 | 77 | 58499 | 7053295 | 120.57 | 4.47 |
| 38 | 864 | 32811 | 37.98 | 3.46 | 78 | 64234 | 7916409 | 123.24 | 4.42 |
| 39 | 982 | 28921 | 39.63 | 3.54 | 79 | 70488 | 8810353 | 124.99 | 4.30 |
| 40 | 1113 | 46213 | 41.52 | 3.66 | 80 | 77312 | 9860123 | 127.54 | |

TABLE OF $z(h)$

| $h$ | $q(h)$ | $z(h)$ | $z(h)2^{-h}$ | $\sum z(h)2^{-h}$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1.000000 | 1.000000 |
| 1 | 1 | 1 | 0.500000 | 1.500000 |
| 2 | 1 | 2 | 0.500000 | 2.000000 |
| 3 | 2 | 4 | 0.500000 | 2.500000 |
| 4 | 2 | 8 | 0.500000 | 3.000000 |
| 5 | 3 | 15 | 0.468750 | 3.468750 |
| 6 | 4 | 28 | 0.437500 | 3.906250 |
| 7 | 5 | 51 | 0.398438 | 4.304688 |
| 8 | 6 | 96 | 0.375000 | 4.679688 |
| 9 | 8 | 162 | 0.316406 | 4.996094 |
| 10 | 10 | 302 | 0.294922 | 5.291016 |
| 11 | 12 | 506 | 0.247070 | 5.538086 |
| 12 | 15 | 913 | 0.222900 | 5.760986 |
| 13 | 18 | 1509 | 0.184204 | 5.945190 |
| 14 | 22 | 2722 | 0.166138 | 6.111328 |
| 15 | 27 | 4339 | 0.132416 | 6.243744 |
| 16 | 32 | 7849 | 0.119766 | 6.363510 |
| 17 | 38 | 12459 | 0.095055 | 6.458565 |
| 18 | 46 | 21887 | 0.083492 | 6.542057 |
| 19 | 54 | 34721 | 0.066225 | 6.608282 |
| 20 | 64 | 61176 | 0.058342 | 6.666624 |
| 21 | 76 | 94817 | 0.045212 | 6.711836 |
| 22 | 89 | 166763 | 0.039759 | 6.751596 |
| 23 | 104 | 258428 | 0.030807 | 6.782403 |
| 24 | 122 | 448453 | 0.026730 | 6.809133 |
| 25 | 142 | 691043 | 0.020595 | 6.829727 |
| 26 | 165 | 1199147 | 0.017869 | 6.847596 |
| 27 | 192 | 1825810 | 0.013603 | 6.861199 |
| 28 | 222 | 3175164 | 0.011828 | 6.873028 |
| 29 | 256 | 4823668 | 0.008985 | 6.882013 |
| 30 | 296 | 8245366 | 0.007679 | 6.889692 |
| 31 | 340 | 12570653 | 0.005854 | 6.895545 |
| 32 | 390 | 21611259 | 0.005032 | 6.900577 |
| 33 | 448 | 32414428 | 0.003774 | 6.904351 |
| 34 | 512 | 55627306 | 0.003238 | 6.907589 |
| 35 | 585 | 83671722 | 0.002435 | 6.910024 |
| 36 | 668 | 142505471 | 0.002074 | 6.912097 |
| 37 | 760 | 214103771 | 0.001558 | 6.913655 |
| 38 | 864 | 364227805 | 0.001325 | 6.914980 |
| 39 | 982 | 542624438 | 0.000987 | 6.915967 |
| 40 | 1113 | 926297112 | 0.000842 | 6.916809 |

