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CERTAIN CURVATURE CHARACTERIZATIONS OF AFFINE HYPERSURFACES

BY

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1. Introduction. Let A^{n+1} be an $(n+1)$-dimensional, $n \geq 2$, affine space considered as a homogeneous space under the action of the unimodular affine group $\mathrm{ASL}(n+1,\mathbb{R})$. We denote by $(\widetilde{\nabla},\widetilde{\Theta})$ the natural equiaffine structure on A^{n+1}, i.e. the standard torsion-free connection $\widetilde{\nabla}$ and the volume element $\widetilde{\Theta}$ given by the determinant which is parallel with respect to this connection.

Suppose that M is a non-degenerate hypersurface in A^{n+1} with the affine normal ξ and with induced equiaffine structure (∇,Θ) (we refer to [23] and [31] for the construction of ξ and (∇,Θ)). Thus we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi\,, \tag{1.1}$$

$$\widetilde{\nabla}_X \xi = -SX \tag{1.2}$$

for all vector fields X and Y tangent to M, where h is a non-degenerate symmetric bilinear form and S a (1, 1)-tensor field on M. The tensor field S is called the *affine shape operator*. The fundamental equations of M in A^{n+1} (i.e. the equations of Gauss, Codazzi and Ricci) are (see [23], [24], [22]):

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY\,, \tag{1.3}$$

$$C(X,Y,Z) = C(Y,X,Z)\,, \tag{1.4}$$

$$(\nabla_X S)Y = (\nabla_Y S)X\,, \tag{1.5}$$

$$h(X,SY) = h(SX,Y)\,, \tag{1.6}$$

where R is the curvature tensor of ∇ defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and the $(0,3)$-tensor C is given by

$$C(X,Y,Z) = (\nabla h)(Y,Z;X) = Xh(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\,,$$

X,Y,Z begin vector fields tangent to M. The tensor field C is called the *cubic form* of M.

A non-degenerate hypersurface M in A^{n+1} is said to be an *affine hypersphere* if the equality

$$S = \lambda I, \qquad \lambda \in \mathbb{R}\,,$$

holds on M, where I denotes the identity $(1,1)$-tensor field on M. If $\lambda \neq 0$ (resp. $\lambda = 0$), then a hypersurface M is called a *proper affine hypersphere* (resp. an *improper affine hypersphere*).

The basic definitions and formulas are given in Section 2. In Section 3 we obtain some results on non-degenerate hypersurfaces M in A^{n+1} satisfying certain curvature conditions imposed on its cubic form. In Section 4 we consider a curvature condition imposed on the tensor R of M. These subjects are a continuation of the investigations presented in [1] and [31], respectively. In these sections we also consider curvature conditions imposed on the generalized curvature tensor R^* defined in [27]. Moreover, in Section 4 we consider affine-quasi-umbilical hypersurfaces M in A^{n+1}. This class of hypersurfaces was introduced in [26]. We prove (see Theorem 4.8) that such hypersurfaces in A^{n+1}, $n \geq 4$, are characterized by the vanishing on M of the Weyl curvature tensor $W(R^*)$ corresponding to the tensor R^*.

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2. Pseudosymmetry curvature conditions on affine hypersurfaces. Let M be a connected n-dimensional, $n \geq 2$, smooth Riemannian manifold with a not necessarily definite metric h. We denote by ∇, R, $\overline{R}$, $\operatorname{Ricc}(R)$ and $W(R)$ the Levi-Cività connection, the curvature tensor, the Riemann–Christoffel curvature tensor, the Ricci tensor and the Weyl conformal curvature tensor of (M,h), respectively. Denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M.

For a symmetric (0, 2)-tensor field D on M define the endomorphism $X \wedge_D Y$ of $\mathfrak{X}(M)$ by

$$(X \wedge_D Y)Z = D(Y,Z)X - D(X,Z)Y\,, \tag{2.1}$$

where $X, Y, Z \in \mathfrak{X}(M)$. The endomorphism $X \wedge_h Y$ will be denoted simply by $X \wedge Y$. Further, for a (1, 3)-tensor field B on M satisfying

$$B(X,Y,Z) = -B(Y,X,Z)\,, \tag{2.2}$$

we denote by $B(X,Y)$ the endomorphism of $\mathfrak{X}(M)$ defined by

$$B(X,Y)Z = B(X,Y,Z)\,.$$

We extend the endomorphisms $X \wedge_D Y$ and $B(X,Y)$ to derivations $(X \wedge_D Y)\cdot$ and $B(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that they commute with contractions and

$$(X \wedge_D Y)\cdot f = 0\,, \qquad B(X,Y)\cdot f = 0\,,$$

for any function f on M. Now, for any (l,k)-tensor field T on M we define the $(l,k+2)$-tensors $B\cdot T$ and $Q(D,T)$ by

$$\begin{aligned}
(B\cdot T)&(\omega^1,\dots,\omega^l,X_1,\dots,X_k;X,Y)\\
&= (B(X,Y)\cdot T)(\omega^1,\dots,\omega^l,X_1,\dots,X_k)\\
&= -\sum_{i=1}^{l} T(\omega^1,\dots,B(X,Y)\cdot\omega^i,\dots,\omega^l,X_1,\dots,X_k)\\
&\quad -\sum_{j=1}^{k} T(\omega^1,\dots,\omega^l,X_1,\dots,B(X,Y)\cdot X_j,\dots,X_k)\,,
\end{aligned}$$

$$\begin{aligned}
Q(D,T)&(\omega^1,\dots,\omega^l,X_1,\dots,X_k;X,Y)\\
&= -((X\wedge_D Y)\cdot T)(\omega^1,\dots,\omega^l,X_1,\dots,X_k)\\
&= \sum_{i=1}^{l} T(\omega^1,\dots,(X\wedge_D Y)\cdot\omega^i,\dots,\omega^l,X_1,\dots,X_k)\\
&\quad +\sum_{j=1}^{k} T(\omega^1,\dots,\omega^l,X_1,\dots,(X\wedge_D Y)\cdot X_j,\dots,X_k)
\end{aligned}$$

respectively, where $X, Y, X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\omega^1,\dots,\omega^l$ are real-valued 1-forms on M.

Two (l,k)-tensor fields T_1 and T_2 on M are *pseudosymmetric related* with respect to a $(1,3)$-tensor B satisfying (2.2) and a symmetric $(0,2)$-tensor D if

(∗) $B\cdot T_1$ and $Q(D,T_2)$ are linearly dependent at every point of M.

In the special case when $T_1 = T_2 = T$, we say that the tensor field T is *pseudosymmetric* with respect to B and D. A tensor field T on M will be called *semisymmetric* with respect to a $(1,3)$-tensor B satisfying (2.2) if $B\cdot T$ vanishes on M.

