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REPRESENTATIONS OF JORDAN ALGEBRAS AND SPECIAL FUNCTIONS

BY

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Introduction. This paper is concerned with the action of a special formally real Jordan algebra U on an Euclidean space E, with the decomposition of E under this action and with an application of this decomposition to the study of Bessel functions on the self-adjoint homogeneous cone Ω associated to U.

The special formally real Jordan algebras are classified: they are the $m\times m$ Hermitian matrices $H_m(\mathbb{F})$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$) endowed with the symmetric product

$$A\circ B=\frac{1}{2}(AB+BA) \tag{1}$$

and the vector space $\mathbb{U}_q=\mathbb{R}+V$ (V is a q-dimensional real vector space) equipped with the product

$$(\lambda,u)\circ(\mu,v)=(\lambda\mu+B(u,v),\lambda v+\mu u)$$

where $\lambda,\mu\in\mathbb{R}$, $u,v\in V$ and B is a symmetric bilinear positive form on V. The associated cones are given by the positive definite matrices and by the light cones respectively.

For a special formally real Jordan algebra U there exists a Euclidean space E and a Jordan algebra injective homomorphism $\phi:U\to\mathrm{Sym}^+(E)$ of U into the formally real Jordan algebra of the self-adjoint endomorphisms of E endowed with the product (1) (the references for the results on Jordan algebras needed in this paper are [1], [6], [5], [2], [3]). For the case $H_m(\mathbb{F})$ we take $E=M_{m,h}(\mathbb{F})$ (the $m\times h$ matrices on $\mathbb{F}$); $\phi(U)E$ is the matrix product. For $\mathbb{U}_q$ we can take $E=C_q$, the Clifford algebra associated to V and consider the imbedding of $\mathbb{U}_q$ in C_q (so that $\phi(U)E$ is a product in C_q). Observe that $\mathbb{U}_2$ is isomorphic to $H_2(\mathbb{R})$ and we can choose $E=M_{2,h}(\mathbb{R})$ in place of C_2. One of the purposes of this paper is to show that a related fact is true in general; we shall prove that if a special formally real Jordan

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algebra U with rank m acts on a Euclidean space E in the described way, then E can be written as an $m\times h$ matrix, so that $\phi(U)E$ is a matrix product which formally extends the Hermitian matrix case.

The main part of the paper deals with the Bessel functions introduced in [3]. That paper ended with an asymptotic formula for the Bessel functions on Ω, which was proved for particular choices of E and by algebra-by-algebra arguments. Here we prove the result for general E and without classification theory. The proof uses the stationary phase method, which needs an imbedding of U in E and an explicit description of a basis of the orthogonal complement.

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Notation. In this paper U will always be a simple n-dimensional special formally real Jordan algebra with rank m and the symbol $\circ$ will denote the product in a Jordan algebra. P is the quadratic representation $P(x) = 2L^2(x)-L(x^2)$, where $L(x)y = x\circ y$, also let $P(x,y) = L(x)L(y)+L(y)L(x)-L(x\circ y)$. Let e be the identity of U and let $\{c_1,\ldots,c_m\}$ be an orthonormal system of primitive idempotents ($c_i\circ c_j = 0$ for $i\neq j$, $c_i\circ c_i = c_i$, $c_1+\ldots+c_m = e$, m maximal). We have the Pierce decomposition of U relative to the previous set of idempotents:

$$U = \bigoplus_{i\le j} U_{i,j}$$

where $U_{i,i} = \mathbb{R}c_i$, $U_{i,j} = L(c_i)L(c_j)U$ for $i\neq j$. The $U_{i,j}$'s have the same (real) dimension d. We fix an orthonormal basis

$$\{c_j\}_{1\le j\le m}\cup\{u^s_{i,j}\}_{1\le i<j\le m\,,1\le s\le d}\,,$$

where any $u^s_{i,j}$ belongs to $U_{i,j}$. We write $U^s_{i,j}$ for the space $\mathbb{R}u^s_{i,j}$. $\mathrm{Tr}(x)$ will denote the trace of an element x in U.

