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VECTOR-VALUED CALDERÓN–ZYGMUND THEORY APPLIED TO TENT SPACES

BY

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0. Introduction. The concept of "tent space" was introduced by R. Coifman, Y. Meyer and E. M. Stein in [3] and [4]. These spaces seem well adapted for the study of a variety of questions in Harmonic Analysis. The theory developed in [3] and [4] uses a functional which maps functions on $\mathbb{R}_+^{n+1}$ into functions on $\mathbb{R}^n$, given by

$$A_q(f)(x) = \Big\{ \int\limits_{\Gamma(x)} |f(y,t)|^q \, d\alpha(y,t)/t^n \Big\}^{1/q}$$

where $1 < q < \infty$, $\Gamma(x)$ is the cone of aperture 1 whose vertex is $x \in \mathbb{R}^n$, and $d\alpha(y,t) = dy\,dt/t$. The *tent space* $T_p^q(\alpha), 1 \le p, q < \infty$ is defined as the space of functions f such that $A_q(f) \in L^p(\mathbb{R}^n)$.

In this note we study tent spaces $T_q^p(\alpha)$ for different measures α. Our purpose is twofold:

First, we show that the boundedness of an operator T from L^p into $T_q^p(\alpha)$ is equivalent to the boundedness of a related operator $\mathbf{S}$ from L^p to the Bochner–Lebesgue space L_A^p where A is an L^q-space; in some cases the operator $\mathbf{S}$ behaves as a vector-valued Calderón–Zygmund operator (see Theorem 1). The proof of this theorem says that, in some sense, $T_q^p(\alpha)$ is a subspace of L_A^p.

Secondly, in the case that μ is a Carleson measure we show that some operators, associated to particular kernels, are bounded from L^p into $T_q^p(\mu)$. This is applied to the Poisson integral (see Theorems 2 and 3). The method can be extended to vector-valued functions, and then some maximal operators fall under its scope (see Theorem 4).

The organization of this paper is as follows: in Section 1 we introduce some notations and state the main results, in Section 2 some technical re-

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sults related to Carleson measures are presented, in Section 3 we give some applications, and Section 4 is devoted to the proofs.

1. Notations and main results. $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0,\infty)$ will be the usual upper half-space in $\mathbb{R}^{n+1}$. We shall denote by $\Gamma(x)$ the cone of aperture 1, $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. Given a cube Q in $\mathbb{R}^n$, we shall denote by $\widehat{Q}$ the *tent* over Q, i.e. if Q has center x and side length l, then $\widehat{Q} = \{(y,t) : |x-y| + t \le l\}$. A positive measure μ on $\mathbb{R}^{n+1}_+$ will be called a *Carleson measure* if there exists a constant C such that $\mu(\widehat{Q}) \le C|Q|$, for every cube Q in $\mathbb{R}^n$ (where $|Q|$ stands for the Lebesgue measure of Q in $\mathbb{R}^n$). Replacing balls with cubes leads to an equivalent definition.

For A, B Banach spaces, let $\mathcal{L}(A,B)$ stand for the set of bounded linear operators from A into B. We shall denote by $L^p_A(\mathbb{R}^n; dx)$, $1 \le p < \infty$, the Bochner–Lebesgue space of A-valued strongly measurable functions f defined on $\mathbb{R}^n$ such that $\int \|f(x)\|^p_A\, dx < \infty$. Analogously, we define $L^p_B(\mathbb{R}^{n+1}_+; d\mu)$. Sometimes, we shall write $L^p_A\,(dx)$ or $L^p_B\,(d\mu)$, for short. $l^r(A)$, $1 < r < \infty$, stands for the usual space of A-valued r-summable sequences.

The space $H^1_A(\mathbb{R}^n; dx)$ can be defined in terms of A-valued atoms in the usual way (see [5]). In [2] it was proved that the Riesz transforms R_j are defined in $L^1_A(\mathbb{R}^n; dx)$ if the space A is U.M.D., and in this case $H^1_A(\mathbb{R}^n; dx) = \{f \in L^1_A(\mathbb{R}^n; dx) : R_j f \in L^1_A(\mathbb{R}^n; dx), 1 \le j \le n\}$.

