COLLOQUIUM MATHEMATICUM

VOL. LXII | 1991 | FASC. 2

SOME PROBLEMS IN MEASURE THEORY

BY

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0. Introduction. In his 1946 paper [33], classical by now, E. Szpilrajn (E. Marczewski) presented a review of some open problems and questions in measure theory. The problems considered there were rather important for further development of the theory. Later attempts by various authors led to affirmative or negative solutions of many of those problems. As a rule, the solution of one or another of those questions was connected with the appearance of new methods and was accompanied by rapid progress in measure theory. The material contained in [33] was quite large, despite its fairly small volume. The following topics were included: measure extensions; uniqueness of invariant measures; relations with measurable and non-measurable cardinals; Hausdorff measures; existence of separable supports of Borel measures in metrizable spaces; properties of subsets of the real line that are small in the Borel sense, and others. All those problems were of general nature and came out of the "inner" development of measure theory. We remark that several applications of measure theory in such branches of mathematics as functional analysis, probability theory, integration theory and so on, were just touched upon in Marczewski's paper.

Over forty years have elapsed since that time, and the situation in measure theory has considerably changed: the scope of problems has continuously expanded and the interest of the theory itself has been changing one way or another. Nevertheless, it must be emphasized that general aspects of the theory still play an important part and still constitute its core. As before, the development of measure theory is stimulated by systematic employment of purely set-theoretic methods and exposition of links with related fields of mathematics. Among those fields worth mentioning are combinatorial theory of infinite sets (including the theory of measurable and non-measurable cardinals), descriptive set theory with its methods (it suffices to recall a collection of theorems about measurable selectors and the well-known Choquet capacity theorem), general topology (recall in particular the important role of the Montgomery operation in the study of Borel measures in metric spaces), general theory of convex sets (recall especially the classical Hahn–Banach extension theorem, proved with the help of the

Axiom of Choice and having numerous applications in integration theory) and others. Undoubtedly, those fields will have a strong influence on the development of measure theory.

The present paper contains a small review of some problems in measure theory, in our opinion having certain interest and possibly admitting non-trivial applications to different fields. In choosing our material we were motivated in the first place by the desire to survey, like in [33], those problems which are sufficiently simple in formulation on the one hand and touch important points of the theory on the other. We are mainly concerned with certain aspects of such topics as: non-measurable sets in metric spaces, quasi-invariant and invariant measures in infinite-dimensional topological vector spaces, properties of Borel measures in various topological spaces, properties of cylindrical and Borel σ-algebras, classes of families of probability measures. We need not add that the choice of problems presented here was inspired by the author's interests and preferences.

To begin with, let us recall certain simple definitions and notions that we shall use in the sequel.

By a *measure space* we mean a triple $(\mathbf{E}, \mathcal{S}, \mu)$ where $\mathbf{E}$ is a set, $\mathcal{S}$ some σ-algebra of subsets of $\mathbf{E}$ and μ a measure defined on $\mathcal{S}$. A measure μ is said to be *non-zero* (*non-degenerate*, *non-trivial*) if it is not identically equal to zero. A measure μ is called σ-*finite* if $\mathbf{E}$ can be covered by a countable family of μ-measurable sets each of which has a finite μ measure. Furthermore, μ is called *diffuse* if the measure of every one-element subset of $\mathbf{E}$ is zero.

Suppose that $\mathbf{E}$ is equipped with some topology and that $\mathcal{S}$ coincides with the σ-algebra of all Borel sets relative to that topology. In this case the measure μ is called a *Borel measure* on $\mathbf{E}$. A Borel measure μ is said to be *Radon* if it is defined on a Hausdorff space $\mathbf{E}$ and

$$\mu(X) = \sup\{\mu(K) : K \subseteq X \ \& \ K \text{ is compact}\}$$

for each Borel subset $X \subseteq \mathbf{E}$.

