ON FINITE MINIMAL NON-\(p\)-SUPERSOLUBLE GROUPS

BY

FERNANDO TUCCILLO (NAPLES)

If \(\mathfrak{F}\) is a class of groups, then a minimal non-\(\mathfrak{F}\)-group (a dual minimal non-\(\mathfrak{F}\)-group resp.) is a group which is not in \(\mathfrak{F}\) but any of its proper subgroups (factor groups resp.) is in \(\mathfrak{F}\). In many problems of classification of groups it is sometimes useful to know structure properties of classes of minimal non-\(\mathfrak{F}\)-groups and dual minimal non-\(\mathfrak{F}\)-groups. In fact, the literature on group theory contains many results directed to classify some of the most remarkable among the aforesaid classes. In particular, V. N. Semenchuk in [12] and [13] examined the structure of minimal non-\(\mathfrak{F}\)-groups for \(\mathfrak{F}\) a formation, proving, among other results, that if \(\mathfrak{F}\) is a saturated formation, then the structure of finite soluble, minimal non-\(\mathfrak{F}\)-groups can be determined provided that the structure of finite soluble, minimal non-\(\mathfrak{F}\)-groups with trivial Frattini subgroup is known.

In this paper we use this result with regard to the formation of \(p\)-supersoluble groups (\(p\) prime), starting from the classification of finite soluble, minimal non-\(p\)-supersoluble groups with trivial Frattini subgroup given by N. P. Kontorovich and V. P. Nagrebetskii ([10]). The second part of this paper deals with non-soluble, minimal non-\(p\)-supersoluble finite groups. The problem is reduced to the case of simple groups. We classify the simple, minimal non-\(p\)-supersoluble groups, \(p\) being the smallest odd prime divisor of the group order, and provide a characterization of minimal simple groups.

All the groups considered are finite.

1. Some preliminary results. We provide some preliminary results; some of them are implicitly contained in the papers mentioned above, so we omit their proofs.

1.1. Let \(G\) be a minimal non-\(p\)-supersoluble group with \(p = \min \pi(G)\). Then \(G\) is soluble.

Proof. If \(p > 2\), then \(|G|\) is odd and so \(G\) is soluble ([5]). If \(p = 2\), the statement follows from a theorem of Ito ([9]), if we recall that 2-supersolvability is equivalent to 2-nilpotency.
1.2. Let $G$ be a minimal non-$p$-supersoluble group. If $O_p(G) \not\leq \Phi(G)$, then $G$ is soluble.

1.3. Let $G$ be a minimal non-$p$-supersoluble group with a normal Sylow subgroup. Then $G$ is soluble and $G_p$ is the only normal Sylow subgroup of $G$.

1.4. Let $G$ be a soluble, minimal non-$p$-supersoluble group. Then $|\pi(G)| \leq 3$. Moreover, if $|\pi(G)| = 3$ then $G_p \triangleleft G$ and $p = \max \pi(G)$.

1.5. Let $G$ be a soluble, minimal non-$p$-supersoluble group without normal Sylow subgroups. Then $p = \max \pi(G)$.

The following propositions can be obtained using techniques similar to those used in [1], [4], [13].

1.6. Let $G$ be a minimal non-$p$-supersoluble group and $G_p \triangleleft G$. Then:
   (i) $G_p/\Phi(G_p)$ is minimal normal in $G/\Phi(G_p)$;
   (ii) if $M$ is a maximal subgroup of $G$ whose index is a power of $p$, then $M = \Phi(G_p)G_p'$;
   (iii) there exists a supersoluble immersion of $\Phi(G_p)$ in $G$;
   (iv) $\Phi(G_p) \leq Z(G_p)$ (and so the class of $G_p$ is $\leq 2$);
   (v) the exponent of $G_p'$ is $\leq p$;
   (vi) the exponent of $G_p'$ is $p$ if $p \neq 2$, and is $\leq 4$ if $p = 2$.

1.7. Let $G$ be a minimal non-$p$-supersoluble group and $G_p < G$. If $K$ is a $p$-complement of $G$, then:
   (i) $K \cap C_G(G_p/\Phi(G_p)) = K \cap \Phi(G) = \Phi(K) \cap \Phi(G)$;
   (ii) $K/K \cap \Phi(G)$ is minimal non-abelian or cyclic primary;
   (iii) $\Phi(G) = \Phi(G_p) \times (\prod_{q \neq p} O_q(G))$;
   (iv) $\Phi(G_p) \leq Z(G)$.

1.8. Let $G$ be a soluble, minimal non-$p$-supersoluble group without normal Sylow subgroups. With $\pi(G) = \{p, q\}$ ($p > q$) we have:
   (i) $G$ has no subgroup of index $q$;
   (ii) $G$ has only one subgroup $M$ of index $p$;
   (iii) $O_p(G) = M_p$.