Given a positive measure μ on $\mathbb{R}^{n+1}_+$ and $1 \le q < \infty$, we define (see [4]) the following functional over B-valued functions on $\mathbb{R}^{n+1}_+$:

$$A_q(f)(x) = \Big\{ \int\limits_{\Gamma(x)} \|f(y,t)\|^q_B\, d\mu(y,t)/t^n \Big\}^{1/q}, \quad x \in \mathbb{R}^n .$$

The *tent space* $T^p_{q,B}\,(d\mu)$, $1 \le p, q < \infty$, is defined as the space of B-valued strongly measurable functions f such that $A_q(f) \in L^p(\mathbb{R}^n)$. $T^p_{q,B}\,(d\mu)$ is equipped with the norm $\|f\|_{T^p_{q,B}(d\mu)} = \|A_q(f)\|_{L^p}$.

In the following, we shall denote by B^p_q the space $L^p_A(\mathbb{R}^n; dx)$ where A is $L^q_B(\mathbb{R}^{n+1}_+; d\mu/t^n)$. Now we state the main results.

THEOREM 1. *Let μ be either a Carleson measure or the $dx\,dt/t$ measure on $\mathbb{R}^{n+1}_+$, A, B Banach spaces and $1 \le p < \infty$, $1 < q < \infty$. Then the following are equivalent*:

(i) *An operator T is bounded from $L^p_A(\mathbb{R}^n; dx)$ into $T^p_{q,B}\,(d\mu)$.*

(ii) *The operator* $\mathbf{S}$ *given by $\mathbf{S}f(x)(y,t) = Tf(y,t)\chi_{\Gamma(x)}(y,t)$ is bounded from $L^p_A(\mathbb{R}^n; dx)$ into B^p_q.*

Moreover, *if T has an associated kernel $K(x,y,t)$ in the sense of Theorem* 2 *below satisfying* (K.1) *and* (K.2) *then* $\mathbf{S}$ *has an associated*

$\mathcal{L}(A, L^q_B(\mathbb{R}^{n+1}_+; d\mu/t^n))$*-valued kernel* (*in the sense of standard vector-valued theory of singular integrals*, *see* [6]) *given by*

$$\mathbf{K}(x,z)(a)\{(y,t)\} = K(y,z,t)(a)\chi_{\Gamma(x)}(y,t)\,,$$

$a \in A$, $x, z \in \mathbb{R}^n$, $(y,t) \in \mathbb{R}^{n+1}_+$, *and satisfying*

(K.3) *If f is an A-valued function with compact support contained in a cube Q*, *then*

$$\mathbf{S}f(x) = \int\limits_{\mathbb{R}^n} \mathbf{K}(x,z)f(z)\,dz \qquad \textit{for } x \notin Q\,.$$

(K.4) *If* $|x - z'| > 2|z - z'|$ *then*

$$\|\mathbf{K}(x,z) - \mathbf{K}(x,z')\| \le C\frac{|z-z'|}{|x-z'|^{n+1}}\,.$$

THEOREM 2. *Let A and B be Banach spaces and μ a Carleson measure on $\mathbb{R}^{n+1}_+$. Let T be a bounded linear operator from $L^{p_0}_A(\mathbb{R}^n; dx)$ into $L^{p_0}_B(\mathbb{R}^{n+1}_+; d\mu)$ for some p_0, $1 < p_0 \le \infty$. Suppose that there exists an $\mathcal{L}(A,B)$-valued function K in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \setminus \{(x,x,t) : x \in \mathbb{R}^n,\ t \ge 0\}$ such that*:

(K.1) *For any pair $(x,t) \in \mathbb{R}^{n+1}_+$, the function $y \mapsto K(x,y,t)$ is locally integrable and if f is a function in $L^{p_0}_A(\mathbb{R}^n; dx)$ with compact support contained in a cube Q, then*

$$Tf(x,t) = \int\limits_{\mathbb{R}^n} K(x,y,t)f(y)\,dy \qquad \textit{for } (x,t) \notin \widehat{Q}\,.$$

(K.2) *There exists $\alpha > 0$ such that*

$$\|K(x,y,t) - K(x,y',t)\|_{\mathcal{L}(A,B)} \le C\frac{|y-y'|t^{\alpha}}{(|x-y'|+t)^{n+1+\alpha}} \qquad \textit{for } |x-y'| + t > 2|y-y'|\,.$$

Then:

(i) *T maps $L^p_A(\mathbb{R}^n; dx)$ into $T^p_{q,B}(d\mu)$ for $1 < p, q \le p_0$, $q < \infty$.*
(ii) *T maps $H^1_A(\mathbb{R}^n)$ into $T^1_{q,B}(d\mu)$ for $1 < q \le p_0$, $q < \infty$.*
(iii) *T maps $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ into $T^p_{q,l^r(B)}(d\mu)$ for $1 < p,q \le r \le p_0$, $r < \infty$.*
(iv) *T maps $H^1_{l^r(A)}(\mathbb{R}^n)$ into $T^1_{q,l^r(B)}(d\mu)$, $1 < q \le r \le p_0$, $r < \infty$.*

2. Some technical results. In [7] the following are proved:

THEOREM A. *Let A and B be Banach spaces*, *μ a Carleson measure on $\mathbb{R}^{n+1}_+$. Let T be a bounded linear operator from $L^{p_0}_A(\mathbb{R}^n; dx)$ into*

$L^{p_0}_B(\mathbb{R}^{n+1}_+; d\mu)$ *for some* $p_0, 1 < p_0 \le \infty$. *Assume that* T *has an associated kernel* K *satisfying* (K.1) *of Theorem* 2 *and*

$$\text{(K.2}'\text{)} \qquad \|K(x,y,t) - K(x,y',t)\|_{\mathcal{L}(A,B)} \le C\frac{|y-y'|}{(|x-y'|+t)^{n+1}} \quad \text{for } |x-y'|+t > 2|y-y'|\,.$$

Then:

(i) T *maps* $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ *into* $L^p_{l^r(B)}(\mathbb{R}^{n+1}_+; d\mu)$, $1 < p \le r \le p_0$.

(ii) T *maps* $H^1_{l^r(A)}(\mathbb{R}^n)$ *into* $L^1_{l^r(B)}(\mathbb{R}^{n+1}_+; d\mu)$, $1 < r \le p_0$.

Remark 1. When we speak about boundedness of an operator T from $L^p_{l^r(A)}$ into $L^p_{l^r(B)}$ (or $T^p_{q,l^r(B)}(d\mu)$) we mean that the assignment $(f_1, f_2, \ldots) \mapsto (Tf_1, Tf_2, \ldots)$ (where the f_i are A-valued functions) is bounded from $L^p_{l^r(A)}$ into $L^p_{l^r(B)}$ (or $T^p_{q,l^r(B)}(d\mu)$). Observe that the $\mathcal{L}(l^r(A), l^r(B))$-valued kernel associated to this new operator is given by $\overline{K}(x,y,t) = K(x,y,t)\otimes\mathrm{Id}$, and so $\|\overline{K}(x,y,t)\| = \|K(x,y,t)\|$. Therefore this operator is of the same type as T and its kernel satisfies the same bounds.

Theorem B. *Let* A *and* B *be Banach spaces. Let* T *be a linear operator which is bounded from* $L^\infty_A(\mathbb{R}^n; w(x)\,dx)$ *into* $L^\infty_B(\mathbb{R}^{n+1}_+; dv)$ *for every pair* (w,v) *which satisfies condition* C_1, *i.e.* $\sup\{v(\widehat{Q})/|Q| : Q \ni x\} \le Cw(x)$, x-*a.e.* (*see* [7]). *Assume that* T *has an associated kernel* K *satisfying* (K.1) *of Theorem* 2 *and* (K.2′) *of Theorem* A. *Then the following vector-valued inequalities hold for any Carleson measure* μ:

(i) T *maps* $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ *into* $L^p_{l^r(B)}(\mathbb{R}^{n+1}_+; d\mu)$ *for* $1 < p,\ r < \infty$.

(ii) T *maps* $H^1_{l^r(A)}(\mathbb{R}^n)$ *into* $L^1_{l^r(B)}(\mathbb{R}^{n+1}_+; d\mu)$ *for* $1 < r < \infty$.

Remark 2. If in the last theorem A is U.M.D., then (ii) can be written as

$$\Big\|\Big\{\sum_{j=0}^{\infty}\|Tf_j\|_B^r\Big\}^{1/r}\Big\|_{L^1(d\mu)} \le C\sum_{i=0}^{n}\Big\|\Big\{\sum_{j=0}^{\infty}\|R_i f_j\|_A^r\Big\}^{1/r}\Big\|_{L^1\,(dx)}\,,$$

where $R_0 f = f$ and $R_i, i = 1, \ldots, n$, denote the Riesz transforms.