A topological space $\mathbf{E}$ is called *Polish* if $\mathbf{E}$ is homeomorphic to a complete separable metric space. It is well known that any two uncountable Polish spaces are Borel isomorphic. In other words, such spaces are identical from the point of view of Borel structure.

Let g be a mapping. By $\operatorname{dom} g$ we denote the domain of g. We say that g is a *functional* if $\operatorname{dom} g$ is a family of functions.

The symbol ω stands, as usual, for the cardinality of the set of positive integers; $\mathbf{c}$ denotes the cardinality continuum. Let α be any infinite cardinal number. It is called *measurable in the wide sense* if a non-degenerate σ-finite diffuse measure can be defined on the σ-algebra of all subsets of α, and *measurable in the narrow sense* if there exists a diffuse measure with range $\{0, 1\}$, defined on the σ-algebra of all subsets of α.

Let $\mathbf{E}$ be a basic set and let τ be a topology on $\mathbf{E}$. Recall that the least cardinality of a basis for τ is called the *weight* of $\mathbf{E}$.

Let $(\mathbf{E}, \mathcal{S}, \mu)$ be a measure space. The inner and the outer measure associated with μ are denoted by μ_* and μ^* respectively. A set $X \subseteq \mathbf{E}$ is called *μ-massive* in $\mathbf{E}$ if $\mu_*(\mathbf{E} \setminus X) = 0$.

1. Non-measurable subsets of metric spaces and generalized integrals. Let $(\mathbf{E}, \varrho)$ be a metric space, $\mathcal{B}(\mathbf{E})$ its Borel σ-algebra and μ a σ-finite (in particular, probability) measure on $\mathcal{B}(\mathbf{E})$. Under certain restrictions imposed on the topological weight of $\mathbf{E}$ one can show the existence in $\mathbf{E}$ of a closed separable support for μ. More precisely, the following well-known result holds.

THEOREM 1.1. *Suppose that the topological weight of the space* $(\mathbf{E}, \varrho)$ *is not measurable in the wide sense. Then for every σ-finite Borel measure μ in* $\mathbf{E}$ *there exists a closed separable set $F(\mu) \subseteq$* $\mathbf{E}$ *with the property that* $\mu(\mathbf{E} \setminus F(\mu)) = 0$. (Thus μ is *concentrated* on $F(\mu)$ or, in other words, $F(\mu)$ is the *separable support* of μ.)

A proof of this theorem can be found, for example, in [2] or [20]. An easy corollary to Theorem 1.1 is

THEOREM 1.2. *Let* $(\mathbf{E}, \varrho)$ *be a complete metric space whose topological weight is not measurable in the wide sense. Then every σ-finite Borel measure μ on* $\mathbf{E}$ *is concentrated on some σ-compact set $F(\mu) \subseteq$* $\mathbf{E}$.

Theorem 1.2 follows from Theorem 1.1 and the fact that in a complete separable metric space every σ-finite Borel measure is Radon.

Theorem 1.1 represents itself a fairly general abstract result about σ-finite Borel measures in metric spaces. However, it can also be successfully applied to quite special cases. For one of such applications to Gaussian measures in $\mathbb{R}^\omega$ (the space of all real sequences), see [5]. Here we are going to consider its application to some questions concerning non-measurability of various sets in complete metric spaces of cardinality continuum.

Recall that a subset X of a Hausdorff topological space $\mathbf{E}$ is called *universally measurable* (with respect to the class of all σ-finite Borel measures on $\mathbf{E}$) if, for every σ-finite Borel measure μ on $\mathbf{E}$, we have $X \in \operatorname{dom} \overline{\mu}$ where $\overline{\mu}$ denotes the standard completion of μ. The following example describes a quite large class of universally measurable sets.