1.9. Let $G$ be a soluble, minimal non-$p$-supersoluble group without normal Sylow subgroups. Using the notation of 1.8 with $K = N_G(G_q)$ and $P = \Phi(M_p)(K \cap M_p)$, we have:
   (i) $M_p/P$ is minimal normal in $G/P$;
   (ii) $\Phi(G) = P \times O_q(G)$;
   (iii) $\Phi(G) \leq K$ (and so $P = K \cap M_p$);
   (iv) $K/\Phi(G)$ is minimal non-abelian;
   (v) $P \leq Z(M)$ (and so the class of $M_p$ is $\leq 2$);
   (vi) $M_p'$ has exponent $\leq p$;
(vii) if $K_p = P(c)$, then $P = \langle c^p \rangle \times Q$ with $Q$ elementary abelian and $M_p = \Omega(M_p)\langle c^p \rangle$ where $\Omega(M_p) = \{x \in M_p \mid x^p = 1\}$.

2. Classification of the soluble, minimal non-$p$-supersoluble groups with trivial Frattini subgroup ([10]). We report this classification, modifying the notation of [10] according to that used in Section 1.

(A) Let $p, q, s$ be primes such that $q \mid p - 1$ and $s \neq q$. Let $K = K_qK_s$ be the subgroup of $GL(s, p)$ defined as follows (equating indexes modulo $s$):

$$K_s = \langle [\gamma_i \delta_{i,j}]_{1 \leq i,j \leq s} \rangle$$

where $\gamma_i = 1$ for $i = 1, \ldots, s - 1$ and $\gamma_s$ is of order $s^k \mid p - 1$ ($k \geq 0$). If $s \mid q - 1$, then

$$K_q = \langle [m^{t-i-1} \delta_{i,j}]_{1 \leq i,j \leq s} \rangle$$

where $m$ is an $q$th root of unity, $2 \leq t \leq q - 1$ and $t^s \equiv 1 \pmod{q}$.

If $s \nmid q - 1$, then

$$K_q = \bigwedge_{i=0}^{r-1} [m_i \delta_{i,j}]_{1 \leq i,j \leq s}$$

where $r = \exp(q, s)$, $m_i^{q^t} = 1$ ($i = 1, \ldots, s$; $t = 0, \ldots, r - 1$) and $m_{i+r} = m_i \delta_{i+1,j}$ ($i = 1, \ldots, s$), $x^r - \beta_k x^{r-1} - \ldots - \beta_1$ being the minimal polynomial over GF$(q)$ of an element of GF$(q^n) \times$ of order $s$. The holomorph of an elementary abelian group of order $p$ by $K$ will be denoted by $\Gamma(p, q, s)$.

(B) Let $p, q$ be primes and $h$ an integer such that $q^h \mid p - 1$. Let $K = \langle a, b \rangle$ be the $q$-subgroup of $GL(q, p)$ defined as follows (equating indexes modulo $q$):

$$a = [m^{(1+q^{h-1})^r-1} \delta_{i,j}]_{1 \leq i,j \leq q}, \quad b = [\delta_{i,j+1}]_{1 \leq i,j \leq q}$$

where $m$ is a primitive $q$th root of unity. The holomorph of an elementary abelian group of order $p^q$ by $K$ will be denoted by $\Delta(p, q, h)$.

(C) Let $p, q$ be primes such that $q \mid p - 1$ and $q \neq 2$. Let $K$ be the subgroup (extraspecial of order $q^3$) of $GL(q, p)$ defined as follows (equating indexes modulo $q$): $K = \langle a, b \rangle$ where

$$a = [m^{l-1} \delta_{i,j}]_{1 \leq i,j \leq q}, \quad b = [\delta_{i,j+1}]_{1 \leq i,j \leq q}$$

with $m^q = 1$ and $l$ a primitive $q$th root of unity. The holomorph of an elementary abelian group of order $p^3$ by $K$ will be denoted by $\Delta(p, q)$.

(D) Let $p$ be a prime such that $4 \mid p - 1$ and let $K$ be the subgroup
(≈ Q₈) of GL(2, p) defined by

\[ K = \left\langle \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \]

where \( m \) is a primitive 4th root of unity. The holomorph of an elementary abelian group of order \( p^2 \) by \( K \) will be denoted by \( \Theta(p) \).

(E) Let \( p, q \) be different primes and \( m \) a positive integer. With \( n = \exp(p, q^m) \), \( A(p, q, m) \) will denote the holomorph of the additive group of the Galois field \( GF(p^n) \) by the subgroup \( \tau \) of order \( q^m \) of the Singer cycle of \( GL(n, p) \cong \text{Aut} GF(p^n)(+) \); i.e. \( x\tau = \lambda x \ (x \in GF(p^n)) \) where \( \lambda \) is a primitive \( q^m \)th root of unity in \( GF(p^n) \).