The following result, which we state for further reference, is a consequence of Theorem B.

Proposition 1. *The following conditions are equivalent*:

(i) μ *is a Carleson measure on* $\mathbb{R}^{n+1}_+$.

(ii) *For* $1 < r, p < \infty$

$$\Big\|\Big\{\sum_{j=0}^{\infty}|A_1(f_j)|^r\Big\}^{1/r}\Big\|_{L^p\,(dx)} \le C\Big\|\Big\{\sum_{j=0}^{\infty}|f_j|^r\Big\}^{1/r}\Big\|_{L^p\,(d\mu)}\,.$$

(iii) *For* $1 < q < r$, $p < \infty$

$$\Big\|\Big\{\sum_{j=0}^{\infty}|A_q(f_j)|^r\Big\}^{1/r}\Big\|_{L^p\,(dx)} \le C\Big\|\Big\{\sum_{j=0}^{\infty}|f_j|^r\Big\}^{1/r}\Big\|_{L^p\,(d\mu)}\,.$$

(iv) *The operator* $Tf(x,t) = t^{-n}\int_{B(x;t)} f(y)\,dy$ *is bounded from* $L^p_{l^r}\,(dx)$ *into* $L^p_{l^r}\,(d\mu)$ *for* $1 < r$, $p < \infty$ (*where* $B(x;t)$ *is the ball centered at* x *with radius* t).

(v) *For* $1 \le q \le p < \infty$, $T^p_p\,(d\mu) \subseteq T^p_q\,(d\mu)$.

Proof. To show that (ii)⇔(iii) it is enough to observe that for any q with $1 \le q < \infty$ and f positive, $A_1(f)(x) = \{A_q(f^{1/q})(x)\}^q$. On the other hand, applying Fubini's theorem we have

$$\int A_1 f(x)g(x)\,dx = \int f(y,t)Tg(y,t)\,d\mu(y,t) \quad \text{for } f(x,t) \text{ and } g(x) \text{ positive,}$$

and this identity gives us (ii)⇔(iv).

In order to prove (i)⇒(iv) observe that the operator T can be majorized by the maximal operator $\mathfrak{M}$ introduced by Fefferman and Stein, which satisfies the vector-valued inequalities from $L^p_{l^r}(\mathbb{R}^n;\, dx)$ into $L^p_{l^r}(\mathbb{R}^{n+1}_+;\, d\mu)$ for $1 < p$, $r < \infty$, as a consequence of Theorem B (see [7]).

For the converse, take $B = B(z;s)$ and $(x,t) \in \widehat{B}$; then $B(x;t) \subset B(z;s)$. Now, if $(x,t) \in \widehat{B}$ and $f = \chi_{B(z;s)}$, we have

$$Tf(x,t) = t^{-n}\int\limits_{B(x;t)} \chi_{B(z;s)}(y)\,dy \ge t^{-n}\int\limits_{B(x;t)} \chi_{B(x;t)}(y)\,dy = c_n$$

and therefore

$$\begin{aligned}\mu(\widehat{B}) &\le \mu(\{(x,t) : Tf(x,t) \ge c_n\}) \le c'_n \int |Tf(x,t)|^p\,d\mu \\ &\le C\int |f|^p\,dx \le C|B|\,.\end{aligned}$$

Finally, we shall show that (iii)⇒(v)⇒(i). If we assume (iii), then for $1 \le q < p$ we have

$$\|A_q(f)\|_{L^p\,(dx)} \le C\|f\|_{L^p\,(d\mu)} = c_n\|f\|_{T^p_p\,(d\mu)}\,,$$

where in the last identity we have used the fact $L^p\,(d\mu) = T^p_p\,(d\mu)$ (see Lemma 2 in Section 4).