A tensor field B of type $(1,3)$ on M is said to be a *generalized curvature tensor* [21] if

$$B(X_1,X_2,X_3)+B(X_2,X_3,X_1)+B(X_3,X_1,X_2)=0\,,$$

$B(X_1,X_2,X_3)=-B(X_2,X_1,X_3), \overline{B}(X_1,X_2,X_3,X_4)=\overline{B}(X_3,X_4,X_1,X_2)$, where $\overline{B}(X_1,X_2,X_3,X_4)=h(B(X_1,X_2,X_3),X_4)$ and $X_1,\ldots,X_4\in\mathfrak{X}(M)$. The *Ricci tensor* $\mathrm{Ricc}(B)$ of B is the trace of the linear mapping $X_1\longmapsto B(X_1,X_2,X_3)$. If $n=\dim M\geq 3$ then we can define the *Weyl curvature tensor* $W(B)$ of B by

$$\begin{aligned}W(B)(X_1,\ldots,X_4)&=\overline{B}(X_1,\ldots,X_4)+\frac{K(B)}{(n-1)(n-2)}h((X_1\wedge X_2)X_3,X_4)\\&\quad-\frac{1}{n-2}(h((X_1\wedge_{\mathrm{Ricc}(B)}X_2)X_3,X_4)-h((X_1\wedge_{\mathrm{Ricc}(B)}X_2)X_4,X_3))\,,\end{aligned}$$

where $K(B)$ is the scalar curvature of B.

For a generalized curvature tensor B we define the tensor $Z(B)$ by

$$Z(B)(X_1,\ldots,X_4)=\overline{B}(X_1,\ldots,X_4)-\frac{K(B)}{n(n-1)}h((X_1\wedge X_2)X_3,X_4)\,.$$

The following proposition gives examples of pseudosymmetric related tensors.

PROPOSITION 2.1. *If B is a $(1,3)$-tensor field on a manifold (M,h) of the form*

$$B(X,Y,Z)=\overline{A}(Y,Z)AX-\overline{A}(X,Z)AY\,, \tag{2.3}$$

where A is a $(1,1)$-tensor field on M and $\overline{A}$ is the symmetric $(0,2)$-tensor field defined by $\overline{A}(X,Z)=h(X,AZ)$, then

$$B\cdot\overline{B}=Q(\mathrm{Ricc}(B),\overline{B}) \tag{2.4}$$

on M.

The above assertion is an immediate consequence of (2.3) and the definitions of $\overline{B}$, $B\cdot\overline{B}$ and $Q(\mathrm{Ricc}(B),\overline{B})$.

Manifolds satisfying (2.4) were considered in [14]. For instance, it was proved (see [14], Theorem 1) that if B is a generalized curvature tensor field on M, then (2.4) is satisfied at each point of M at which there exists a non-zero covector w satisfying

$$\begin{aligned}w(X_1)B(X_2,X_3,X_4)+w(X_2)B(X_3,X_1,X_4)&\\+w(X_3)B(X_1,X_2,X_4)&=0\,,\end{aligned} \tag{2.5}$$

where $X_1,\ldots,X_4$ are tangent vectors at x. Examples of manifolds fulfilling (2.4) for $B=R$ or $B=W(R)$ are given in [14] and [2]. Furthermore, in [8] it was proved that any conformally flat manifold M of dimension ≥ 4

satisfying (2.4) with $B = R$ is pseudosymmetric. A Riemannian manifold M is said to be *pseudosymmetric* if the Riemann–Christoffel curvature tensor $\overline{R}$ is pseudosymmetric with respect to the curvature tensor R and the metric tensor h [12]. Recently, pseudosymmetric manifolds were studied in [3]–[5], [10]–[12], [15]–[16], and [18]–[20].

Ricci-pseudosymmetric manifolds and Weyl-pseudosymmetric manifolds can be defined in a similar manner. Such manifolds were investigated in [6], [7], [9], [13], [17] and [25]. Certain properties of pseudosymmetric (with respect to R and h) generalized curvature tensors were also obtained in [11], [12] and [16].

A $(0,k)$-tensor field T on M is said to be *totally symmetric* (cf. [1]) if

$$T(X_1,\dots,X_k) = T(X_{\sigma(1)},\dots,X_{\sigma(k)})$$

for any permutation σ of $\{1,\dots,k\}$ and $X_1,\dots,X_k \in \mathfrak{X}(M)$.

Let now M be a non-degenerate hypersurface of the affine space A^{n+1}. The following proposition states that on M there exist tensor fields which are pseudosymmetric related with respect to the tensors R and h.

PROPOSITION 2.2. *On any non-degenerate hypersurface M in A^{n+1} we have*

$$R \cdot h = Q(h, \overline{S})\,, \tag{2.6}$$

$$R \cdot \overline{S} = Q(h, \overline{S}^2)\,, \tag{2.7}$$

$$R \cdot C = Q(h, CS)\,, \tag{2.8}$$

where CS is the $(0,3)$-tensor fields defined by

$$CS(X,Y,Z) = C(SX,Y,Z) \tag{2.9}$$

and $\overline{S}$ and $\overline{S}^2$ are the $(0,2)$-tensor fields defined by

$$\overline{S}(X,Y) = h(SX,Y) \quad \text{and} \quad \overline{S}^2(X,Y) = \overline{S}(SX,Y)\,,$$

respectively.

The above proposition is an immediate consequence of the following proposition:

PROPOSITION 2.3. *Let T be a totally symmetric $(0,k)$-tensor field, $k \geq 1$, on a non-degenerate hypersurface M in A^{n+1}. Then*

$$R \cdot T = Q(h, TS)$$

on M, where TS is the $(0,k)$-tensor field defined by

$$TS(X_1,\dots,X_k) = T(SX_1, X_2,\dots,X_k)\,.$$

Proof. This is a consequence of the Gauss equation (1.3) and the definition of pseudosymmetric related tensors.

Moreover, it can be easily noted that the following proposition is also true.

Proposition 2.4. *Any tensor field on an affine hypersphere M in A^{n+1} is pseudosymmetric with respect to R and h.*

We give another example of pseudosymmetric related tensors. On a non-degenerate hypersurface M in A^{n+1} consider the generalized curvature tensor R defined by ([27])

$$R^*(X,Y,Z) = R(X,Y)SZ = \overline{S}(Y,Z)SX - \overline{S}(X,Z)SY \,. \tag{2.10}$$

The equality (2.10) implies

$$\begin{aligned} \overline{R}^*(X_1,\ldots,X_4) &= h(R^*(X_1,X_2,X_3),X_4) \\ &= \overline{S}(X_1,X_4)\overline{S}(X_2,X_3) - \overline{S}(X_1,X_3)\overline{S}(X_2,X_4)\,. \end{aligned} \tag{2.11}$$

From Proposition 2.1 we easily obtain the following corollary.