EXAMPLE 1.1. Let $(\mathbf{E}, \varrho)$ be a complete separable metric space and X an analytic (Suslin) subset of $\mathbf{E}$. Then both X and $\mathbf{E} \setminus X$ are universally measurable in $\mathbf{E}$. This follows directly from the known Choquet capacity theorem (see [25], [7] or [3], Chap. 9, for the proof). Suppose now that $Y \subseteq \mathbf{E}$ is the image of $\mathbf{E} \setminus X$ under a continuous mapping. The problem of the universal measurability of Y is unsolvable within the standard axiomatics of set theory. Indeed, this can be obtained immediately from the following assertions.

1) Gödel's constructivity axiom (see, e.g., [30] and [1]) implies the existence of an analytic subset X of the real line $\mathbb{R}$ and a continuous mapping $f : \mathbb{R} \setminus X \to \mathbb{R}$ such that $f(\mathbb{R} \setminus X)$ is not Lebesgue measurable. In particular, the set $f(\mathbb{R} \setminus X)$ is not universally measurable in $\mathbb{R}$.

2) Under the assumption of Martin's axiom along with the negation of the Continuum Hypothesis, if $Y \subseteq \mathbf{E}$ is the image of $\mathbf{E} \setminus X$ via a continuous mapping then Y is universally measurable in $(\mathbf{E}, \varrho)$.

Let $\mathbf{E}$ be a Hausdorff topological space and let X be a subset of $\mathbf{E}$. We say that X is *absolutely non-measurable* (relative to the class of all non-degenerate σ-finite diffuse Borel measures in $\mathbf{E}$) if $X \notin \operatorname{dom} \overline{\mu}$ for every non-degenerate σ-finite diffuse Borel measure μ on $\mathbf{E}$ (as before, $\overline{\mu}$ stands for the completion of μ).

Thus we can see that, in a Hausdorff topological space, universally measurable sets and absolutely non-measurable sets are antipodal in nature.

For any complete separable metric space $(\mathbf{E}, \varrho)$ of cardinality continuum, the problem of existence, in $\mathbf{E}$, of an absolutely non-measurable subset is solved in the affirmative via a classical construction due to F. Bernstein [24], which we recall here. First, recall the following general lemma in set theory.

LEMMA 1.1. *Let* $\mathbf{E}$ *be an infinite set and let* $(X_i)_{i \in I}$ *be a family of subsets of* $\mathbf{E}$ *such that* $\operatorname{card}(I) \leq \operatorname{card}(\mathbf{E})$ *and* $(\forall i \in I)\,(\operatorname{card}(X_i) = \operatorname{card}(\mathbf{E}))$. *Then there exists a set* $X \subseteq \mathbf{E}$ *with*

$$(\forall i \in I)\,(\operatorname{card}(X \cap X_i) = \operatorname{card}((\mathbf{E} \setminus X) \cap X_i) = \operatorname{card}(\mathbf{E})).$$

The proof of this lemma can be carried out by transfinite induction (which gives the construction of the required set X). As a direct application of Lemma 1.1 we get the following classical result due to Bernstein.

THEOREM 1.3. *Let* $(\mathbf{E}, \varrho)$ *be an uncountable complete separable metric space. There is a subset* X *of* $\mathbf{E}$ *such that both* X *and* $\mathbf{E} \setminus X$ *intersect every non-empty perfect subset of* $\mathbf{E}$; *consequently*, $\operatorname{card}(X) = \operatorname{card}(\mathbf{E} \setminus X) = \mathbf{c}$.