On the other hand, if we take a ball $B = B(x_0;r)$, we have

$$\begin{aligned}r^{n(1-p/q)}\mu(\widehat{B})^{p/q} &= r^{-np/q}\int\limits_B \mu(\widehat{B})^{p/q}\,dx \\ &= r^{-np/q}\int\limits_B\Big(\int\limits_{\Gamma(x)} |\chi_{\widehat{B}}(y,t)|^q\,d\mu(y,t)\Big)^{p/q}\,dx\end{aligned}$$

$$\leq \int_B \Big(\int_{\Gamma(x)} |\chi_{\widehat{B}}(y,t)|^q \, d\mu(y,t)/t^n \Big)^{p/q} dx$$

$$\leq \int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |\chi_{\widehat{B}}(y,t)|^q \, d\mu(y,t)/t^n \Big)^{p/q} dx \,,$$

and by (v) this is less than

$$\int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |\chi_{\widehat{B}}(y,t)|^p \, d\mu(y,t)/t^n \Big) \, dx = c_n \mu(\widehat{B}) \,.$$

3. Applications. Our first application deals with operators of Poisson type.

THEOREM 3. *Let ϕ be a measurable function on $\mathbb{R}^n$ such that there exists $\alpha > 0$ with*

(a) $|\phi(x)| \leq C(|x| + A)^{-n-\alpha}$ *and* (b) $|\nabla\phi(x)| \leq C(|x| + B)^{-n-1-\alpha}$

where A, B, C are constants independent of x. For the function $\Phi(x,t) = t^{-n}\phi(x/t)$, $t \geq 0$, the operator

$$\Phi f(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) f(y) \, dy \,,$$

and for any Carleson measure μ, the following vector-valued inequalities hold:

$$\Big\| \Big\{ \sum_{j=0}^{\infty} |\Phi(f_j)|^r \Big\}^{1/r} \Big\|_{T_q^p(d\mu)} \leq C_{p,q,r} \Big\| \Big\{ \sum_{j=0}^{\infty} |f_j|^r \Big\}^{1/r} \Big\|_{L^p\,(dx)} \quad \textit{for } 1 < p,q,r < \infty \,, \tag{3.1}$$

$$\Big\| \Big\{ \sum_{j=0}^{\infty} |\Phi(f_j)|^r \Big\}^{1/r} \Big\|_{T_q^1(d\mu)} \leq C_{q,r} \sum_{i=0}^{n} \Big\| \Big\{ \sum_{j=0}^{\infty} |R_i f_j|^r \Big\}^{1/r} \Big\|_{L^1\,(dx)} \quad \textit{for } 1 < q,r < \infty \,. \tag{3.2}$$

Proof. Observe that $|\Phi f(x,t)| \leq \|f\|_\infty \|\phi\|_1$, and thus for any pair (v,w) satisfying condition C_1 (see Theorem B), Φ maps $L^\infty(\mathbb{R}^n; w(x)\,dx)$ into $L^\infty(\mathbb{R}^{n+1}_+; d\mu)$. Moreover, it is easy to check from condition (b) that if $|x-y'| + t > 2|y-y'|$ then

$$|\Phi(x-y,t) - \Phi(x-y',t)| = t^{-n} |\phi((x-y)/t - \phi((x-y')/t)|$$
$$\leq C \frac{|y-y'| t^\alpha}{(|x-y'|+t)^{n+1+\alpha}} \,,$$

and so $\Phi(x-y,t)$ satisfies (K.2) of Theorem 2 and, in particular, (K.2′) of Theorem B. Therefore, Φ is bounded from $L^p_{l^r}(\mathbb{R}^n; dx)$ into $L^p_{l^r}(\mathbb{R}^{n+1}_+; d\mu)$

for $1 < p, r < \infty$ and any Carleson measure μ.

Summarizing, Φ satisfies the hypothesis of Theorem 2 with $A = B = l^r$, $1 < r < \infty$, and any p_0 with $1 < p_0 < \infty$, and Theorem 3 is a consequence of Theorem 2.

R e m a r k 3. In the case of positive linear operators, extensions to vector-valued functions are trivial. Therefore, if ϕ is positive then the vector-valued inequalities (3.1) and (3.2) remain true for $1 \le r \le \infty$. This is the case for the Poisson kernel $P(x,t) = \Phi(x,t)$, where $\phi(x) = P(x) = c_n(|x|^2+1)^{-(n+1)/2}$ with $c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$.

The next application can be viewed as Zo's maximal theorem (see [9]):

THEOREM 4. *Let μ be a Carleson measure and ϕ a measurable function in $\mathbb{R}^{n+1}_+$ such that*

(a) $\int_{\mathbb{R}^n} |\phi(x,t)|\,dx \le A < \infty, \forall t \ge 0$,
(b) $|\nabla_x \phi(x,t))| \le Ct^\alpha/(|x|+t)^{n+1+\alpha}$ *for some* $\alpha > 0$.