Corollary 2.5. *Let M be a non-degenerate hypersurface in A^{n+1}. Then*

$$R^* \cdot \overline{R}^* = Q(\mathrm{Ricc}(R^*), \overline{R}^*)$$

on M.

A non-degenerate hypersurface M in A^{n+1} is said to be an *affine Einstein hypersurface* ([27]) if $\mathrm{Ricc}(R^*)$ is proportional to h. Thus Corollary 2.5 yields

Corollary 2.6. *The curvature tensor $\overline{R}^*$ of a non-degenerate affine Einstein hypersurface in A^{n+1} is pseudosymmetric with respect to R^* and h.*

To end this section, we prove some lemmas.

Lemma 2.7 ([8], Theorem 3.5). *Let B be a generalized curvature tensor on a Riemannian manifold (M,h), $n \geq 4$, with a not necessarily definite metric h. Moreover, suppose the Weyl curvature tensor $W(B)$ vanishes on M. Then*

$$B \cdot \overline{B} = Q(\mathrm{Ricc}(B), \overline{B})$$

on M if and only if at each point of M the Ricci tensor $\mathrm{Ricc}(B)$ has the form

$$\mathrm{Ricc}(B) = \alpha h + \beta a \otimes a\,, \quad \alpha,\beta \in \mathbb{R}\,,$$

where a is a covector.

Lemma 2.8. *Let A and B be two symmetric $(0,2)$-tensors at a point x of a Riemannian manifold (M,h) with a not necessarily definite metric h.*

(i) *If $Q(A,B) = 0$ at x then A and B are linearly dependent.*
(ii) *If $A \circ B = B \circ A$ at x and*

$$\alpha Q(h,A) + \gamma Q(A,B) + \beta Q(h,B) = 0\,, \qquad \alpha,\beta,\gamma \in \mathbb{R}\,,\ \gamma \neq 0\,, \tag{2.12}$$

at x then the tensors

$$A - \frac{1}{n}\operatorname{tr}(A)h \quad \textit{and} \quad B - \frac{1}{n}\operatorname{tr}(B)h$$

are linearly dependent, where $A \circ B$ is the $(0,2)$-tensor with the local components $(A \circ B)_{rs} = h^{pq}A_{rp}B_{qs}$.

Proof. (i) The proof was given in [8] (see the proof of Lemma 3.4).

(ii) Contracting the equality

$$\alpha Q(h,A)_{rstu} + \gamma Q(A,B)_{rstu} + \beta Q(h,B)_{rstu} = 0$$

with h^{ru} we obtain

$$\alpha\Big(A - \frac{1}{n}\operatorname{tr}(A)h\Big) + \gamma\Big(\frac{1}{n}\operatorname{tr}(A)B - \frac{1}{n}\operatorname{tr}(B)A\Big) + \beta\Big(B - \frac{1}{n}\operatorname{tr}(B)h\Big) = 0\,,$$

which yields

$$\alpha Q(h,A) + \gamma Q\Big(\frac{1}{n}\operatorname{tr}(A)h, B\Big) - \gamma Q\Big(\frac{1}{n}\operatorname{tr}(B)h, A\Big) + \beta Q(h,B) = 0\,.$$

Next, subtracting the above equality from (2.12) we get

$$\gamma Q\Big(A - \frac{1}{n}\operatorname{tr}(A)h, B - \frac{1}{n}\operatorname{tr}(B)h\Big) = 0\,.$$

Now (i) completes the proof.

LEMMA 2.9. *Let (M,h), $\dim M \ge 4$, be a Riemannian manifold with a not necessarily definite metric h. If a symmetric $(0,2)$-tensor A satisfies at $x \in M$ the condition*

$$\begin{aligned} A((X_1 \wedge_A X_2)X_3, X_4) &= \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)h((X_1 \wedge X_2)X_3, X_4) \\ &\quad + \frac{1}{n-2}(h((X_1 \wedge_{a\otimes a} X_2)X_3, X_4) \\ &\quad - h((X_1 \wedge_{a\otimes a} X_2)X_4, X_3))\,, \quad \tau, \varrho \in \mathbb{R}\,, \end{aligned} \tag{2.13}$$

then $A = \alpha h + \beta b \otimes b$, $\alpha, \beta \in \mathbb{R}$, at x, where $\|a\|^2$ is the square of the norm of the covector a.

Proof. (2.13) can be written in the form

$$\begin{aligned} A_{ru}A_{st} - A_{rt}A_{su} &= \frac{1}{(n-1)(n-2)}((n-2)\tau - \|a\|^2\varrho)(h_{ru}h_{st} - h_{rt}h_{su}) \\ &\quad + \frac{1}{n-2}\varrho(h_{ru}a_s a_t + h_{st}a_r a_u - h_{rt}a_s a_u - h_{su}a_r a_t)\,. \end{aligned} \tag{2.14}$$

Contracting this with h^{ru} we obtain

$$\operatorname{tr}(A)A_{st} - A^2{}_{st} = \tau h_{st} + \varrho a_s a_t\,, \tag{2.15}$$

where A_{ru}, $A^2{}_{ru}$, h_{ru}, and a_r are the local components of the tensors A, $A^2 = A \circ A$, h and a, respectively.

Next, transvecting (2.15) with $A^r{}_p = A_{qp}h^{rq}$ we obtain

$$\begin{aligned} A^2{}_{ru}A_{st} - A^2{}_{rt}A_{su} &= \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)(A_{ru}h_{st} - A_{rt}h_{su}) \\ &\quad + \frac{1}{n-2}\varrho(A_{ru}a_s a_t - A_{rt}a_s a_u + h_{st}P_r a_u - h_{su}P_r a_t)\,, \end{aligned} \tag{2.16}$$

where $P_r = a_p A^p{}_r$. Further, contracting the above relation with h^{st}, we get

$$\operatorname{tr}(A)A^2{}_{ru} - A^3{}_{ru} = \tau A_{ru} + \varrho P_r a_u\,, \qquad A^3{}_{ru} = A^2{}_{pr}h^{pq}A_{qu}\,.$$

From this it follows immediately that

$$\varrho P_r = \varrho\lambda a_r\,, \qquad \lambda \in \mathbb{R}\,.$$

Using the above equality and (2.15) we can write (2.16) in the form

$$\begin{aligned} &\operatorname{tr}(A)(A_{ru}A_{st} - A_{rt}A_{su}) - \tau(h_{ru}A_{st} - h_{rt}A_{su}) - \beta(a_r a_u A_{st} - a_r a_t A_{su}) \\ &\quad = \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)(A_{ru}a_s a_t - A_{rt}a_s a_u) \\ &\qquad + \frac{1}{n-2}\varrho(a_s a_t A_{ru} - a_s a_u A_{rt} + \lambda(a_r a_u h_{st} - a_r a_t h_{su}))\,. \end{aligned}$$