2.1. Theorem (Kontorovich–Nagrebetskiǐ [10]). Let \( p \) be a prime. A group \( G \) is soluble, minimal non-\( p \)-supersoluble with \( \Phi(G) = 1 \) if and only if \( G \) is isomorphic to one of the following groups:

(A) \( \Gamma(p, q, s) \) with \( s = p \) or \( s \mid p - 1 \);
(B) \( \Delta(p, q, h) \);
(C) \( \Delta(p, q) \);
(D) \( \Theta(p) \);
(E) \( \Lambda(p, q, m) \) with \( q^m - 1 \mid p - 1 \) and \( q^m \nmid p - 1 \).

3. Structure of the soluble, minimal non-\( p \)-supersoluble groups

3.1. Let \( G \) be a soluble, minimal non-\( p \)-supersoluble group without normal Sylow subgroups. With the notation of 1.9 we have \( P = \Phi(M_p) \langle \gamma \rangle \).

Proof. Without loss of generality, assume \( O_q(G) = 1 \) \( (q \neq p) \). Since \( P \leq Z(M) \) (see 1.9) and \( G/P \cong \Gamma(p, q) \) (Theorem 2.1), we can assume, with \( |P| = p^n \ (n \geq 0) \), that \( G_q = \bigtimes_{t=0}^{r-1} (a_t) \) where

\[ a_t = \begin{bmatrix} [m_{i+1}\delta_{i,j}]_{1 \leq i,j \leq p} & 0 \\ 0 & [\delta_{i,j}]_{1 \leq i,j \leq n} \end{bmatrix} \]

and

\[ c = \begin{bmatrix} [\delta_{i,j+1}]_{1 \leq i,j \leq p} & [\lambda_{i,j}]_{1 \leq i \leq p, 1 \leq j \leq n} \\ 0 & [\gamma_{i,j}]_{1 \leq i,j \leq n} \end{bmatrix} \]

with

\[ c^{-1}a_t c = a_{t+1} \ (t = 0, \ldots, r - 2), \quad c^{-1}a_r c = a_1^{\beta_1} \cdots a_{r-1}^{\beta_{r-1}} \]

and with the same notation as in (A) of Section 2. From (3.1) it follows that, for each \( t = 0, \ldots, r - 1 \),

\[ [\delta_{i,j-1}]_{1 \leq i,j \leq p} [(m_{i+1} - 1)\delta_{i,j}]_{1 \leq i,j \leq p}[\lambda_{i,j}]_{1 \leq i \leq p, 1 \leq j \leq n} = 0, \]

from which we deduce

\[(m_{i+t+1} - 1)\lambda_{i+1,j} = 0 \]
for each: $i = 1, \ldots, p; j = 1, \ldots, n; t = 0, \ldots, r - 1$ (equating indexes modulo $p$). Since the indexes are arbitrary, as in (A) of Section 2 we conclude that $\lambda_{ij} = 0$ for each $i = 1, \ldots, p$ and $j = 1, \ldots, n$. Thus $G$ splits on $P$ and so, as $P = \Phi(G)$, we get $P = 1$.

3.2. Let $G$ be a minimal non-$p$-supersoluble group such that $G/\Phi(G)$ is one of the groups (A), (B), (C) of Theorem 2.1. Then $O_p(G)$ is abelian.

Proof. Without loss of generality, assume $O_p(G) = 1$. We examine separately the different cases.

Case 1: $G/\Phi(G) \simeq \Gamma(p, q, s)$ and $\exp(q, s) = r > 1$. We can assume (see 3.1 and 1.6, 1.9) $G = O_p(G)G_q(c)$ where $O_p(G) = \langle x_1, \ldots, x_s, c^{p^r} \rangle$, with $\varepsilon = 0$ if $s \neq p$, $\varepsilon = 1$ if $s = p$, and $\langle x_1, \ldots, x_s \rangle$ of exponent $p$; $G_q = \bigtimes_{i=0}^{r-1} \langle a_i \rangle$

\begin{equation}
\alpha_i^{-1} x_i = x_i^{m_i+t} y_{i,t} \quad (i = 1, \ldots, s; t = 0, \ldots, r - 1; y_{i,t} \in \Phi(O_p(G))(c^{p^r}))
\end{equation}

and

\begin{equation}
c^{-1} x_i c = x_{i-1} z_i \quad (i = 1, \ldots, s - 1; z_i \in \Phi(O_p(G))(c^{p^r}))
\end{equation}

with the same notation as in (A) of Section 2. As $\Phi(O_p(G))(c^{p^r}) \leq Z(O_p(G)G_q)$ (see 1.9), we have

\begin{equation}
a_{i+1}^{-1} [x_i, x_j] a_i = [x_i, x_j]^{m_i+t} m_j+t
\end{equation}

for each $i, j = 1, \ldots, s$ and $t = 0, \ldots, r - 1$. It follows that if $[x_i, x_j] \neq 1$ then

\begin{equation}
m_{i+t} m_{j+t} \equiv 1 \pmod{p}
\end{equation}

for each $t = 0, \ldots, r - 1$. On the other hand, from (3.3) it follows that, for every integer $k$, we have (equating indexes modulo $s$)

\begin{equation}
c^k[x_i, x_j] c^{-k} = [x_{i+k}, x_{j+k}]^\beta \quad (0 \leq \beta \leq p - 1)
\end{equation}

for each $i, j = 1, \ldots, s$; we deduce that if $[x_i, x_j] \neq 1$ then also $[x_{i+k}, x_{j+k}] \neq 1$, and so, by (3.4), we obtain

\begin{equation}
m_{i+k} m_{j+k} \equiv 1 \pmod{p}
\end{equation}

for each integer $k$ (equating indexes modulo $s$).