Then the operator

$$\mathfrak{M}_\phi f(x,t) = \sup_{\delta>0} \Big|\delta^{-n} \int\limits_{\mathbb{R}^n} \phi((x-y)/\delta, t/\delta) f(y)\,dy\Big|$$

satisfies the vector-valued inequalities (3.1) *and* (3.2).

P r o o f. Let S be the linear operator defined by

$$Sf(x,t) = \Big\{\delta^{-n} \int\limits_{\mathbb{R}^n} \phi((x-y)/\delta, t/\delta) f(y)\,dy\Big\}_{\delta>0}.$$

By (a) it is clear that S is bounded from $L^\infty(\mathbb{R}^n; w(x)\,dx)$ into $L^\infty_{l^\infty}(\mathbb{R}^{n+1}_+; dv)$ for any pair (v,w) satisfying C_1; moreover, S is given by an $\mathcal{L}(\mathbb{C}, l^\infty) \equiv l^\infty$-valued kernel $K(x,y,t) = \{\delta^{-n}\phi((x-y)/\delta, t/\delta)\}_{\delta>0}$ which satisfies (K.2) since ϕ satisfies (b). Therefore, by Theorem B, S is bounded from $L^p_{l^r}(\mathbb{R}^n; dx)$ into $L^p_{l^r(l^\infty)}(\mathbb{R}^{n+1}_+; d\mu)$ for $1 < p, r < \infty$ and from $H^1_{l^r}(dx)$ into $L^1_{l^r(l^\infty)}(\mathbb{R}^{n+1}_+; d\mu)$. Thus Theorem 2 applies, and S is bounded from $L^p_{l^r}(\mathbb{R}^n; dx)$ into $T^p_{q,l^r(l^\infty)}\,d\mu)$ for $1 < p, q, r < \infty$ and from $H^1_{l^r}(dx)$ into $T^1_{q,l^r(l^\infty)}(d\mu)$, $1 < q, r < \infty$. The result follows by observing that $\|Sf(x,t)\|_{l^\infty} = \mathfrak{M}_\phi f(x,t)$.

COROLLARY 1. *Given ε, $0 < \varepsilon < 1$, we define the maximal operator $\mathfrak{M}_\varepsilon f(x,t) = \sup |Q|^{-1} \int_Q |f(y)|\,dy$, where the supremum is taken over the cubes in $\mathbb{R}^n$ containing x and having side length $l(Q)$ such that $\varepsilon l(Q) \le t \le l(Q)$. Then $\mathfrak{M}_\varepsilon$ satisfies the vector-valued inequalities* (3.1). (Observe that in the limiting case $\varepsilon = 0$ this operator is $\mathfrak{M}$.)

Proof. Take a differentiable function ϕ_ε on $\mathbb{R}^{n+1}_+$ such that if Q_0 is the unit cube in $\mathbb{R}^n$, $A = Q_0 \times ([-1,1] - [-\varepsilon,\varepsilon])$, $B = 2Q_0 \times [-2,2]$ then $\chi_A \le \phi_\varepsilon \le \chi_B$ and $|\nabla_x \phi_\varepsilon(x,t))| \le C_\varepsilon t^\alpha/(|x|+t)^{n+1+\alpha}$ for some $\alpha > 0$. Finally, observe that $\mathfrak{M}_\varepsilon f(x,t) \le \mathfrak{M}_{\phi_\varepsilon} f(x,t)$ and apply Theorem 4 (notice that we have used cubes instead of balls and therefore some constants depending only on the dimension and ε should appear in the last inequality).