This, by symmetrization in r and s, yields

$$\begin{aligned} \left(\frac{n-2}{n-1}\tau + \frac{1}{n-2}\varrho\|a\|^2\right)Q(h,A)_{rstu} &+ \frac{n-3}{n-2}\varrho Q(a\otimes a, A)_{rstu} \\ &+ \frac{1}{n-2}\lambda\varrho Q(a\otimes a, h)_{rstu} = 0\,. \end{aligned} \tag{2.17}$$

If $\varrho \neq 0$ then from Lemma 2.8(ii) it follows that the tensors $A - \frac{1}{n}\operatorname{tr}(A)h$ and $a \otimes a - \frac{1}{n}\|a\|^2 h$ are linearly dependent. So A has the required form. If $\varrho = 0$ then (2.17) yields $\tau Q(h,A) = 0$, whence we get $A - \frac{1}{n}\operatorname{tr}(A)h = 0$ or $\tau = 0$. In the second case, i.e. when $\tau = 0$, (2.14) implies that $\operatorname{rank}(A) = 1$. The last remark completes the proof.

3. Curvature conditions imposed on the cubic form. Let M be non-degenerate hypersurface in the affine space A^{n+1}, $n \geq 2$. Let A be the $(0,4)$-tensor field on M defined by

$$\begin{aligned} &A(X_1, X_2, X_3, X_4) \\ &\quad = (\nabla C)(X_1, X_2, X_3; X_4) - (\nabla C)(X_1, X_2, X_4; X_3)\,, \end{aligned} \tag{3.1}$$

where $X_1, \dots, X_4$ are vector fields tangent to M. Note that in virtue of (1.3), A satisfies the condition

$$A = R \cdot h \,. \tag{3.2}$$

From this and (2.6) it follows that A vanishes if and only if the tensor $\overline{S}$ is proportional to h (i.e. ∇C is totally symmetric). Non-degenerate hypersurface in A^{n+1} with ∇C and $\nabla^2 C$ totally symmetric were considered in [1]. ∇C and $\nabla^2 C$ are both totally symmetric if and only if $C = 0$ or $S = 0$ ([1], Theorem 1). In the special case when $\nabla C = 0$ and $C \neq 0$, then $S = 0$, ∇ is flat, the Pick invariant of M vanishes and h is a hyperbolic metric with zero Ricci tensor ([1], Corollary). Of course, if $\nabla^2 C$ is totally symmetric, then $R \cdot C = 0$ (i.e. C is semisymmetric with respect to R). As a generalization of the semisymmetry of C with respect to R, we can consider the pseudosymmerty of C with respect to R and h. Note that $Q(h, C)$ vanishes at $x \in M$ if and only if C vanishes at x. Thus

$$R \cdot C = L_C Q(h, C) \tag{3.3}$$

on the set U_C of all points of M at which C is non-zero, where L_C is a function defined on U_C.

Proposition 3.1. *Let M be a non-degenerate hypersurface in A^{n+1}. Then C satisfies* (3.3) *on U_C if and only if*

$$C((S - L_C I)X, Y, Z) = 0 \tag{3.4}$$

on U_C, where X, Y, Z are vector fields tangent to U_C.

Proof. If (3.4) holds on U_C then (3.3) is also satisfied. This is an immediate consequence of (2.8). Assume now that (3.3) holds on U_C. Let $U \subset U_C$ be a coordinate neighbourhood. We can write (3.3) in the form

$$\begin{aligned} &- C_{pst} R^p{}_{rvw} - C_{rpt} R^p{}_{svw} - C_{rsp} R^p{}_{tvw} \\ &\quad = L_C (h_{rw} C_{vst} + h_{sw} C_{rvt} + h_{tw} C_{rsv} - h_{rv} C_{wst} - h_{sv} C_{rwt} - h_{tv} C_{rsw}) \,, \end{aligned}$$

where $R^p{}_{rvw}$, C_{rst} and h_{rw} are the local components of R, C and h, respectively. Applying (1.3) to the above equality, we get

$$\begin{aligned} &h_{rw} V_v{}^p C_{pst} + h_{sw} V_v{}^p C_{prt} + h_{tw} V_v{}^p C_{prs} \\ &\qquad - h_{rv} V_w{}^p C_{pst} - h_{sv} V_w{}^p C_{prt} - h_{tv} V_w{}^p C_{prs} = 0 \,, \end{aligned}$$

where $V_v{}^p = S_v{}^p - L_C \delta^p_v$. The above relation, by contraction with h^{rw}, yields

$$(n+1) V_v{}^p C_{pst} = h_{sv} V^{pq} C_{pqt} + h_{tv} V^{pq} C_{pqs} \,, \tag{3.5}$$

whence, by contraction with h^{st} and making use of the apolarity condition $h^{pq} C_{pqs} = 0$, we obtain

$$V^{pq} C_{pqt} = 0 \,, \tag{3.6}$$

where $V^{pq} = h^{ps}V_s{}^q$ and h^{ps} are the local components of h^{-1}. Substituting (3.6) into (3.5) we obtain (3.4) on U, which completes the proof.

Now we will consider non-degenerate hypersurfaces M in A^{n+1} which have a tensor field A pseudosymmetric with respect to R and h.

LEMMA 3.2. *Let M be a non-degenerate hypersurface in A^{n+1}. Then the tensor $Q(h,A)$ vanishes at $x \in M$ if and only if $\overline{S}$ is proportional to h at x.*

Proof. Of course, if $\overline{S}$ is proportional to h at x then both A and $Q(h,A)$ vanish at x. Assume now that $Q(h,A)$ vanishes at x. Let $U \subset M$ be a coordinate neighbourhood of x. Thus, at x,

$$\begin{aligned} &h_{rw}A_{vstu} + h_{sw}A_{rvtu} + h_{tw}A_{rsvu} + h_{uw}A_{rstv} \\ &-h_{rv}A_{wstu} - h_{sv}A_{rwtu} - h_{tv}A_{rswu} - h_{uv}A_{rstw} = 0\,. \end{aligned} \tag{3.7}$$

Contracting this with h^{tw} and h^{uv} we obtain

$$(n-2)A_{rsvu} = A_{vsur} + A_{vrus} - h_{sv}h^{pq}A_{pruq} - h_{rv}h^{pq}A_{psuq}\,, \tag{3.8}$$

$$h^{pq}A_{psrq} = -h^{pq}A_{prsq}\,. \tag{3.9}$$

Similarly, contracting (3.7) with h^{rw} and h^{uv} and using (3.9) we find

$$2h^{pq}A_{pstq} = h^{pq}A_{pqts}\,. \tag{3.10}$$

On the other hand, from (2.6) and (3.2) it follows that

$$h^{pq}A_{pqts} = 0\,. \tag{3.11}$$

Now (3.8), by (3.11) and (3.10), takes the form

$$(n-1)(h_{ru}\overline{S}_{vs} - h_{rv}\overline{S}_{us} + h_{us}\overline{S}_{rv} - h_{sv}\overline{S}_{ru}) = 2(h_{rs}\overline{S}_{uv} - h_{uv}\overline{S}_{rs})\,,$$

whence, by contraction with h^{rs}, we obtain our assertion.