Now, suppose $O_p(G)$ is non-abelian. As $c^{p^r} \in Z(G)$ and $G/\Phi(G) \simeq \Gamma(p, q, s)$, for any $i = 1, \ldots, s$ there exists $j = 1, \ldots, s$ such that (3.5) holds. As $k$ is arbitrary, it follows that $m_i^2 \equiv 1 \pmod{p}$ for each $i = 1, \ldots, s$ and
so, as \( s \neq 2 \), we have \( q = 2 \). We can then assume
\[
m_1 \equiv \ldots \equiv m_h \equiv 1 \pmod{p} \quad (1 \leq h \leq s - 1).
\]
As \( m, m_j \equiv -1 \pmod{p} \) (\( i = 1, \ldots, h; \ j = h + 1, \ldots, s \)), it follows that
\[
[x_i, x_j] = 1 \text{ for each } i = 1, \ldots, h \text{ and } j = h + 1, \ldots, s, \text{ from which we get}
\]
\[
1 = e^{[x_i, x_j]c_k} = [x_i+k, x_j+k]
\]
for every integer \( k \) and for each \( i = 1, \ldots, h \) and \( j = h + 1, \ldots, s \). It follows, obviously, that \( [x_i, x_j] = 1 \) for each \( i, j = 1, \ldots, s \) and so \( O_p(G) \) is abelian, which contradicts the hypothesis.

**Case** 2: \( G/\Phi(G) \simeq I(p, q, s) \) and \( s \mid p - 1 \). In this case \( O_p(G) = G_p = \langle x_1, \ldots, x_s \rangle \) is of exponent \( p \), \( G_q = \langle a \rangle \) is of order \( q \) and we can assume
\[
a^{-1}x_ia = x_1^{m^{i-1}}y_i \quad (i = 1, \ldots, s; \ y_i \in \Phi(G_p))
\]
with the same notation as in (A) of Section 2. As \( \Phi(G_p) \leq Z(G) \), we have
\[
a^{-1}[x_i, x_j]a = [x_i, x_j] = [x_i, x_j]^{m^{i+j-2}} \quad (i, j = 1, \ldots, s).
\]
It follows that if \( [x_i, x_j] \neq 1 \) then
\[
m^{i+j-2} \equiv 1 \pmod{p},
\]
from which, as \( m \neq 1 \pmod{p} \) and so \( \exp(m, p) = q \), we obtain \( t^{i+j-2} \equiv 0 \pmod{q} \), which is false, since \( 2 \leq t \leq q - 1 \). Thus \( [x_i, x_j] = 1 \) for each \( i, j = 1, \ldots, s \), that is, \( G_p \) is abelian.

**Case** 3: \( G/\Phi(G) \simeq \Delta(p, q, s) \). As in the previous case, \( O_p(G) = G_p = \langle x_1, \ldots, x_s \rangle \) is of exponent \( p \). Moreover, \( G_q = \langle a, b \mid a^q = b^q = 1, b^{-1}ab = a^{1+q^{k-1}} \rangle \) and we can assume
\[
a^{-1}x_ia = x_1^{m(1+q^{k-1})^{j-1}}y_i \quad (i = 1, \ldots, q; \ y_i \in \Phi(G_p)),
\]
\[
b^{-1}x_1b = x_{i-1}z_i \quad (i = 1, \ldots, q; \ z_i \in \Phi(G_p)),
\]
with the same notation as in (B) of Section 2. As \( \Phi(G_p) \leq Z(G) \), we have
\[
a^{-1}[x_i, x_j]a = [x_i, x_j] = [x_i, x_j]^{m(1+q^{k-1})^{i-1}+1+q^{k-1})^{j-1}},
\]
from which, if \( [x_i, x_j] \neq 1 \) (\( i < j \)), we obtain
\[
m(1+q^{k-1})^{i-1}+(1+q^{k-1})^{j-1} \equiv 1 \pmod{p}
\]
and so, as \( \exp(m, p) = q^h \), we get \( (1+q^{k-1})^{i-1} + 1 \equiv 0 \pmod{q^h} \), therefore, obviously, \( q = 2 \). Thus \( G_p = \langle x_1, x_2 \rangle \), and from (3.6) we get
\[
[x_1, x_2] = b^{-1}[x_1, x_2]b = [x_2, x_1] = [x_1, x_2]^{-1},
\]
hence \( [x_1, x_2] = 1 \), that is, \( G_p \) is abelian.
Case 4: $G/\Phi(G) \simeq \Delta(p,q)$. As in the previous cases, $O_p(G) = G_p = \langle x_1, \ldots, x_q \rangle$ is of exponent $p$, $G_q$ is extraspecial of order $q^3$ and exponent $q$, and we can assume, if $G_q = \langle a,b \rangle$,
\[
\begin{align*}
    a^{-1}x_ia & = x_i^{\nu_i-1}y_i \\ 
    b^{-1}x_ib & = x_{i-1}z_i 
\end{align*} 
\]
with the same notation as in (C) of Section 2. As $\Phi(G_p) \leq Z(G)$, we have
\[
[a,b]^{-1}[x_i,x_j][a,b] = [x_i,x_j] = [x_i,x_j]^l. 
\]
It follows that if $[x_i,x_j] \neq 1$ then $l^2 \equiv 1 \pmod{p}$, which is false, since $l \neq 1$ and $q \neq 2$. Thus $[x_i,x_j] = 1$, so $G_p$ is abelian.