4. Proofs

LEMMA 1. *Let μ be a Carleson measure. Let $\varepsilon, b > 0$ and define $\Gamma^b(x) = \Gamma(x) \cap \{(y,t) \in \mathbb{R}^{n+1}_+ : t > b\}$ and $\Gamma_b(x) = \Gamma(x) \cap \{(y,t) \in \mathbb{R}^{n+1}_+ : t < b\}$. Then*

$$\text{(i)} \quad \int_{\Gamma^b(x)} t^{-n-\varepsilon}\, d\mu(y,t) \le Cb^{-\varepsilon}, \qquad \text{(ii)} \quad \int_{\Gamma_b(x)} t^{-n+\varepsilon}\, d\mu(y,t) \le Cb^{\varepsilon}.$$

Proof. Let $\Gamma^b_j = \Gamma(x) \cap \{(y,t) \in \mathbb{R}^{n+1}_+ : 2^{j-1}b < t \le 2^j b\}$ and $\Gamma^j_b = \Gamma(x) \cap \{(y,t) \in \mathbb{R}^{n+1}_+ : 2^{-j}b < t \le 2^{-j+1}b\}$. We have

$$\int_{\Gamma^b(x)} t^{-n-\varepsilon}\, d\mu(y,t) = \sum_{j=1}^{\infty} \int_{\Gamma^b_j} t^{-n-\varepsilon}\, d\mu(y,t) \le \sum_{j=1}^{\infty} (2^j b)^{-n-\varepsilon} \mu(\widehat{B}(x;2^j b))$$
$$\le Cb^{-\varepsilon} \sum_{j=1}^{\infty} (2^j)^{-\varepsilon} \le Cb^{-\varepsilon}.$$

Part (ii) is analogous.

LEMMA 2. *Let μ be a positive measure on $\mathbb{R}^{n+1}_+$ and A a Banach space. Then $T^p_{p,A}(d\mu) = L^p_A(\mathbb{R}^{n+1}_+; d\mu)$ for $1 < p < \infty$.*

Proof. By Fubini's theorem,

$$\|A_p(f)\|^p_{L^p(dx)} = \int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |f(y,t)|^p\, d\mu(y,t)/t^n \Big)\, dx$$
$$= \int_{\mathbb{R}^{n+1}_+} |f(y,t)|^p \Big(\int_{\mathbb{R}^n} \chi_{\Gamma(x)}(y,t)\, dx/t^n \Big)\, d\mu(y,t)/t^n$$
$$= c_n \int_{\mathbb{R}^n} |f(y,t)|^p\, d\mu(y,t).$$

Proof of Theorem 1. By the definition of the norm in $T^p_{q,B}(d\mu)$ we have

$$\{\|Tf\|_{T^p_{q,B}(d\mu)}\}^p = \int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} \|Tf(y,t)\|^q_B\, d\mu(y,t)/t^n \Big)^{p/q} dx$$

$$= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{n+1}_+} \|Tf(y,t)\|_B^q \chi_{\Gamma(x)}(y,t)\, d\mu(y,t)/t^n \Big)^{p/q} dx$$

$$= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{n+1}_+} \|\mathbf{S}f(x)(y,t)\|_B^q \, d\mu(y,t)/t^n \Big)^{p/q} dx$$

$$= \int_{\mathbb{R}^n} \{\|\mathbf{S}f(x)\|_{L^q_B\,(d\mu/t^n)}\}^p \, dx\,.$$

Suppose that f is a function in $L^p_A(\mathbb{R}^n;\, dx)$ with compact support contained in Q and $x \notin Q$. It is clear that if $(y,t) \in \Gamma(x)$ then $(y,t) \notin \widehat{Q}$, and therefore using (K.1) we obtain

$$\mathbf{S}f(x)(y,t) = Tf(y,t)\chi_{\Gamma(x)}(y,t) = \Big\{ \int_{\mathbb{R}^n} K(y,z,t)f(z)\, dz \Big\} \chi_{\Gamma(x)}(y,t)\,,$$

which is (K.3).

Assume now that $|x-z'| > 2|z-z'|$ and $a \in A$. Then

$$\{\|\mathbf{K}(x,z)(a) - \mathbf{K}(x,z')(a)\|_{L^q_B\,(d\mu/t^n)}\}^q$$
$$= \int_{\Gamma(x)} \|K(y,z,t)(a) - K(y,z',t)(a)\|_B^q \, d\mu(y,t)/t^n$$
$$\le \|a\| \Big(\int_{\Gamma^{|x-z'|}(x)} + \int_{\Gamma_{|x-z'|}(x)} \Big) \|K(y,z,t) - K(y,z',t)\|^q \, d\mu(y,t)/t^n$$
$$= \|a\|\{I_1 + I_2\}\,.$$