From the last lemma it follows that if a tensor field A is pseudosymmetric with respect to R and h then

$$R \cdot A = L_A Q(h,A) \tag{3.12}$$

on the set U_A of all points of M at which A is non-zero, where L_A is a function defined on U_A.

THEOREM 3.3. *Let M be a non-degenerate hypersurface in A^{n+1} with a tensor field A pseudosymmetric with respect to R and h.*

(i) *If* $\dim M \geq 3$ *then M is an affine hypersphere.*
(ii) *If* $\dim M = 2$ *and h is positive definite then M is an affine sphere.*
(iii) *If* $\dim M = 2$ *and h is indefinite then M is an affine Einstein surface.*

Proof. Let U_A be the set defined above. We note that (3.12) can be written in the following form (on a coordinate neighbourhood $U \subset U_A$):

$$h_{rw}V_v{}^pA_{pstu} + h_{sw}V_v{}^pA_{prtu} - h_{rv}V_w{}^pA_{pstu} - h_{sv}V_w{}^pA_{prtu} \tag{3.13}$$

$$-h_{tw}V_v{}^pA_{rsup} + h_{uw}V_v{}^pA_{rstp} + h_{tv}V_w{}^pA_{rsup} - h_{uv}V_w{}^pA_{rstp} = 0\,,$$

where $V_w{}^p = S_w{}^p - L_A\delta^p_w$. Contracting (3.13) with h^{tw} we obtain

$$(n-2)V_v{}^pA_{rsup} = V_v{}^pA_{psru} + V_v{}^pA_{prsu} + h_{rw}E_{su} + h_{sv}E_{ru}\,, \tag{3.14}$$

where $E_{rs} = V^{pq}A_{prsq}$ and $V^{pq} = V_r{}^qh^{rp}$. Similarly, contracting (3.13) with h^{rw} and h^{uv}, we find

$$nV_v{}^pA_{pstu} = V_v{}^pA_{tsup} - V_v{}^pA_{ustp} + h_{sv}V^{pq}A_{pqtu} - h_{tv}E_{su} + h_{uv}E_{st}\,, \tag{3.15}$$

$$V^{pq}A_{pqts} = 2E_{st}\,. \tag{3.16}$$

On the other hand, from (2.6) and (3.2) it follows that

$$V^{pq}A_{pqts} = 0\,. \tag{3.17}$$

Applying this and (3.16) in (3.14) and (3.15) we find

$$(n-2)V_v{}^pA_{rsup} = V_v{}^pA_{psru} + V_v{}^pA_{prsu}\,, \tag{3.18}$$

$$nV_v{}^pA_{pstu} = V_v{}^pA_{tsup} - V_v{}^pA_{ustp}\,, \tag{3.19}$$

respectively. Moreover, using (2.6) and (3.2), we can express A_{pstu} in the form

$$A_{pstu} = h_{pu}V_{ts} + h_{su}V_{pt} - h_{pt}V_{us} - h_{st}V_{pu}\,, \tag{3.20}$$

where $V_{up} = h_{pq}V_u{}^q$. Now (3.18) and (3.19) take the forms

$$(n-1)(V_{vr}V_{us} + V_{vs}V_{ur} - h_{ur}V^2{}_{sv} - h_{us}V^2{}_{rv}) = 2(V_{uv}V_{rs} - h_{rs}V^2{}_{ur})\,, \tag{3.21}$$

$$(n+1)(V_{uv}V_{ts} - V_{tv}V_{us} + h_{us}V^2{}_{tv} - h_{ts}V^2{}_{uv}) = 0\,, \tag{3.22}$$

respectively. From the last equality, by contraction with h^{us}, we obtain

$$V^2{}_{tv} = \frac{1}{n}\lambda V_{tv}\,, \qquad \lambda = h^{pq}V_{pq}\,. \tag{3.23}$$

Further, contracting (3.21) with h^{us}, we get

$$\lambda\left(V_{rs} - \frac{1}{n}\lambda h_{rs}\right) = 0\,. \tag{3.24}$$

From (3.20) and the definition of the set U_A it follows that $V_{rs} - \frac{1}{n}\lambda h_{rs}$ is non-zero at every point of U_A. Thus (3.24) yields $\lambda = 0$ and, by (3.23), we also have $V^2{}_{rs} = 0$. Now (3.21) and (3.22) turn into

$$(n-1)(V_{vr}V_{us} + V_{vs}V_{ur}) = 2V_{uv}V_{rs}\,,$$

$$V_{uv}V_{ts} = V_{us}V_{tv}\,, \tag{3.25}$$

respectively. But from the last two equalities and the assumption $n \geq 3$ it follows immediately that V_{us} must vanish on U_A, i.e. the set U_A is empty. This completes the proof of (i).

(ii) From (3.25) it follows that

$$V_{rs} = \lambda\psi_r\psi_s\,, \qquad \lambda \in \mathbb{R} - \{0\}\,,$$

at every $x \in U$. Together with (3.20), this yields

$$\psi^r A_{rstu} = \lambda\psi^r\psi_r(\psi_t h_{us} - \psi_u h_{ts})\,,$$
$$\psi^r A_{ustr} = \psi_u V_{ts} + \psi_s V_{tu} - \lambda\psi^r\psi_r(\psi_s h_{ut} + \psi_u h_{st})\,.$$

Now (3.19) turns into

$$\lambda\psi^r\psi_r(\psi_t h_{us} - \psi_u h_{ts}) = 0\,.$$

Thus, we see that U_A must be empty, which completes the proof of (ii).

(iii) The relation (3.25) can be written in the form

$$\begin{aligned}\overline{S}_{ur}\overline{S}_{ts} - \overline{S}_{us}\overline{S}_{tr} &= L_A(h_{ur}\overline{S}_{ts} + h_{ts}\overline{S}_{ur} - h_{us}\overline{S}_{tr} - h_{tr}\overline{S}_{us}) \\ &\quad - {L_A}^2(h_{ur}h_{ts} - h_{us}h_{tr})\,,\end{aligned}$$

which, by the identity (cf. [9], Lemma 2(iii))

$$h_{ur}\overline{S}_{ts} + h_{ts}\overline{S}_{ur} - h_{us}\overline{S}_{tr} - h_{tr}\overline{S}_{us} = \operatorname{tr}(S)(h_{ur}h_{ts} - h_{us}h_{tr})\,,$$

turns into

$$\overline{S}_{ur}\overline{S}_{ts} - \overline{S}_{us}\overline{S}_{tr} = L_A(\operatorname{tr}(S) - L_A)(h_{ur}h_{ts} - h_{us}h_{tr})\,.$$

Thus, the tensor $\operatorname{Ricc}(R^*)$ is proportional to h on U_A. But this completes the proof of (iii).