3.3. Remark. Proposition 3.2 assures that if $G$ is soluble, minimal non-$p$-supersoluble without normal Sylow subgroups then $O_p(G) = M_p$ is abelian (the notation is that of 1.9). As $M_p = \Omega(M_p)(e^p)$ we then have $M_p = N \times \langle e^p \rangle$ where $N$ is elementary abelian of order $p^s$. Let now $p$ and $q$ be primes such that $q \mid p - 1$, let $K = K_qK_p$ ($K_q \triangleleft K$) be a minimal non-abelian group and let $\psi = \pi\sigma$ be the homomorphism $K \to GL(p,p)$, where $\pi$ and $\sigma$ are respectively the canonical homomorphism $K \to K/\Phi(K_p)$ and the immersion of $K/\Phi(K_p)$ in $GL(p,p)$ considered in (A) of Section 2. If $N$ is an elementary abelian group of order $p^n$, let $G$ be the semidirect product $K \ltimes_o N$. Then $G$ is soluble, minimal non-$p$-supersoluble and without normal Sylow subgroups. Such a semidirect product will be denoted by $G^*(p,q,n)$, where $p^n = |K_q|$ (if $n = 1$, then $G^*(p,q,n) = G(p,q)$).

The following proposition provides the structure of the soluble, minimal non-$p$-supersoluble groups without normal Sylow subgroups in terms of $G^*(p,q,n)$.

3.4. Let $G$ be a group without normal Sylow subgroups. Then $G$ is soluble and minimal non-$p$-supersoluble if and only if $G/O_q(G) \simeq G^*(p,q,n)$ and $O_q(G) = \Phi(G)_q$ ($\pi(G) = \{p,q\}$, $p > q$).

Proof. The condition is obviously sufficient. Let now $G$ be soluble, minimal non-$p$-supersoluble without normal Sylow subgroups. We have (see 1.9 and Theorem 2.1) $G/\Phi(G) \simeq G(p,q)$, $\Phi(G) = \Phi(M_p)(e^p) \times O_q(G)$ (see 3.1) and so, by 3.3, $\Phi(G) = \langle e^p \rangle \times O_q(G)$. Again by Remark 3.3, we get $\Omega(M_p) = N \times \langle e^p \rangle$ ($N$ elementary abelian of order $p^n$, $o(e) = p^s$). Arguing as in the proof of 3.1, with $O_q(G) = 1$ and supposing $n > 1$, we can assume, as $e^p \in Z(G)$, that
\[
G_q(e^p) = \langle d \rangle \left( \bigotimes_{i=0}^{r-1} (a_i) \right) \leq \text{Aut}(M_p) = GL(p+1,p), 
\]
where
\[ a_t = \begin{bmatrix} m_{i+j+1} & \delta_{i,j+1} \\ 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} \delta_{i,j+1} & \lambda_i \\ 0 & 1 \end{bmatrix} \]
and
\[ d^{-1}a_t d = a_{t+1} \quad (t = 0, \ldots, r - 2), \quad d^{-1}a_{r-1} d = a_0^\beta \cdots a_{r-1}^\beta, \]
with the same notation as in (A) of Section 2. Arguing exactly as in the proof of 3.1 we obtain \( \lambda_i = 0 \) for each \( i = 1, \ldots, p \), and so, obviously, \( G \simeq \Gamma^*(p, q, n) \).

3.5. Remark. The statement of 3.2 is not true if \( G/\Phi(G) \simeq \Theta(p) \) or \( \Lambda(p, q, n) \), as the following examples show.