The set X in Theorem 1.3 is usually called a *Bernstein set*. It is not difficult to show that every such set is absolutely non-measurable in $\mathbf{E}$. Indeed, let μ be an arbitrary non-degenerate σ-finite diffuse Borel measure in $\mathbf{E}$. Suppose that X is measurable with respect to the completion $\overline{\mu}$ of μ. Then obviously we have

$$\overline{\mu}(X) + \overline{\mu}(\mathbf{E} \setminus X) = \overline{\mu}(\mathbf{E}) > 0,$$

so at least one of the numbers $\overline{\mu}(X)$ and $\overline{\mu}(\mathbf{E} \setminus X)$ is strictly positive. Without loss of generality we can assume that $\overline{\mu}(X) > 0$. Since μ is a Radon measure, there is a compact set $K \subseteq X$ for which $\overline{\mu}(K) > 0$. In particular, K is uncountable, and it is clear that K contains a non-empty perfect subset of $\mathbf{E}$, which contradicts the fact that X is a Bernstein set in $\mathbf{E}$. This concludes the proof of the absolute non-measurability of X.

From the above discussion it follows that in any complete separable metric space of cardinality continuum there exist absolutely non-measurable subsets. A natural question arises: how essential is the assumption that the space is separable? To answer this question we need the following simple lemma from the theory of metric spaces.

LEMMA 1.2. *Let* $(\mathbf{E}, \varrho)$ *be an arbitrary non-empty complete metric space without isolated points. Then there is a subset of* $\mathbf{E}$ *which is homeomorphic to the Cantor set* 2^ω (*in particular*, *this subset is an uncountable complete separable subspace of* $\mathbf{E}$).

For a proof of Lemma 1.2 see, e.g., [24].

Let now $(\mathbf{E}, \varrho)$ be a complete metric space of cardinality continuum. By $(X_i)_{i \in I}$ we denote the injective family of all uncountable closed separable subsets of $\mathbf{E}$. It is easy to see that $\mathrm{card}(I) \leq \mathbf{c}$ and $(\forall i \in I)(\mathrm{card}(X_i) = \mathbf{c})$. Note that the set I of indices can be empty. Further, using Lemma 1.1 we get a set $X \subseteq \mathbf{E}$ such that

$$(\forall i \in I)(\mathrm{card}(X \cap X_i) = \mathrm{card}((\mathbf{E} \setminus X) \cap X_i) = \mathbf{c}).$$

Moreover, in view of Lemma 1.2 we see that both X and $\mathbf{E} \setminus X$ intersect any non-empty perfect subset of $\mathbf{E}$. In other words, the following generalization of Theorem 1.3 is true.

THEOREM 1.4. *If* $(\mathbf{E}, \varrho)$ *is a complete metric space of cardinality continuum then there is a subset* X *of* $\mathbf{E}$ *such that both* X *and* $\mathbf{E} \setminus X$ *intersect every non-empty perfect subset of* $\mathbf{E}$ *and* $\mathrm{card}(X) = \mathrm{card}(\mathbf{E} \setminus X) = \mathbf{c}$.

It is interesting to note that, unlike the Bernstein set in Theorem 1.3, the set X in Theorem 1.4 is not necessarily absolutely non-measurable in $\mathbf{E}$. This follows directly from the following result.

THEOREM 1.5. *The following conditions are equivalent*:

1) $\mathbf{c}$ *is not measurable in the wide sense*;

2) *any complete metric space* $(\mathbf{E}, \varrho)$ *of cardinality* $\mathbf{c}$ *contains an absolutely non-measurable subset.*

For the proof of Theorem 1.5 it suffices to make use of Theorem 1.1 together with the reasoning used to establish the absolute non-measurability of every Bernstein set in an uncountable complete separable metric space. We add here that if $(\mathbf{E}, \varrho)$ is an arbitrary complete metric space of cardinality $\mathbf{c}$ whose weight is not measurable in the wide sense, then there exists an absolutely non-measurable subset of $\mathbf{E}$.

EXAMPLE 1.2. Let ω_1 denote the first uncountable cardinal. Consider the Hilbert space $l^2(\omega_1)$ with orthonormal base of cardinality ω_1. Clearly $\mathrm{card}(l^2(\omega_1)) = \mathbf{c}$ and the topological weight of $l^2(\omega_1)$ equals ω_1. By a classical theorem of Ulam (cf. [26]), the cardinal ω_1 is non-measurable in the wide sense. Hence $l^2(\omega_1)$ contains absolutely non-measurable subsets. A similar argument works for any Hilbert space $l^2(\omega_\alpha)$ where ω_α is an infinite cardinal not greater than $\mathbf{c}$ and less than the first weakly inaccessible cardinal.

