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SOME MODELS OF PLANE GEOMETRIES AND A FUNCTIONAL EQUATION

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Let M_p be the family of all straight lines in $\mathbb{R}^2$ which are parallel to the y axis and of all curves of the form $y = p(x+\alpha)+\beta$, where p is a fixed function and α, β run over $\mathbb{R}$. Notice that, if $p(x) = x^2$, then the map

$$(x, y) \mapsto (x, y + x^2)$$

is a bijection of $\mathbb{R}^2$ which maps the family of all straight lines onto the family M_p. A similar isomorphism exists whenever p is a polynomial function of degree 2. The converse is also true:

THEOREM. *If there exists a continuous bijection ϕ of $\mathbb{R}^2$ onto itself which induces a map of the family of all straight lines onto M_p, then p must be a polynomial function of degree* 2.

This theorem was shown in [2] under the additional condition that p is differentiable. Our purpose is to improve a part of the proof of [2] in order to show the above theorem as stated. Our proof is based on the solution of a functional equation related to the functional equation of Cauchy $f(x+y) = f(x) + f(y)$.

Proof. Since the set $\{(x, y) : y = p(x)\}$ is the image of some straight line by the homeomorphism ϕ, the function p must be continuous. It is shown in [2] that there exists a constant γ such that the function $q(x) = p(x+\gamma)$ satisfies the functional equation

$$q(x) - q(x+t) = (q(1) - q(1+t) - q(0) + q(t))x + q(0) - q(t)\,.$$

From there on the differentiability of q was used. Now we will argue without this assumption.

First we show that, by the supposition of the theorem, neither p nor q can be of the form $ax + b$. In fact, if $p(x) = ax + b$, then M_p consists of all vertical lines and all lines with slope a. But then not every pair of points of $\mathbb{R}^2$ lies on one curve of M_p. So M_p could not be isomorphic to the family of all straight lines.

Thus it suffices to prove that if q is continuous and not linear then q is a polynomial function of degree 2. Since the above equation is invariant under addition of constants to q we can assume without loss of generality that $q(0) = 0$. Now we can rewrite the equation as follows:

$$q(x) + q(t) - q(x+t) = (q(1) - q(1+t) + q(t))x\,. \tag{1}$$

Since the left side of (1) is symmetric in x, t the right side is also symmetric, i.e.,

$$(q(1) - q(1+t) + q(t))x = (q(1) - q(1+x) + q(x))t\,.$$

Thus

$$q(1) - q(1+t) + q(t) = ct\,, \tag{2}$$

where $c = 2q(1) - q(2)$. If $c = 0$ then, by (1) and (2), $q(x) + q(t) = q(x+t)$. Hence by continuity and the theorem of Cauchy q is linear, which was ruled out above. Thus $c \neq 0$. Since (1) is homogeneous in q we can assume without loss of generality that $c = 1$. So, by (1) and (2),

$$q(x) + q(t) = q(x+t) + xt\,. \tag{3}$$

Now, add $\frac{1}{2}(x^2 + t^2)$ to both sides of (3). So we get

$$f(x+t) = f(x) + f(t)\,,$$

where $f(x) = q(x) + x^2/2$. Then, by the theorem of Cauchy, $f(x) = ax$, and it follows that $q(x) = -x^2/2 + ax$. This concludes the proof.

We are indebted to János Aczél and Roman Ger for simplifying our original way of solving (3). They pointed out also that, by well known refinements of the theorem of Cauchy (see [1], [3]), the above argument proves also that all solutions of (3) which are measurable or have the property of Baire must be of the form $-x^2/2 + ax$.

One can ask for extensions of our theorem to higher dimensions. The simplest such question is the following. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and N_p be the family of surfaces which consists of all planes in $\mathbb{R}^3$ which are parallel to the z axis and of all surfaces of the form $z = p(x+\alpha, y+\beta) + \gamma$.

Problem. Characterize those functions p for which N_p is continuously isomorphic to the family of all planes in $\mathbb{R}^3$.

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