3.5.1. Example. Let \( P \) be an extraspecial group of order \( p^3 \) and exponent \( p \), with \( 4 \mid p - 1 \). With \( P = \langle x_1, x_2 \rangle \), let \( \langle \sigma, \tau \rangle \simeq Q_8 \) be the subgroup of \( \text{Aut} P \) defined as follows:
\[ x_1^\sigma = x_1^m, \quad x_2^\sigma = x_2^{m-1}, \quad x_1^\tau = x_2, \quad x_2^\tau = x_1^{-1}, \]
where \( m \) is a primitive 4th root of unity. The holomorph of \( P \) by \( \langle \sigma, \tau \rangle \) is minimal non-\( p \)-supersoluble and its Sylow \( p \)-subgroup is not abelian. Such a holomorph will be denoted by \( \Theta^*(p) \).

3.5.2. Example. Let \( P \) be as in the previous example and \( 4 \nmid p - 1 \). Let \( \sigma \) be the automorphism of \( P \) defined as follows:
\[ x_1^\sigma = x_2[x_1, x_2]^{n_1}, \quad x_2^\sigma = x_1^{-1}[x_1, x_2]^{n_2}, \]
with \( n_1 \) and \( n_2 \) integers (between 0 and \( p - 1 \)). The holomorph of \( P \) by \( \langle \sigma \rangle \) is minimal non-\( p \)-supersoluble and its Sylow \( p \)-subgroup is not abelian. Such a holomorph will be denoted by \( \Lambda^*(p, n_1, n_2) \).

3.5.3. Example. Further examples of soluble, minimal non-\( p \)-supersoluble groups whose Sylow \( p \)-subgroups are not abelian are all minimal non-\( p \)-nilpotent groups with \( G_p \) non-abelian. As a minimal non-\( p \)-supersoluble group is minimal non-\( p \)-nilpotent if and only if \( G/\Phi(G) \simeq \Lambda(p, q, 1) \), a minimal non-\( p \)-nilpotent group with \( O_2(G) = 1 \) \((q \neq p)\) will be denoted by \( \Lambda^*(p, q) \). The structure of minimal non-\( p \)-nilpotent groups is well known (\([11]\)).

3.6. Let \( G \) be a minimal non-\( p \)-supersoluble group such that \( G/\Phi(G) \simeq \Theta(p) \). Then \( G/O_2(G) \) is isomorphic either to \( \Theta(p) \) or to \( \Theta^*(p) \).

Proof. Let \( O_2(G) = O_2'(G) = 1 \). We have \( G_p = \langle x_1, x_2 \rangle \) of exponent \( p \), \( G_2 = \langle a, b \rangle \simeq Q_8 \), and we can assume
\[ a^{-1}x_1a = x_1^{m_1}y^{n_1}, \quad b^{-1}x_1b = x_2^{n_3} \]
\[ a^{-1}x_2a = x_2^{m_2}y^{n_2}, \quad b^{-1}x_2b = x_1^{-1}y^{n_4} \quad (y = [x_1, x_2]), \]
where $m$ is a primitive 4th root of unity and $n_i$ ($i = 1, \ldots , 4$) are integers (between 0 and $p - 1$). Since $y \in Z(G)$, we get
\[
a^{-2}x_1a^2 = x_1^{-1}y^{(m+1)n_1} = b^{-2}x_1b^2 = x_1^{-1}y^{n_3+n_4} = (ab)^{-2}x_1(ab)^2
\]
and
\[
a^{-2}x_2a^2 = x_2^{-1}y^{(m-1)n_2} = b^{-2}x_2b^2 = x_2^{-1}y^{n_4-n_3} = (ab)^{-2}x_2(ab)^2
\]
It follows that if $y \neq 1$ then $n_1, \ldots , n_4$ is a solution of the linear system
\[
(m + 1)\xi_1 - \xi_3 - \xi_4 = 0, \\
(m^{-1} + 1)\xi_2 + \xi_3 - \xi_4 = 0, \\
\xi_1 + m\xi_2 + (m - 1)\xi_3 = 0, \\
m^{-1}\xi_1 + m^{-1}\xi_2 + \xi_3 - m^{-1}\xi_4 = 0,
\]
which, as its matrix is non-singular, has only the trivial solution; hence, obviously, $G \simeq \Theta(p)$ or $\Theta^*(p)$.

3.7. Let $G$ be a minimal non-$p$-supersoluble group such that $G/\Phi(G) \simeq \Lambda(p,q,m)$ with $m > 1$. Then either $G/O_q(G) \simeq \Lambda(p,q,m)$ or $G/O_2(G) \simeq \Lambda^*(p,n_1,n_2)$.