THEOREM 3.4. *Let M be an affine Einstein surface in the affine space A^3. If the set U_A is non-empty and if $(\operatorname{tr}(S))^2 = 2\operatorname{tr}(S^2)$ on U_A then A is pseudosymmetric with respect to R and h and L_A is defined by $2L_A = \operatorname{tr}(S)$.*

Proof. Since M is an Einstein affine surface we have, on a coordinate neighbourhood $U \subset U_A$,

$$\operatorname{tr}(S)\overline{S}_{ts} - \overline{S}^2{}_{ts} = \tfrac{1}{2}((\operatorname{tr}(S))^2 - \operatorname{tr}(S^2))h_{ts}\,.$$

We put $V_{rs} = \overline{S}_{rs} - L_A h_{rs}$, $L_A = \frac{1}{2}\operatorname{tr}(S)$. Now we can easily verify that

$$h^{pq}V_{pq} = 0\,,\ V^2{}_{ur} = 0\,,\ V_{ur}V_{ts} - V_{us}V_{tr} = 0\,,\ V_{vr}V_{us} + V_{vs}V_{ur} = 2V_{uv}V_{rs}\,.$$

Further, using the above equalities, we can express the tensor field $R \cdot A - L_A Q(h, A)$ in the form

$$(R \cdot A - L_A Q(h, A))_{rstuvw} = 2V_{rs}(h_{uw}V_{vt} + h_{tv}V_{uw} - h_{tw}V_{uv} - h_{uv}V_{tw})\,,$$

which implies $(R \cdot A - L_A Q(h, A))_{rstuvw} = 0$, completing the proof.

4. Affine hypersurfaces with pseudosymmetric curvature tensors. Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 3$. In [31]

(Proposition 2) it was proved that the curvature tensor R of M is semisymmetric (more precisely: $R \cdot R = 0$ on M) if and only if M is an affine hypersphere. This fact will be used in the proof of the following theorem.

THEOREM 4.1. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 3$. If the curvature tensor R is pseudosymmetric with respect to R and h, then M is an affine hypersphere.*

Proof. We remark that $Q(h,R)$ vanishes at $x \in M$ if and only if the shape operator S is proportional to the identity transformation at x. Denote by U_R the set of all points of M at which $Q(h,R)$ is non-zero. Thus, on U_R,

$$R \cdot R = L_R Q(h,R)\,, \tag{4.1}$$

where L_R is a function defined on U_R. Let $U \subset U_R$ be a coordinate neighbourhood. We write (4.1) in the form

$$\begin{aligned}
&V_{rv}(h_{st}V_{uw} - h_{us}V_{tw}) - V_{rw}(h_{st}V_{uv} - h_{us}V_{tv})\\
&+ V_{rt}(h_{sw}V_{uv} - h_{sv}V_{uw} + h_{uw}V_{sv} - h_{uv}V_{sw})\\
&- V_{ru}(h_{sw}V_{tv} - h_{sv}V_{tw} + h_{tw}V_{sv} - h_{tv}V_{sw})\\
&+ V^2{}_{rv}(h_{us}h_{tw} - h_{ts}h_{uw}) - V^2{}_{rw}(h_{us}h_{tv} - h_{ts}h_{uv})\\
&+ L_R h_{tr}(h_{sw}V_{uv} - h_{sv}V_{uw} + h_{uw}V_{sv} - h_{uv}V_{sw})\\
&- L_R h_{ur}(h_{sw}V_{tv} - h_{sv}V_{tw} + h_{tw}V_{sv} - h_{tv}V_{sw}) = 0\,,
\end{aligned}$$

where $V_{rt} = h_{rp}V_t{}^p$, $V_t{}^p = S_t{}^p - L_R\delta_t^p$ and $V^2{}_{rt} = h_{rq}V_r{}^pV_p{}^q$. This, by contractions with h^{st}, h^{uv} and h^{rw}, gives

$$\begin{aligned}
&(n-2)(V_{uw}V_{rv} - V_{uv}V_{rw} + h_{uv}V^2{}_{rw} - h_{uw}V^2{}_{rv})\\
&\qquad + L_R(h_{rw}V_{uv} - h_{rv}V_{uw} + h_{uw}V_{rv} - h_{uv}V_{rw}) = 0\,,
\end{aligned} \tag{4.2}$$

$$n(n-2)V^2{}_{rw} - (nL_R + (n-2)\operatorname{tr}(V))V_{rw} + L_R\operatorname{tr}(V)h_{rw} = 0\,, \tag{4.3}$$

$$n\operatorname{tr}(V^2) = (\operatorname{tr}(V))^2\,, \tag{4.4}$$

where $\operatorname{tr}(V) = h^{pq}V_{pq}$ and $\operatorname{tr}(V^2) = h^{pq}V^2{}_{pq}$. From (4.2), by antisymmetrization and symmetrization with respect to u and r, respectively, we obtain

$$V_{rw}V_{uv} - V_{rv}V_{uw} = h_{uv}V^2{}_{rw} - h_{uw}V^2{}_{rv} + h_{rw}V^2{}_{uv} - h_{rv}V^2{}_{uw}\,, \tag{4.5}$$

$$\begin{aligned}
&2L_R(h_{uw}V_{rv} - h_{uv}V_{rw} + h_{rw}V_{uv} - h_{rv}V_{uw})\\
&\qquad + (n-2)(h_{rv}V^2{}_{uw} - h_{rw}V^2{}_{uv} + h_{uv}V^2{}_{rw} - h_{uw}V^2{}_{rv}) = 0\,,
\end{aligned} \tag{4.6}$$

respectively. Contracting (4.5) and (4.6) with h^{rw} and h^{uw} we get

$$V^2{}_{st} = \frac{1}{n-1}(\operatorname{tr}(V)V_{st} - \operatorname{tr}(V^2)h_{st})\,, \tag{4.7}$$

$$(4.8)\qquad {V^2}_{st} = \frac{2}{n-2} L_R V_{st} + \frac{1}{n}\left(\operatorname{tr}(V^2) - \frac{2}{n-2} L_R \operatorname{tr}(V)\right) h_{st}\,,$$

respectively. Next comparing the right sides of (4.3) and (4.7) and using (4.4) we obtain

$$\left(\frac{1}{n-2} L_R - \frac{1}{n(n-1)}\operatorname{tr}(V)\right)\left(V_{rs} - \frac{1}{n}\operatorname{tr}(V) h_{rs}\right) = 0\,.$$

Of course, the tensor field $V - \frac{1}{n}\operatorname{tr}(V)h$ is non-zero at every point of U_R. By the last equality, we have on U

$$\frac{1}{n-2} L_R = \frac{1}{n(n-1)}\operatorname{tr}(V)\,.$$

Now, applying this and (4.4) in (4.8), we obtain

$${V^2}_{st} = \frac{1}{n(n-1)}\operatorname{tr}(V)\left(2V_{st} + \frac{n-3}{n}\operatorname{tr}(V) h_{st}\right),$$

which, together with (4.7), yields $\operatorname{tr}(V) = 0$ and, in consequence, $L_R = 0$. Thus, (4.1) turns into $R \cdot R = 0$. On the other hand, in view of Proposition 2 of [31], $R \cdot R = 0$ implies that S is proportional to the identity transformation. Thus we see that the set U_R must be empty. This completes the proof.