There exists an N-dimensional Euclidean space E with the following property. Let $\mathrm{Sym}(E)$ be the space of self-adjoint endomorphisms of E and $\mathrm{Sym}^+(E)$ the same space when endowed with the Jordan product (1). $\mathrm{Sym}^+(E)$ is a formally real Jordan algebra [1,XI] and there exists a Jordan algebra injective homomorphism $\phi : U\to \mathrm{Sym}^+(E)$ such that $\phi(e) = \mathrm{id}$ ([2]). Let $Q : E\to U$ be the quadratic form satisfying $(\phi(x)\xi,\xi) = (x, Q(\xi))$ for any $x\in U$ and $\xi\in E$; we denote by ψ the associated bilinear form. We write E_i for the subspace $\phi(c_i)E$ of E ($1\le i\le m$).

Let $\Omega = \exp U$ be the homogeneous self-adjoint cone associated to U. Then $Q : E\to\overline{\Omega}$. We ask E to satisfy $Q(E) = \overline{\Omega}$. The set $\Sigma = \{\xi\in E : Q(\xi) = e\}$ is called the *Stiefel manifold* and the following polar decomposition holds a.e. [3]:

$$E = \Omega\times\Sigma\,.$$

Preliminary results. We begin with an elementary fact whose proof will be omitted.

LEMMA 1. *The subspaces* $E_j = \phi(c_j)E$ $(1 \le j \le m)$ *of* E *are mutually orthogonal and satisfy the direct sum decomposition* $E = \bigoplus_{1\le j\le m} E_j$. *As a consequence*, *for any* $\xi \in E$, $\phi(c_i)\phi(c_j)\xi = 0$ *provided* $i \ne j$.

LEMMA 2. *Let* $\xi_i \in E_i$ *and* $\xi_j \in E_j$ $(i, j = 1, \dots, m)$. *Then* $\psi(\xi_i, \xi_j) \in U_{i,j}$. *Moreover*, $Q(\xi_i) = \|\xi_i\|^2 c_i$.

Proof. By [3, Lemma 1] one knows that $Q(\phi(u)\xi) = P(u)Q(\xi)$, which by linearization implies

$$\psi(\phi(x)\xi, \phi(y)\eta) + \psi(\phi(y)\xi, \phi(x)\eta) = P(x, y)\psi(\xi, \eta)\,. \tag{2}$$

Now let $x = c_i$, $y = c_j$, $\xi = \xi_i$, $\eta = \xi_j$; then by (2) and Lemma 1, $\psi(\xi_i, \xi_j) = P(c_i, c_j)\psi(\xi_i, \xi_j)$, which by [1,VII,2] implies the result. In particular, $Q(\xi_i) = \lambda c_i$ with $\lambda = \mathrm{Tr}(\lambda c_i) = \mathrm{Tr}(Q(\xi_i)) = (Q(\xi_i), e) = (\xi_i, \xi_i) = \|\xi_i\|^2$.

LEMMA 3. *Let* $\xi \in E$ *and suppose* $Q(\xi) \in U_{j,j}$. *Then* $\xi \in E_j$.

Proof. Write

$$\xi = \sum_{1\le i\le m} \phi(c_i)\xi = \sum_{1\le i\le m} \xi_i\,.$$

Then by Lemma 2,

$$\begin{aligned} Q(\xi) &= \psi\Big(\sum_{1\le i\le m} \xi_i, \sum_{1\le i\le m} \xi_i\Big) \\ &= \sum_{1\le i\le m} Q(\xi_i) + 2\sum_{h<k} \psi(\xi_h, \xi_k) = \sum_{1\le i\le m} Q(\xi_i)\,. \end{aligned}$$

The assumption and Lemma 1 now imply $Q(\xi_i) = 0$ for $i \ne j$ and Lemma 2 again implies $\xi_i = 0$ for $i \ne j$. Therefore $\xi \in E_j$.

LEMMA 4. *Let* $\{u^s_{i,j}\}_{1\le s\le d}$ *be an orthonormal basis of* $U_{i,j}$ $(1 \le i < j \le m)$. *Then*

$$u^s_{i,j} \circ u^t_{i,j} = \delta_{s,t}(c_i + c_j)/2$$

(*Kronecker's* δ).

Proof. We know [1,VIII] that $U_{i,j} \circ U_{i,j} \subseteq U_{i,i} + U_{j,j}$ and that $c_i \circ u_{i,j} = \frac{1}{2}u_{i,j}$ for any $u_{i,j} \in U_{i,j}$ $(i \ne j)$. Then the associativity of the inner product

$$(u^s_{i,j} \circ u^s_{i,j}, c_i) = (u^s_{i,j}, u^s_{i,j} \circ c_i)$$

implies the result.