Proof. Let $O_q(G) = 1$. As $m > 1$, and so $\exp(p,q^m) = q$, we have $G_p = \langle x_1\ldots,x_q \rangle$ and, if $G_p$ is abelian, then $G \simeq \Lambda(p,q,m)$. Let now $G_p$ be non-abelian and so (see 1.6) special of exponent $p$. Then (see, for instance [6], Th. 6.5) $q^m$ divides $p^r + 1$ for some integer $r \leq q/2$. As $m > 1$, we get $q = 2$ and so $G_p$ is extraspecial of order $p^3$ (and exponent $p$). We can then assume $G_p = \langle x_1, x_2 \rangle$, $G_2 = \langle b \rangle$ with
\[
b^{-1}x_1b = x_2y^{n_1}, \\
b^{-1}x_2b = x_1^{n_2}x_2^{\beta_2}y^{n_2} \\
(y = [x_1, x_2]),
\]
where $n_1$ and $n_2$ are integers (between 0 and $p - 1$) and $x^2 - \beta_2x - \beta_1 \in GF(p)[x]$ is the minimal polynomial of an element $\lambda$ of $GF(p^2)$ of order $2^m$. We have obviously $\lambda^2 \in GF(p)$. On the other hand, as $y \in Z(G)$, we get
\[
[x_1, x_2] = b^{-1}[x_1, x_2]b = [x_2, x_1^{\beta_1}] = [x_1, x_2]^{-\beta_1},
\]
from which we deduce $\beta_1 = -1$. It follows that $m = 2$ and so $G \simeq \Lambda^*(p,n_1,n_2)$.

From the results of this section a theorem follows that provides the structure of soluble, minimal non-$p$-supersoluble groups.

3.8. Theorem. Let $p$ be a prime. A group $G$ is soluble and minimal non-$p$-supersoluble if and only if $O_{p'}(G) = \Phi(G)p'$ and $G/O_{p'}(G)$ is isomor-
phic to one of the following groups:

- (a) $\Gamma(p,q,s), s \mid p - 1$;
- (b) $\Gamma^*(p,q,n)$;
- (c) $\Delta(p,q,h)$;
- (d) $\Delta(p,q)$;
- (e) $\Theta(p)$;
- (f) $\Theta^*(p)$;
- (g) $A(p,q,m), q^{m-1} \mid p - 1, q^m \nmid p - 1, m > 1$;
- (h) $A^*(p,n_1,n_2)$;
- (i) $A^*(p,q)$.

4. Non-soluble, minimal non-$p$-supersoluble groups. Minimal non-$p$-supersoluble groups are not necessarily soluble. For instance, $\text{PSL}(2,p)$ ($p$ prime $> 3$) is minimal non-$p$-supersoluble. Now we show how the study of non-soluble, minimal non-$p$-supersoluble groups can be reduced to that of simple groups.

From 1.2 we deduce immediately the following proposition.

**4.1.** Let $G$ be a non-soluble, minimal non-$p$-supersoluble group. Then $F(G) = \Phi(G)$.

**4.2.** Let $G$ be a non-soluble, minimal non-$p$-supersoluble group. Then $G/\Phi(G)$ is simple.

*Proof.* Let $G$ be a counterexample of least order and so $\Phi(G) = 1$. We have, by 4.1, $O_p(G) = 1$. If $N$ is a minimal normal subgroup of $G$, from this it follows that $N$ is a $p'$-group. If $G = MN$ ($M$ maximal in $G$) we find that $G$ is $p$-supersoluble: a contradiction.

**4.3.** Let $G$ be a non-soluble, minimal non-$p$-supersoluble group. Then $O_p(G) \leq Z(G)$.

*Proof.* If $C = C_G(O_p(G)) < G$, we have, by 4.2, $C \leq \Phi(G)$. Let $M$ be a maximal subgroup of $G$. Since there exists a supersoluble immersion of $O_p(G)$ in $M$, we conclude that $M/C$ is supersoluble; hence $G/C$ is soluble, which contradicts the hypothesis.

From 4.2 and 4.3 we immediately deduce the following theorem.

**4.4. Theorem.** Let $G$ be a non-soluble group and let $p$ be an odd prime. Then $G$ is minimal non-$p$-supersoluble if and only if $G/\Phi(G)$ is simple, minimal non-$p$-supersoluble and $O_p(G) \leq Z(G)$.

The next results provide a classification of simple, minimal non-$p$-supersoluble groups if $p$ is the smallest odd prime that divides the order of the group. In the proof of one of the propositions we use the classification of the finite simple groups.

**4.5.** Let $G$ be a minimal non-3-supersoluble group. Then all proper subgroups of $G$ are soluble.
Proof. Let $G$ be a counterexample of least order. Since $G/\Phi(G)$ is, as $G$, minimal non-3-supersoluble, we have obviously $\Phi(G) = 1$ and so $O_3(G) = 1$, and, by 4.1, $O_3(G) = 1$. If $N$ is a minimal normal subgroup of $G$ and $N \neq G$, we have obviously $3 \nmid |N|$, which is false, as $O_3(G) = 1$. Thus $G$ is simple. Since $G$ is not a minimal simple group, let $H$ be a proper simple non-abelian subgroup of $G$. As $H$ is 3-supersoluble, we have $3 \nmid |H|$, and therefore $H$ is isomorphic to a Suzuki group $S_z(2^{2n+1})$. Thus the proper simple non-abelian subgroups of $G$ are isomorphic to Suzuki groups; from this, using the classification of the finite simple groups (see for instance [2]), it follows that $G$ itself is a Suzuki group and so $G$ is 3-supersoluble, since $3 \nmid |G|$: a contradiction.