Let B be a generalized curvature tensor field on a non-degenerate hypersurface M in A^{n+1}, $n \geq 2$. The tensor $Q(h, \overline{B})$ vanishes at $x \in M$ if and only if $Z(\overline{B}) = 0$ at x (cf. [4], Lemma 1.1(iii)). Denote by $U_{\bar{B}}$ the set of all points of M at which $Z(\overline{B}) = 0$. If $\overline{B}$ is pseudosymmetric with respect to R and h then

$$(4.9)\qquad R \cdot \overline{B} = L_{\bar{B}} Q(h, \overline{B})$$

on $U_{\bar{B}}$, where $L_{\bar{B}}$ is a function defined on $U_{\bar{B}}$. We can easily prove the following property of generalized curvature tensors which are pseudosymmetric with respect to R and h.

Lemma 4.2. *Let B be a generalized curvature tensor field on non-degenerate hypersurface M in A^{n+1}, $n \geq 2$. Then (4.9) holds on $U_{\bar{B}}$ if and only if*

$$(4.10)\qquad \overline{B}((S - L_{\bar{B}} I)X_1, X_2, X_3, X_4) = \frac{1}{n-1} h((X_4 \wedge_D X_3)X_2, X_1)$$

on $U_{\bar{B}}$, where D is a $(0,2)$-tensor field on $U_{\bar{B}}$ defined by

$$(4.11)\qquad D(X_1, X_2) = \sum_{i=1}^{n} \varepsilon_i \varepsilon_j (\overline{S} - L_{\bar{B}} h)(E_i, E_j) \overline{B}(E_i, X_1, X_2, E_j)$$

for any local orthonormal basis $\{E_1, \ldots, E_n\}$.

Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 2$. We define on M the following tensors:

$$V(X_1, X_2) = \overline{S}(X_1, X_2) - Lh(X_1, X_2)\,, \tag{4.12}$$

$$V^2(X_1, X_2) = \sum_{i=1}^{n} \varepsilon_i V(X_1, E_i)V(X_2, E_i)\,, \tag{4.13}$$

$$V^3(X_1, X_2) = \sum_{i=1}^{n} \varepsilon_i V^2(X_1, E_i)V(X_2, E_i)\,, \tag{4.14}$$

$$\begin{aligned} F(X_1, X_2, X_3, X_4) &= -\frac{1}{n-1}h((X_1 \wedge_E X_2)X_3, X_4) \\ &+ (V^2 + LV)((X_1 \wedge_V X_2)X_3, X_4) + L(V^2 + LV)((X_1 \wedge X_2)X_3, X_4)\,, \end{aligned} \tag{4.15}$$

for any orthonormal basis $\{E_1, \ldots, E_n\}$, where L is a function on M and

$$E = -V^3 - 2LV^2 + (\operatorname{tr}(V^2) + L\operatorname{tr}(V) - L^2)V + L(\operatorname{tr}(V^2) + L\operatorname{tr}(V))h\,.$$

Now using (4.12) and (4.13) we can write the curvature tensor $\overline{R}^*$, the Ricci tensor $\operatorname{Ricc}(R^*)$ and the scalar curvature $K(R^*)$ in the following form:

$$\begin{aligned} \overline{R}^*(X_1, X_2, X_3, X_4) &= V((X_1 \wedge_V X_2)X_3, X_4) + L^2h((X_1 \wedge X_2)X_3, X_4) \\ &\quad + L(h((X_1 \wedge_V X_2)X_3, X_4) - h((X_1 \wedge_V X_2)X_4, X_3))\,, \end{aligned} \tag{4.16}$$

$$\operatorname{Ricc}(R^*) = -V^2 + (\operatorname{tr}(V) + (n-2)L)V + L(\operatorname{tr}(V) + (n-1)L)h\,, \tag{4.17}$$

$$K(R^*) = -\operatorname{tr}(V^2) + (\operatorname{tr}(V))^2 + 2(n-1)L\operatorname{tr}(V) + n(n-1)L^2\,, \tag{4.18}$$

respectively. We note that the tensor D corresponding to the tensor $\overline{R}^*$ satisfies the equation

$$D = E\,. \tag{4.19}$$

If $n = \dim M \geq 3$ then we can define the Weyl curvature tensor $W(R^*)$ of R^* ([26]) by

$$\begin{aligned} W(R^*)&(X_1, X_2, X_3, X_4) \\ &= \overline{R}^*(X_1, X_2, X_3, X_4) + \frac{1}{(n-1)(n-2)}K(R^*)h((X_1 \wedge X_2)X_3, X_4) \\ &\quad - \frac{1}{n-2}(h((X_1 \wedge_{\operatorname{Ricc}(R^*)} X_2)X_3, X_4) - h((X_1 \wedge_{\operatorname{Ricc}(R^*)} X_2)X_4, X_3))\,. \end{aligned}$$

This, by making use of (4.16)–(4.18), turns into

$$W(R^*)(X_1, X_2, X_3, X_4) = -\frac{1}{n-2}\operatorname{tr}(V)(h((X_1 \wedge_V X_2)X_3, X_4) \tag{4.20}$$

$$- h((X_1 \wedge_V X_2)X_4, X_3)) + V((X_1 \wedge_V X_2)X_3, X_4)$$
$$+ \frac{1}{(n-1)(n-2)}((\mathrm{tr}(V))^2 - \mathrm{tr}(V^2))h((X_1 \wedge X_2)X_3, X_4)$$
$$+ \frac{1}{n-2}(h((X_1 \wedge_{V^2} X_2)X_3, X_4) - h((X_1 \wedge_{V^2} X_2)X_4, X_3)) .$$

The Weyl curvature tensor corresponding to the curvature tensor R of a hypersurface in A^{n+1} was constructed in [30]. Of course, the two tensors are different in general.

We denote by $U_{\bar{R}^*}$ the set all points of M at which $Z(R^*) \neq 0$. If $\overline{R}^*$ is pseudosymmetric with respect to R and h then

$$R \cdot \overline{R}^* = L_{\bar{R}^*} Q(h, \overline{R}^*) \tag{4.21}$$

on $U_{\bar{R}^*}$, where $L_{\bar{R}^*}$ is a function on $U_{\bar{R}^*}$.