LEMMA 5. *Let* $u_{i,j}$ *be a normalized vector in* $U_{i,j}$ $(i \ne j)$. *Then for* $\xi_i \in E_i$ *the mapping*

$$\xi_i \to \phi(\sqrt{2}u_{i,j})\xi_i$$

is an inner product space isomorphism between E_i and E_j.

Proof. First we show that $\phi(u_{i,j})E_i \subseteq E_j$. By Lemma 3 it is enough to prove that $Q(\phi(u_{i,j})E_i) \subseteq U_{j,j}$. Indeed, suppose $\xi_i \in E_i$, $\|\xi_i\| = 1$; then by [3, Lemma 1], and Lemmas 2 and 3

$$\begin{aligned} Q(\phi(\sqrt{2}u_{i,j})\xi_i) &= 2P(u_{i,j})Q(\xi_i) = 2P(u_{i,j})c_i \\ &= 4u_{i,j} \circ (u_{i,j} \circ c_i) - 2(u_{i,j} \circ u_{i,j}) \circ c_i \\ &= 2u_{i,j} \circ u_{i,j} - (c_i + c_j) \circ c_i = (c_i + c_j) - c_i\,. \end{aligned}$$

To complete the proof we need to show that $\|\phi(\sqrt{2}u_{i,j})\xi_i\| = \|\xi_i\|$ for any $\xi_i \in E_i$. Indeed, by Lemmas 2 and 3,

$$\begin{aligned} \|\phi(\sqrt{2}u_{i,j})\xi_i\|^2 &= 2(\phi(u_{i,j})\xi_i, \phi(u_{i,j})\xi_i) \\ &= 2(\phi(u_{i,j} \circ u_{i,j})\xi_i, \xi_i) = (\phi(c_i + c_j)\xi_i, \xi_i) \\ &= (c_i + c_j, Q(\xi_i)) = (c_i + c_j, \|\xi_i\|^2 c_i) = \|\xi_i\|^2\,. \end{aligned}$$

A characterization of the Stiefel manifold. Lemma 2 and the identity $Q(\xi) = \sum_{1\le i\le m} Q(\xi_i) + 2\sum_{i<j} \psi(\xi_i, \xi_j))$ provide a simple characterization of the Stiefel manifold Σ.

PROPOSITION. *Let $\xi = \sum_{1\le i\le m} \phi(c_i)\xi = \sum_{1\le i\le m} \xi_i$ belong to E. Then $\xi \in \Sigma$ if and only if $\psi(\xi_i, \xi_j) = \delta_{ij} c_i$.*

An asymptotic formula for Bessel functions. Following [3] we define the Bessel function

$$J(r) = \int_\Sigma e^{-i(\sigma, \phi(\sqrt{r})\sigma_0)}\, d\beta(\sigma)$$

where $\sigma_0 \in \Sigma$ and is fixed once for all, $r \in \Omega$ and the measure has been defined in [3]. The following theorem has been proved in [3] through classification theory and assuming particular choices of E:

THEOREM 1. *Let U be a special formally real Jordan algebra. Let $x = \sum_{1\le j\le m} \lambda_j c_j$ be an element in Ω with distinct eigenvalues $\lambda_1 > \ldots > \lambda_m$ (> 0). Then, as $t \to +\infty$,*

$$\begin{aligned} J((tx)^2) &= \int_\Sigma e^{-it(\phi(x)\sigma, \sigma_0)}\, d\beta(\sigma) \\ &= (2\pi/t)^{(N-n)/2} \sum_\omega (|H(\sigma_\omega)|^{-1/2} e^{i(\pi/4)s(\sigma_\omega) + it(\phi(x)\sigma_\omega, \sigma_0)}) \\ &\quad + O(t^{-((N-n)/2)-1})\,, \end{aligned}$$

where $\sigma_\omega = \sum_{1\le j\le m} \omega_j \phi(c_j)\sigma_0$ ($\omega_j = \pm 1$); $H(\sigma_\omega)$ denotes the Hessian matrix of the function $g(\sigma) = (\phi(x)\sigma, \sigma_0)$ and its determinant takes the *value*

$$|H(\sigma_\omega)| = (-1)^{N-n} \prod_{i<j} \Big(\tfrac{1}{2}(\omega_i\lambda_i + \omega_j\lambda_j)\Big)^d \Big(\prod_{1\le i\le m} \omega_i\lambda_i\Big)^{(N/m)-md+d-1};$$

while $s(\sigma_\omega)$ *denotes the signature of* $H(\sigma_\omega)$ *and is equal to*

$$s(\sigma_\omega) = -\sum_{1\le i\le m} ((N/m) - d(i-1) - 1)\omega_i\,.$$

The proof requires a few lemmas.