TABLE OF $\widehat{s}(h)$ AND $t(h) = \frac{2}{3}(\frac{3^{h/2}}{2^h})\widehat{s}(h)$

| $h$ | $t(h)$ | $\sum_{i=41}^{h} t(i)$ | $\widehat{s}(h)$ | $h$ | $t(h)$ | $\sum_{i=41}^{h} t(i)$ | $\widehat{s}(h)$ |
|---|---|---|---|---|---|---|---|
| 41 | 0.022594 | 0.022594 | 12.34 | 71 | 0.000688 | 0.205860 | 28.11 |
| 42 | 0.020468 | 0.043062 | 12.91 | 72 | 0.000617 | 0.206478 | 29.14 |
| 43 | 0.018116 | 0.061178 | 13.19 | 73 | 0.000539 | 0.207017 | 29.39 |
| 44 | 0.016377 | 0.077555 | 13.77 | 74 | 0.000484 | 0.207501 | 30.47 |
| 45 | 0.014496 | 0.092051 | 14.08 | 75 | 0.000423 | 0.207924 | 30.71 |
| 46 | 0.013106 | 0.105156 | 14.69 | 76 | 0.000379 | 0.208303 | 31.81 |
| 47 | 0.011567 | 0.116724 | 14.98 | 77 | 0.000331 | 0.208634 | 32.05 |
| 48 | 0.010450 | 0.127174 | 15.62 | 78 | 0.000297 | 0.208931 | 33.20 |
| 49 | 0.009216 | 0.136389 | 15.91 | 79 | 0.000259 | 0.209190 | 33.44 |
| 50 | 0.008327 | 0.144716 | 16.60 | 80 | 0.000232 | 0.209422 | 34.60 |
| 51 | 0.007333 | 0.152050 | 16.88 | 81 | 0.000202 | 0.209624 | 34.84 |
| 52 | 0.006614 | 0.158664 | 17.58 | 82 | 0.000181 | 0.209805 | 36.06 |
| 53 | 0.005821 | 0.164485 | 17.86 | 83 | 0.000158 | 0.209963 | 36.28 |
| 54 | 0.005254 | 0.169739 | 18.62 | 84 | 0.000141 | 0.210105 | 37.52 |
| 55 | 0.004616 | 0.174355 | 18.89 | 85 | 0.000123 | 0.210228 | 37.76 |
| 56 | 0.004159 | 0.178514 | 19.65 | 86 | 0.000110 | 0.210339 | 39.04 |
| 57 | 0.003655 | 0.182169 | 19.94 | 87 | 0.000096 | 0.210435 | 39.26 |
| 58 | 0.003293 | 0.185462 | 20.75 | 88 | 0.000086 | 0.210521 | 40.56 |
| 59 | 0.002889 | 0.188351 | 21.01 | 89 | 0.000075 | 0.210596 | 40.79 |
| 60 | 0.002602 | 0.190952 | 21.85 | 90 | 0.000067 | 0.210663 | 42.15 |
| 61 | 0.002281 | 0.193233 | 22.12 | 91 | 0.000058 | 0.210721 | 42.36 |
| 62 | 0.002053 | 0.195287 | 23.00 | 92 | 0.000052 | 0.210773 | 43.73 |
| 63 | 0.001799 | 0.197085 | 23.26 | 93 | 0.000045 | 0.210819 | 43.95 |
| 64 | 0.001618 | 0.198703 | 24.16 | 94 | 0.000041 | 0.210859 | 45.38 |
| 65 | 0.001417 | 0.200120 | 24.43 | 95 | 0.000035 | 0.210895 | 45.59 |
| 66 | 0.001274 | 0.201394 | 25.37 | 96 | 0.000032 | 0.210926 | 47.03 |
| 67 | 0.001115 | 0.202508 | 25.62 | 97 | 0.000027 | 0.210954 | 47.24 |
| 68 | 0.001001 | 0.203510 | 26.59 | 98 | 0.000025 | 0.210978 | 48.94 |
| 69 | 0.000876 | 0.204386 | 26.85 | 99 | 0.000021 | 0.211000 | 48.94 |
| 70 | 0.000787 | 0.205173 | 27.86 | 100 | 0.000019 | 0.211019 | 50.46 |

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