The following lemma is an immediate consequence of Lemma 4.2.

LEMMA 4.3. *Let the tensor field $\overline{R}^*$ of a non-degenerate hypersurface M in A^{n+1}, $n \geq 2$, be pseudosymmetric with respect to R and h. Then on $U_{\bar{R}^*}$ the equalities* (4.21) *and $F = 0$ (with $L = L_{\bar{R}^*}$) are equivalent.*

LEMMA 4.4. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 2$. If the shape operator S of M satisfies*

$$S^2 + \mu S = \varrho I , \qquad \mu, \varrho \in \mathbb{R} , \tag{4.22}$$

at $x \in M$, then $R \cdot \overline{R}^* = -\mu Q(h, \overline{R}^*)$ at x.

Proof. We note that (4.22) can be written at x in the form

$$V^2 - \frac{1}{n}\,\mathrm{tr}(V^2)h = \mu\left(V - \frac{1}{n}\,\mathrm{tr}(V)h\right) , \tag{4.23}$$

where V^2 and V (with $L = -\mu$) are defined by (4.13) and (4.12), respectively. Next, applying (4.23) and (4.14) in (4.15) we see that F vanishes at x. Now Lemma 4.3 completes the proof.

LEMMA 4.5. *Suppose the curvature tensor $\overline{R}^*$ of a non-degenerate hypersurface M in A^{n+1}, $n \geq 2$, satisfies*

$$R \cdot \overline{R}^* = -\mu Q(h, \overline{R}^*) , \qquad \mu \in \mathbb{R} ,$$

at $x \in M$. Then (4.22) *holds at x.*

Proof. From Lemma 4.3 it follows that $F(X_1, X_2, X_3, X_4) = 0$ at x (with $L = -\mu$). From this, by symmetrization with respect to X_3 and X_4, we obtain

$$\begin{aligned}(n-1)(Q(V, V^2) + LQ(h, V^2) + L^2 Q(h, V))\\ -Q(h, V^3) - 2LQ(h, V^2) + (\mathrm{tr}(V^2) + L\,\mathrm{tr}(V) - L^2)Q(h, V) = 0 ,\end{aligned} \tag{4.24}$$

whence we get

$$(4.25)\qquad V^3 = \frac{1}{n}\operatorname{tr}(V^3)h + (n-3)L\Big(V^2 - \frac{1}{n}\operatorname{tr}(V)h\Big)$$
$$+(\operatorname{tr}(V^2)+L\operatorname{tr}(V)+(n-2)L^2)\Big(V-\frac{1}{n}\operatorname{tr}(V)h\Big)+\frac{n-1}{n}(\operatorname{tr}(V)V^2-\operatorname{tr}(V^2)V)\,.$$

Substituting this in (4.24) we find

$$Q(V,V^2) = \frac{1}{n}\operatorname{tr}(V)Q(h,V^2) - \frac{1}{n}\operatorname{tr}(V^2)Q(h,V)\,,$$

whence, in view of Lemma 2.8(ii), it follows that the tensors $V^2 - \frac{1}{n}\operatorname{tr}(V^2)h$ and $V - \frac{1}{n}\operatorname{tr}(V)h$ are linearly dependent. Evidently, our assertion is true when $V - \frac{1}{n}\operatorname{tr}(V)h = 0$. Otherwise

$$V^2 - \frac{1}{n}\operatorname{tr}(V^2)h = \lambda\Big(V - \frac{1}{n}\operatorname{tr}(V)h\Big)\,,\qquad \lambda\in\mathbb{R}\,,$$

at x. The last formula can be written in the form

$$S^2 + (2\mu-\lambda)S = \frac{1}{n}(\operatorname{tr}(S^2) - (\lambda-2\mu)\operatorname{tr}(S))I\,.$$

From Lemma 4.4 it follows that $2\mu-\lambda=\mu$. Thus the above relation yields (4.22), which completes the proof.

Combining the last two lemmas we obtain the following

THEOREM 4.6. *Let M be a non-degenerate hypersurface in A^{n+1}, $n\geq 2$. Then on $U_{\bar{R}^*}$ the equations $R\cdot\overline{R}^* = L_{\bar{R}^*}Q(h,\overline{R}^*)$ and $S^2 - \frac{1}{n}\operatorname{tr}(S^2)I = L_{\bar{R}^*}(S-\frac{1}{n}\operatorname{tr}(S)I)$ are equivalent.*

Using this theorem we can obtain a curvature characterization of affine Einstein hypersurfaces.

COROLLARY 4.7. *Let M be a non-degenerate hypersurface in A^{n+1}, $n\geq 2$. Then M is an affine Einstein hypersurface if and only if $R\cdot\overline{R}^* = \operatorname{tr}(S)Q(h,\overline{R}^*)$ on $U_{\bar{R}^*}$.*

A non-degenerate hypersurface M in A^{n+1}, $n\geq 2$, is said to be *affine-quasi-umbilical* ([26]) if

$$(4.26)\qquad \overline{S} = \alpha h + \beta a\otimes a\,,\qquad \alpha,\beta\in\mathbb{R}\,,$$

at every $x\in M$, where a is a covector at x.

THEOREM 4.8. *Let M be a non-degenerate hypersurface in A^{n+1}, $n\geq 4$. Then M is affine-quasi-umbilical if and only if the Weyl conformal curvature tensor $W(\overline{R}^*)$ vanishes on M.*

Proof. Assume that (4.26) is fulfilled at $x \in M$. We put $V = \overline{S} - Lh$, $L = \alpha$. Thus we have

$$V^2 = \mathrm{tr}(V)V \quad \text{and} \quad \mathrm{tr}(V^2) = (\mathrm{tr}(V))^2 \,.$$

Applying these formulas in (4.20) we easily obtain $W(R^*) = 0$.

Assume now that $W(R^*)$ vanishes at x. From Corollary 2.5 and Lemma 2.7 it follows that

$$\mathrm{Ricc}(R^*) = \tau h + \varrho a \otimes a \,, \qquad \tau, \varrho \in \mathbb{R} \,,$$

at x, where a is a covector. Now, using the above formula and the definitions of $\overline{R}^*$ and $\mathrm{Ricc}(R^*)$, we can rewrite the equality $W(R^*) = 0$ in the form

$$\overline{S}((X_1 \wedge_{\bar{S}} X_2)X_3, X_4) = \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)h((X_1 \wedge X_2)X_3, X_4)$$
$$+ \frac{1}{n-2}\varrho(h((X_1 \wedge_{a\otimes a} X_2)X_3, X_4) - h((X_1 \wedge_{a\otimes a} X_2)X_4, X_3)) \,.$$

Now Lemma 2.9 completes the proof.

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