LEMMA 6. *Suppose that* $(U^s_{i,j}, U^t_{h,k}) = 0$; $1 \le i \le j \le m$; $1 \le s \le d$ *for* $i \ne j$, *no* s *appears for* $i = j$; $1 \le h \le k \le m$; $1 \le t \le d$ *for* $h \ne k$, *no* t *appears for* $h = k$ (*the hypothesis means that the triples* (i,j,s) *and* (h,k,t) *do not coincide*). *Then*

$$(\phi(U^s_{i,j})\sigma_0, \phi(U^t_{h,k})\sigma_0) = 0\,.$$

P r o o f. For $u, v \in U$, (2) implies

$$(\phi(u)\sigma_0, \phi(v)\sigma_0) = (u, \psi(\sigma_0, \phi(v)\sigma_0)) = (u, \tfrac{1}{2}P(e,v)Q(\sigma_0)) = (u,v)\,,$$

which for u and v belonging to $U^s_{i,j}$ and $U^t_{h,k}$ respectively implies the result.

The previous argument also proves the following lemma.

LEMMA 7. *Same hypothesis as in Lemma 6*; *then*

$$(\phi(U^s_{i,j})\sigma_0, \phi(c_h)\phi(U^t_{h,k})\sigma_0) = 0\,.$$

LEMMA 8. *For any* $1 \le i < j \le m$ *and* $1 \le s \le d$ *we have*

$$(\phi(U)\sigma_0, \phi(c_i - c_j)\phi(U^s_{i,j})\sigma_0) = 0\,.$$

(Observe that, if U is the Jordan algebra of real $m \times m$ symmetric matrices and E is the Euclidean space $M_m(\mathbb{R})$ of square real matrices, this lemma simply says that symmetric and skew-symmetric matrices are orthogonal in $M_m(\mathbb{R})$).

P r o o f. Write the Pierce decomposition

$$U = \bigoplus U^t_{h,k}\,, \quad 1 \le h \le k \le m\,,\ 1 \le t \le d \text{ for } h \ne k\,,$$
$$\text{no } t \text{ appears for } h = k\,.$$

If the triples (i,j,s) and (h,k,t) are different we apply Lemma 7. Otherwise, let $u^s_{i,j} \in U^s_{i,j}$. Then by [3, Lemma 1], Lemma 4 and [1,VII]

$$\begin{aligned}(\phi(u^s_{i,j})\sigma_0, \phi(c_i - c_j)\phi(u^s_{i,j})\sigma_0) &= (\sigma_0, \phi(u^s_{i,j})\phi(c_i - c_j)\phi(u^s_{i,j})\sigma_0)\\ &= (\sigma_0, \phi(P(u^s_{i,j})(c_i - c_j))\sigma_0) = (e, P(u^s_{i,j})(c_i - c_j))\\ &= (e, 2u^s_{i,j} \circ (u^s_{i,j} \circ (c_i - c_j)) - (u^s_{i,j} \circ u^s_{i,j})(c_i - c_j)) = 0\,.\end{aligned}$$

LEMMA 9. *Let $u^s_{i,j}$ be a normalized vector in $U^s_{i,j}$ ($1 \le i < j \le m$, $1 \le s \le d$). Then the vectors $\phi(c_i - c_j)\phi(u^s_{i,j})\sigma_0$ are orthonormal in E.*

Proof. By [3, Lemma 1] and Lemma 4

$$Q(\phi(c_i - c_j)\phi(u^s_{i,j})\sigma_0) = P(c_i - c_j)P(u^s_{i,j})Q(\sigma_0) = P(c_i - c_j)(u^s_{i,j} \circ u^s_{i,j}) = \tfrac{1}{2}P(c_i - c_j)(c_i + c_j) = \tfrac{1}{2}(c_i + c_j)\,.$$

By Lemma 2 this implies $\|\phi(c_i - c_j)\phi(u^s_{i,j})\sigma_0\| = 1$.

