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ON THE MAXIMAL CONDITION IN FORMAL POWER SERIES RINGS

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Let R denote a ring with an identity element which satisfies the maximal condition for left ideals: equivalently, every left ideal is finitely generated. We give an indirect proof that the ring of formal power series in n indeterminates over R also satisfies the maximal condition for left ideals. When R is commutative we recover the classical theorem due to Chevalley : see Balcerzyk and Józefiak [1], Theorem 2.2.6, p. 60. Since $R[[X_1, \ldots, X_n]] \simeq R[[X_1, \ldots, X_{n-1}]][[X_n]]$, it is sufficient, by induction, to consider the case of a formal power series ring S in one indeterminate X over R. The proof below uses the facts that S is complete in the SX-adic filtration and that if B is a left ideal of S, $B : SX = \{c \in S \mid Xc \in B\}$ is also a left ideal of S, since X is a central element of S.

THEOREM. *If R satisfies the maximal condition on left ideals, then so does S.*

P r o o f. Assume that S does not satisfy the maximal condition for left ideals. Then the non-empty set T of left ideals of S which are not finitely generated forms an inductive system with respect to the partial order determined by inclusion. Zorn's lemma guarantees the existence of at least one maximal element in T, say B. Now, $X \notin B$, otherwise B/SX (and therefore B) is finitely generated, since $R \simeq S/SX$. Thus, the left ideal $B + SX$ properly contains B and is therefore finitely generated. Hence, a finitely generated left ideal $A \subset B$ exists such that $B + SX = A + SX$ and so

$$B = A + B \cap SX = A + (B : SX)X .$$

It follows from this equation that if $B : SX$ properly contains B, then $B : SX$ (and therefore $(B : SX)X$) is finitely generated, implying that B is finitely generated. Thus, $B : SX = B$ and we obtain $B = A + BX$, where $A = (a_1, \ldots, a_n)S$. If $b \in B$ then $b - b_1 \in BX$ where $b_1 = \sum_{i=1}^{n} s_{1i}a_i$ ($s_{1i} \in S$). Again, there is an element $b_2 = \sum_{i=1}^{n} s_{2i}a_i$ ($s_{2i} \in SX$) such that $b - b_1 - b_2 \in BX^2$. In general, there are elements $b_1, \ldots, b_k, \ldots$ of A such that $b_k = \sum_{i=1}^{n} s_{ki} a_i$ ($s_{ki} \in SX^{k-1}$) and $b - \sum_{i=1}^{k} b_i \in BX^k$. For each $i = 1, \ldots, n$ the sequence of partial sums $\{t_{ki}\}$, where $t_{ki} = \sum_{m=1}^{k} s_{mi}$, is a Cauchy sequence in the SX-adic filtration because $s_{ki} \in SX^{k-1}$ for each $k \geq 1$. Hence, the series $\sum_{k=1}^{\infty} s_{ki}$ converges with sum $s'_i \in S$, since S is complete. Let $b' = \sum_{i=1}^{n} s'_i a_i$; then $b - b' \in BX^k$ for each $k \geq 0$. Thus, $b - b' \in \bigcap_{k=1}^{\infty} SX^k = (0)$ and so $B = A$, a contradiction.

REFERENCES

[1] S. Balcerzyk and T. Józefiak, *Commutative Noetherian and Krull rings*, Ellis Horwood, 1989.

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