If $(y,t) \in \Gamma(x)$, then $|y-x| < t$, and therefore

$$2|z-z'| < |x-z'| \le |x-y| + |y-z'| < t + |y-z'|\,.$$

Thus, by using (K.2) and Lemma 1(i), we have

$$I_1 \le C|z-z'|^q \int_{\Gamma^{|x-z'|}(x)} (t + |y-z'|)^{-q(n+1)} t^{-n} \, d\mu(y,t)$$
$$\le C|z-z'|^q \int_{\Gamma^{|x-z'|}(x)} t^{-q(n+1)} t^{-n} \, d\mu(y,t) \le C \frac{|z-z'|^q}{|x-z'|^{(n+1)q}}\,.$$

On the other hand, by (K.2) and Lemma 1(ii) we have

$$I_2 \le C|z-z'|^q \int_{\Gamma_{|x-z'|}(x)} (t + |y-z'|)^{-q(n+1+\alpha)} t^{-\alpha q - n} \, d\mu(y,t)$$
$$\le C|z-z'|^q \int_{\Gamma_{|x-z'|}(x)} |x-z'|^{-q(n+1+\alpha)} t^{-\alpha q - n} \, d\mu(y,t)$$

$$\leq C\frac{|z-z'|^q|x-z'|^{\alpha q}}{|x-z'|^{(n+1+\alpha)q}} = C\frac{|z-z'|^q}{|x-z'|^{(n+1)q}}\,.$$

This proves (K.4) and the theorem.

Proof of Theorem 2

Case $1 < p \leq q \leq p_0$. By Theorem A and Theorem 1, $\mathbf{S}$ is a bounded linear operator from $L^p_A(\mathbb{R}^n; dx)$ into B^p_p, $1 < p \leq p_0$, with an associated kernel satisfying (K.2′). Thus the standard vector-valued theory of singular integrals can be applied (see [6]), and we conclude that $\mathbf{S}$ is bounded from $L^p_A(\mathbb{R}^n; dx)$ into B^p_q for $1 < p \leq q \leq p_0$ and from $H^1_A(\mathbb{R}^n; dx)$ into B^1_q for $1 < q \leq p_0$, i.e., T is bounded from $L^p_A(\mathbb{R}^n; dx)$ into $T^p_{q,B}(d\mu)$ for $1 < p \leq q \leq p_0$ and from $H^1_A(\mathbb{R}^n; dx)$ into $T^1_{q,B}(d\mu)$ for $1 < q \leq p_0$.

Case $1 < q < p < p_0$. By Lemma 2 and Theorem A, T is bounded from $L^p_A(\mathbb{R}^n; dx)$ into B^p_p, so by Proposition 1, T is bounded from $L^p_A(\mathbb{R}^n; dx)$ into B^p_q.

In order to obtain (iii) and (iv), observe that by Theorem A and Theorem 1, $\mathbf{S}$ is bounded from $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ into $l^r(B)^p_p$ for $1 < p \leq r \leq p_0$. Then, by repeating the arguments above (with $l^r(A)$, $l^r(B)$ instead of A and B) we find that $\mathbf{S}$ is bounded from $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ into $l^r(B)^p_q$ for $1 < p \leq q \leq r \leq p_0$ and from$H^1_{l^r(A)}(\mathbb{R}^n; dx)$ into $l^r(B)^1_q$ for $1 < q \leq r \leq p_0$. This means that T is bounded from $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ into $T^p_{q,l^r(B)}(d\mu)$ for $1 < p \leq q \leq r \leq p_0$ and from $H^1_{l^r(A)}(\mathbb{R}^n; dx)$ into $T^1_{q,l^r(B)}(d\mu)$ for $1 < q \leq r \leq p_0$.

The case $1 < q \leq p \leq r \leq p_0$ follows since by Lemma 2 and Theorem A, T is bounded from $L^p_{l^r(A)}(\mathbb{R}^n; dx)$ into $L^p_{l^r(B)}(\mathbb{R}^{n+1}_+; d\mu)$ for $1 < p \leq r \leq p_0$, and Proposition 1 again concludes the proof.

Remarks. The idea of applying vector-valued Calderón–Zygmund theory in the context of tent spaces can be found in [8] for the space T^p_∞.

The fact that $T^p_q(\alpha)$ is a subspace of L^p_A where A is $L^q(\mathbb{R}^{n+1}_+; d\mu/t^n)$ can be used to develop a general abstract theory of the spaces $T^p_q(\alpha)$. This will appear elsewhere.

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