To prove the orthogonality it enough to show that, say,

$$(\phi(c_i)\phi(u^s_{i,j})\sigma_0, \phi(c_h)\phi(u^t_{h,k})\sigma_0) = 0 \tag{3}$$

when the triples (i,j,s) and (h,k,t) do not coincide. This is a consequence of Lemmas 1 and 7.

Proof of Theorem 1. Let $g(\sigma) = (\phi(x)\sigma, \sigma_0)$ be as in the statement of the theorem. The Hessian of g at the point σ_ω can be computed in the following way. Let γ be a curve on the Stiefel manifold Σ such that $\gamma(0) = \sigma_\omega$ and $\gamma'(0) = a \in (\phi(U)\sigma_\omega)^\perp$. It has been proved in [3, p. 139] that

$$g''(\sigma_\omega)(a,a) = -(\phi(y)a, a)$$

with $\phi(y)\sigma_\omega = \phi(x)\sigma_0$. The isomorphism between the tangent space at σ_0 and the tangent space at σ_ω yields

$$g''(\sigma_\omega)(a,a) = -(\phi(y)b, b)$$

with $a = \sum_{1\le j\le m} \omega_j \phi(c_j) b$ and $b \in (\phi(U)\sigma_w)^\perp$. We therefore need to fix an orthonormal basis of this space.

By Lemma 9 there is a vector space V with orthonormal basis

$$\{\phi(c_i - c_j)\phi(u^s_{i,j})\sigma_0\}_{1\le i<j\le m, 1\le s\le d}$$

Let us put

$$A_j = E_j \cap (V \oplus \phi(U)\sigma_0)\,, \qquad 1 \le j \le m$$

(by Lemma 8, V and $\phi(U)\sigma_0$ are orthogonal). By Lemma 1 and (3)

$$A_j = \phi(\mathbb{R}c_j)\sigma_0 \oplus \bigoplus_{1\le i\le m,\ i\ne j,\ 1\le s\le d} \phi(c_j)\phi(U^s_{i,j})\sigma_0 \qquad 1 \le j \le m\,. \tag{4}$$

Let R_j be the orthogonal complement of A_j in E_j. Then

$$E_j = A_j \oplus R_j\,, \qquad 1 \le j \le m\,.$$

Now we fix an orthonormal basis $\{r^j_i\}$ of R_j which (by moving j and by applying Lemma 1) provides an orthonormal basis of

$$R = \bigoplus_{1\le j\le m} R_j$$

(the dimension of the R_j's will be computed later). Then, by Lemmas 1 and 8,

$$E = (\phi(U)\sigma_0) \oplus V \oplus R$$

and we fix

$$\{r_i^j\} \cup \{\phi(c_i - c_j)\phi(u_{i,j}^s)\sigma_0\} \tag{5}$$

as an orthonormal basis of $V \oplus R = (\phi(U)\sigma_0)^\perp$.

Let b be an element in (5). If b belongs to V then, say, $b = \phi(c_h - c_k)\phi(u_{h,k}^t)\sigma_0$, therefore, by Lemma 9 we get

$$\begin{aligned}(\phi(y)b, b) = (y, Q(b)) &= \sum_{1 \le j \le m} \omega_j \lambda_j (c_j, Q(\phi(c_h - c_k)\phi(u_{h,k}^t)\sigma_0)) \\ &= \sum_{1 \le j \le m} \omega_j \lambda_j (c_j, (c_h + c_k)/2) = (\omega_h \lambda_h + \omega_k \lambda_k)/2\end{aligned}$$

while for b in R we have, say, $b = r_i^k$ ($\in E_k$); then by Lemma 2

$$(\phi(y)b, b) = (y, Q(b)) = \sum_{1 \le j \le m} \omega_j \lambda_j (c_j, c_k) = \omega_k \lambda_k \,.$$

Now we compute the dimensions of the above spaces. We have

$$\dim((\phi(U)\sigma_0) \oplus V) = (m + m(m-1)d/2) + m(m-1)d/2 \,.$$

Therefore

$$\dim R = N - m^2 d + md - m \,.$$

By (4)

$$\dim A_j = 1 + (m-1)d \,, \qquad 1 \le j \le m \,.$$

By Lemma 5, the E_j's have the same dimension N/m. Then

$$\dim R_j = \frac{N}{m} - md + d - 1 \,, \qquad 1 \le j \le m \,.$$

Therefore the Hessian is

$$|H(\sigma_\omega)| = (-1)^{N-n} \prod_{h<k} \left(\tfrac{1}{2}(\omega_h \lambda_h + \omega_k \lambda_k)\right)^d \Big(\prod_{1 \le k \le m} \omega_k \lambda_k\Big)^{(N/m)-md+d-1} \,.$$