4.6. The Suzuki group $S_z(2^{2n+1})$ is minimal non-5-supersoluble if and only if $2n + 1$ is prime.

Proof. If $2n + 1$ is not prime, denote by $2m + 1$ a proper divisor (≠ 1) of $2n + 1$. Then $S_z(2^{2n+1})$ has a subgroup isomorphic to $S_z(2^{2m+1})$ (see for instance [8]), which is not 5-supersoluble. Conversely, let $2n + 1 = q$ be prime. Then (see for instance [8]) the only non-supersoluble subgroups of $S_z(2^q)$ are Frobenius groups whose kernel are 2-groups (of order $2^{q-1}$) and whose complements are cyclic (of order $2^q - 1$). Such groups are obviously 5-supersoluble, and therefore $S_z(2^q)$ is minimal non-5-supersoluble.

4.7. Let $G$ be a minimal non-$p$-supersoluble group, where $p$ is the smallest odd prime divisor of $|G|$. Then all proper subgroups of $G$ are soluble. In particular, if $G$ is simple, then $G$ is a minimal simple group.

Proof. If $p = 3$, the statement follows from 4.5. Let now $p \geq 5$ and let $G$ be a counterexample of least order. By similar arguments to the proof of 4.5 we show that $G$ is simple, and therefore, as $3 \nmid |G|$, $G$ is a Suzuki group $S_z(2^{2n+1})$. We then have $p = 5$. Since $G$ is not a minimal simple group, $2n + 1$ is not prime, which contradicts 4.6.

4.8. Theorem. Let $G$ be a simple non-abelian group. Then $G$ is minimal non-$p$-supersoluble with $p$ the smallest odd prime divisor of $|G|$ if and only if $G$ is isomorphic to one of the following groups:

(i) $PSL(2, 2^q)$, $q$ prime;
(ii) $PSL(2, q)$, $q$ prime $> 3$ and $q^2 + 1 \equiv 0 \pmod{5}$;
(iii) $S_z(2^q)$, $q$ prime (≠ 2).

Moreover, the groups (i)–(iii) are, up to isomorphism, the only simple, minimal non-$s$-supersoluble groups for every odd prime $s$ that divides their order.

Proof. A direct analysis (see for instance [7] and [8]) proves that the groups (i)–(iii) are minimal non-$s$-supersoluble for every odd prime $s$ that divides their order. Let now $G$ be simple and minimal non-$p$-supersoluble,
with \( p = \min \pi(G) \setminus \{2\} \). By 4.7, \( G \) is a minimal simple group. The classification of the minimal simple groups due to J. G. Thompson ([15]) provides, besides the groups (i)–(iii), the groups \( \text{PSL}(2, 3^q) \), \( q \) odd prime, and \( \text{PSL}(3, 3) \).

We can exclude \( \text{PSL}(2, 3^q) \), because a Sylow 3-subgroup of \( \text{PSL}(2, 3^q) \) is minimal normal in its normalizer (see for instance [7]) and therefore the latter is not 3-supersoluble. As far as \( \text{PSL}(3, 3) \) is concerned, if we regard it as an automorphisms group of the projective plane \( \Pi \) over GF(3), the stabilizer \( G_\alpha \left( G_r \right) \) of a point (of a line) of \( \Pi \) is isomorphic to the complete holomorph of an elementary abelian group \( P \) of order \( 3^2 \) (\( \text{Aut } P \cong \text{GL}(2, 3) \) is the stabilizer in \( G_\alpha \left( G_r \right) \) of a line (a point) not containing \( \alpha \) (not belonging to \( r \)). Such subgroups are obviously non-3-supersoluble and so \( \text{PSL}(3, 3) \) is not minimal non-3-supersoluble.

The following theorem provides a characterization of minimal simple groups.

\[ 4.9. \text{ Let } G \text{ be a simple non-abelian group. Then } G \text{ is minimal non-}\ p\text{-supersoluble for every prime } p \geq 5 \text{ that divides its order if and only if } G \text{ is a minimal simple group.} \]

\[ \text{Proof.} \] A direct analysis proves that the minimal simple groups are minimal non-\( p \)-supersoluble for every prime \( p \geq 5 \) that divides their order. Vice versa, let \( G \) be simple and minimal non-\( p \)-supersoluble for every prime \( p \geq 5 \) that divides its order. Let \( \omega = \{2, 3\} \). Then \( H/O_\omega(H) \) is supersoluble for every proper subgroup \( H \) of \( G \). It follows that \( H \) is soluble and therefore \( G \) is a minimal simple group.

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DIPARTIMENTO DI MATEMATICA E APPLICAZIONI “RENATO CACCIOPOLI”
UNIVERSITÀ DI NAPOLI
VIA MEZZOCANNONE, 8
I-80134 NAPOLI, ITALY

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