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A SUBFIELD OF A COMPLEX BANACH ALGEBRA IS NOT NECESSARILY TOPOLOGICALLY ISOMORPHIC TO A SUBFIELD OF $\mathbb{C}$

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The classical Mazur–Gelfand theorem ([1]–[5]) implies that any subfield of a complex Banach algebra A is topologically isomorphic to $\mathbb{C}$, provided it is a linear subspace of A. Here we present a somewhat surprising observation that if F is a subfield of A which is just a subring, and not a subalgebra, it need not be topologically isomorphic to a subfield of $\mathbb{C}$.

Let A be a complex Banach algebra and let F be a subfield of A. Denote by A_0 the smallest closed subalgebra of A containing F. This is a commutative algebra with unit element equal to the unity of F. Thus A_0 has a non-zero multiplicative-linear functional mapping isomorphically F into $\mathbb{C}$. Therefore any subfield of A is isomorphic to a subfield of $\mathbb{C}$ under a continuous isomorphism. We shall show that in certain cases such an isomorphism cannot be a homeomorphic map.

Denote by Q the set of all rational complex numbers, i.e. numbers of the form $\varrho = r_1 + ir_2$ with rational r_1 and r_2. Denote by W the field of all rational functions in a variable t, with coefficients in Q; it contains the subfield of all constant functions, i.e. quotients of elements in Q. This subfield is clearly a dense subset of the complex plane $\mathbb{C}$. Fixing a transcendental number c we obtain an isomorphic imbedding of W into $\mathbb{C}$ given by $w \to w(c)$, $w \in W$ (a function w is uniquely determined by its value $w(c)$ and this value is a well defined complex number, since c is transcendental). One can easily see that each isomorphism h of W into $\mathbb{C}$ is of the form $w \to \widetilde{w}(d)$, where d is a transcendental number given by $d = h(t)$, and $\widetilde{w}$ is either w or $\overline{w}$, depending on whether $h(i) = i$ or $h(i) = -i$. Here $\overline{w}$ is an element of W obtained by replacing in w all coefficients by their complex conjugates.

Take a complex Banach space X, $\dim X > 1$, and take as A the algebra $L(X)$ of all continuous endomorphisms of X. One can easily see that A contains a non-zero operator T satisfying

$$T^2 = 0\,. \tag{1}$$

Define now a subfield of A setting

$$F_0 = \{w(c)I + w'(c)T \in A : w \in W\}\,,$$

where c is a fixed transcendental number and I is the unity of A (the identity operator on X). By (1) we have

$$(w_1(c)I + w_1'(c)T)(w_2(c)I + w_2'(c)T) = w_1(c)w_2(c)I + [w_1(c)w_2(c)]'T\,,$$

thus F_0 is a subring of A; moreover,

$$(w(c)I + w'(c)T)^{-1} = w(c)^{-1}I - \frac{w'(c)}{w(c)^2}T\,,$$

which we check easily using (1). Thus F_0 is a subfield of A. Since the value $w(c)$ uniquely determines w, and hence also $w'(c)$, the map $w(c)I + w'(c)T \to w$ is an isomorphism of F_0 onto W, and so F_0 is isomorphic to a subfield of $\mathbb{C}$. On the other hand, the map $w(c)I + w'(c)T \to (w(c), w'(c))$ is a homeomorphism of F_0 onto a dense subset of $\mathbb{C}^2$ (F_0 is a dense subset of a two-dimensional subspace of A). As observed above, any isomorphism of F_0 into $\mathbb{C}$ is given by

$$h_0 : w(c)I + w'(c)T \to \widetilde{w}(d)\,,$$

where d is some transcendental number. Such a map is never a homeomorphism. The discontinuity of h_0^{-1} follows from the discontinuity of the map $w(c) \to w'(c)$, and the latter can be seen by observing that $w'(c) = 0$ on a dense subset of $\mathbb{C}$ consisting of numbers $w(c)$ for constant functions w, while $w'(c)$ is not identically zero. An alternative proof can be obtained by observing that a dense subset of $\mathbb{C}^2$ cannot be homeomorphic to a subset of $\mathbb{C}$. Thus we have

PROPOSITION. *There exists a complex Banach algebra A and a subfield F of A which is not topologically isomorphic with a subfield of $\mathbb{C}$.*

Remarks. The above construction can be performed in any complex Banach algebra A possessing a nilpotent element T, $T^{n-1} \neq 0$, $T^n = 0$ for some $n > 1$. In this case as the subfield F we take

$$F = \left\{ w(c)I + w'(c)T + \ldots + \frac{w^{(n-1)}(c)}{(n-1)!}T^{n-1} \in A : w \in W \right\}.$$

This is a subfield of A homeomorphic to a dense subset of $\mathbb{C}^n$.

A modified argument gives a similar construction in a real Banach algebra.

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