We now turn to the computation of the signature. Since $\lambda_h > \lambda_k$ (for $h < k$) the sign of $\omega_h \lambda_h + \omega_k \lambda_k$ is the sign of ω_h. Therefore the signature is

$$\begin{aligned}&- \sum_{1 \le i \le m} d(m-i)\omega_i - \left(\frac{N}{m} - md + d - 1\right) \sum_{1 \le i \le m} \omega_i \\ &\qquad\qquad = - \sum_{1 \le i \le m} \left(\frac{N}{m} - d(i-1) - 1\right) \omega_i \,.\end{aligned}$$

By the stationary phase method (see [4]) this ends the proof of the theorem.

A particular matrix realization of E. In this section we use the previous results to write E as an $m \times v$ matrix space (with vector coefficients) so that the action $\phi(U)E$ reduces to a matrix product which coincides with the usual one in the Hermitian case. Such a construction is therefore interesting only for the Jordan algebra $\mathbb{U}_q$ (see the Introduction) and we shall spend a few words on this case.

Let U be a simple special formally real Jordan algebra and let E be a Euclidean space as in the Notation.

Let $x = \bigoplus_{i\le j} x_{i,j}$ belong to U ($x_{i,j} \in U_{i,j}$). We associate to x the $m\times m$ matrix

$$X = [X_{i,j}]_{i,j=1,\ldots,m} \tag{6}$$

where

$$X_{i,j} = \begin{cases} \phi(c_i)\phi(x_{i,j}) & \text{for } i \le j, \\ \phi(c_i)\phi(x_{j,i}) & \text{for } i > j, \end{cases}$$

so that the matrix coefficients are d-dimensional for $i \ne j$ and 1-dimensional for $i = j$.

Let ξ be an element in E. From now on the symbol

$$\operatorname{Span}\Big(\prod \phi(U)\xi\Big)$$

will denote the linear span of the elements $\prod_{u\in A}\phi(u)\xi$, where the product is over any subset of the basis of U.

Now let $E_1 = \phi(c_1)E$ and let $G \subseteq E_1$ such that $\operatorname{Span}(\prod\phi(U)G) = E$ (such a G exists because of Lemma 5). Let g^1 be a unit vector in G and suppose $\operatorname{Span}(\prod\phi(U)g^1) \subsetneq E$; then $\operatorname{Span}(\prod\phi(U)g^1) \not\supseteq G$. Now we choose $g^2 \in G$ orthogonal to $\operatorname{Span}(\prod\phi(U)g^1)$ and we go on until we obtain an orthogonal set $\{g^1,\ldots,g^v\}$ in G. Let $G^h = \operatorname{Span}(\prod\phi(U)g^h)$ ($1 \le h \le v$). Then $(G^h, G^k) = 0$ for $h \ne k$ and we write

$$E = \bigoplus_{1\le h\le v} G^h .$$

Let $G^h_p = G^h \cap E_p = \phi(c_p)G^h$, $1 \le h \le v$, $1 \le p \le m$. Then by Lemma 1

$$E = \bigoplus_{1\le h\le v,\ 1\le p\le m} G^h_p .$$

Now we decompose an element ξ in E as

$$\xi = \bigoplus_{1\le h\le v,\ 1\le p\le m} \xi^h_p \tag{7}$$

and we associate to ξ the $m \times v$ matrix

$$\Xi = \left[\xi_p^h\right]_{1\le h\le v,\ 1\le p\le m} \tag{8}$$

Lemma 5 and a moment's reflection show that (8) depends only on ξ.

We now state a lemma whose easy proof is omitted.

LEMMA 10. *Let ξ_p belong to $E_p = \phi(c_p)E$. Then $\phi(u_{i,j})\xi_p = 0$ for any $u_{i,j} \in U_{i,j}$ (if $i \neq p$ and $j \neq p$).*

The statement of the next theorem follows the notation introduced in this section.

THEOREM 2. *Let $x \in U$, $\xi \in E$, let X be the $m\times m$ matrix associated to x in* (6) *and let Ξ be the $m \times v$ matrix associated to ξ in* (8). *Then $X\Xi$ is the $m \times v$ matrix associated to $\phi(x)\xi$.*

Proof. Let $\xi = \bigoplus_{1\le h\le v,\ 1\le p\le m} \xi_p^h$ as in (7). By linearity it suffices to prove the result for, say, $\xi = \xi_p^h$ (whose matrix Ξ is zero but for the (p,h)-coefficient). By applying Lemmas 1, 5 and 10 we have

$$\begin{aligned}\phi(x)\xi_p^h &= \sum_{i\le j}\phi(x_{i,j})\xi_p^h = \sum_{i\le p}\phi(x_{i,p})\xi_p^h + \sum_{p<i}\phi(x_{p,i})\xi_p^h\\ &= \sum_{i\le p}\phi(c_i)\phi(x_{i,p})\xi_p^h + \sum_{p<i}\phi(c_i)\phi(x_{p,i})\xi_p^h\,.\end{aligned}$$

Any element $\phi(c_i)\phi(x_{i,p})\xi_p^h$ or $\phi(c_i)\phi(x_{p,i})\xi_p^h$ belongs to E_i; then, by definition, each one of them belongs to the corresponding space G_i^h (same i). Hence the matrix associated to $\phi(x)\xi_p^h$ is

$$\Gamma = [\gamma_{i,j}]_{1\le i\le m,\ 1\le j\le v}$$

where $\gamma_{i,j} = 0$ for $j \neq p$ and $\gamma_{i,p} = \phi(c_i)\phi(x_{i,p})\xi_p^h$ for $i \le p$ and $\gamma_{i,p} = \phi(c_i)\phi(x_{p,i})\xi_p^h$ for $i > p$. This ends the proof.

We now describe the above argument for the case $U = H_m(\mathbb{C})$, $E = M_{m,v}(\mathbb{C})$. In this case we fix E_1 to be zero but for the first row and we can choose G to be the subspace of E_1 whose elements have real entries. Now fix $g^1,\ldots,g^v$ as the natural basis of G and the above construction yields $M_{m,v}(\mathbb{C})$.

Now consider the case $U = \mathbb{U}_q = \mathbb{R} + V$, $E = C_q$ (the Clifford algebra associated to V). Let $e_{\hat 1},\ldots,e_{\hat q}$ be an orthonormal basis of V with respect to B (see the Introduction). Then $e_0 = (1,0)$, $e_j = (0, e_{\hat j})$ $(1 \le j \le q)$ give an orthonormal basis of $\mathbb{U}_q$ and $\phi : e_j \to F_j$ denotes the imbedding of $\mathbb{U}_q$ in C_q (see e.g. [3]). Now fix the idempotents $c_1 = (e_0+e_1)/2$, $c_2 = (e_0-e_1)/2$. Then $E = E_1 \oplus E_2$, where

$$E_1 = (F_0+F_1)C_q^1\,, \qquad E_2 = (F_0-F_1)C_q^1\,,$$

where (cf.[3]) C_q^1 is the linear span of the products of F_j's with any $j \neq 1$. Now we follow the argument of this section by fixing $g^1 = F_0 + F_1$. Then a short computation shows that

$$\begin{aligned}\operatorname{Span}\Big(\prod \phi(\mathbb{U}_q) g^1\Big) &= \operatorname{Span}\Big(\prod \phi(\mathbb{U}_q)(F_0 + F_1)\Big) \\ &= (F_0 + F_1)\ {}_eC_q^1 + (F_0 - F_1)\ {}_oC_q^1\end{aligned}$$

where ${}_eC_q^1$ (${}_oC_q^1$) is the subspace of C_q^1 containing the elements obtained by multiplying an even (odd) number of F_j's ($j \neq 0$, $j \neq 1$). Then C_q turns out to be the matrix

$$\begin{bmatrix} (F_0 + F_1)\ {}_eC_q^1 & (F_0 + F_1)\ {}_oC_q^1 \\ (F_0 - F_1)\ {}_oC_q^1 & (F_0 - F_1)\ {}_eC_q^1 \end{bmatrix}.$$

The previous argument shows that (besides C_q) we can take E as an $m \times v$ matrix with vector coefficients.

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