EXAMPLE 1.3. Let $\mathbb{R}$ be the real line with the Euclidean topology τ. We define another topology τ^* on $\mathbb{R}$, putting

$$\tau^* = \{Z : (\exists U)\,(\exists Y)\,(U \in \tau \ \&\ Y \subseteq \mathbb{R} \ \&\ \mathrm{card}(Y) \leq \omega \ \&\ Z = U \setminus Y)\}.$$

It is easy to see that

1) τ^* is a topology on $\mathbb{R}$, strictly stronger that τ;
2) $(\mathbb{R}, \tau^*)$ is non-separable and non-metrizable;
3) the topological weight of $(\mathbb{R}, \tau^*)$ is $\mathbf{c}$;
4) the Borel structures of $(\mathbb{R}, \tau^*)$ and $(\mathbb{R}, \tau)$ coincide;
5) there exist absolutely non-measurable sets in $(\mathbb{R}, \tau^*)$.

Thus we see that in some quite simple cases restrictions on topological weight are not necessary for the existence of an absolutely non-measurable subset in a Hausdorff topological space, since the topological weight of $(\mathbb{R}, \tau^*)$ can be either measurable in the wide sense or not. In this context the following open problem is of considerable interest.

PROBLEM 1.1. Find a topological characterization of Hausdorff topological spaces containing absolutely non-measurable subsets.

The problem above seems interesting even in special cases. For instance, if l^∞ is the non-separable Banach space of all bounded real sequences then the following two statements are equivalent:

1) l^∞ contains an absolutely non-measurable subset;
2) $\mathbf{c}$ is not measurable in the wide sense.

Now, let $(\mathbf{E}, \varrho)$ be an arbitrary separable metric space and let μ be an arbitrary σ-finite diffuse Borel measure in $\mathbf{E}$. Suppose that for a subset $Z \subseteq \mathbf{E}$ we have $\mu^*(Z) > 0$. Then obviously Z is uncountable. It is natural to pose the following question.

PROBLEM 1.2. Is it possible to decompose Z into a collection of subsets Z_i $(i \in I)$ so that $\mathrm{card}(I) > \omega$ and $(\forall i \in I)\,(\mu^*(Z_i) > 0)$?

Without additional set-theoretic assumptions the solution of this problem is not known. Note the following result related to Problem 1.2.

THEOREM 1.6. *If Martin's Axiom holds then for the set Z above there exists a partition* $(Z_i)_{i\in I}$ *such that* $\mathrm{card}(I) = \mathbf{c}$ *and* $(\forall i \in I)\,(\mu^*(Z_i) = \mu^*(Z) > 0)$.

EXAMPLE 1.4. Suppose that $\mathbf{c}$ is measurable in the wide sense, and consider the real line $\mathbb{R}$ with the ordinary Lebesgue measure λ. Then, by a well known result due to K. Kunen, there exists $Z \subseteq \mathbb{R}$ such that $\mathrm{card}(Z) < \mathbf{c}$ and $\lambda^*(Z) > 0$. In particular, Z is not Lebesgue measurable, and it is clear that Z cannot be represented as the union of a family $(Z_i)_{i\in I}$ where $\mathrm{card}(I) = \mathbf{c}$ and $(\forall i \in I)\,(\lambda^*(Z_i) > 0)$.

Now we will consider generalized integrals, in particular some of their properties connected with non-measurability of sets.

Let $(\mathbf{E}, \mathcal{S}, \mu)$ be a measure space. For simplicity we assume that μ is a probability measure and that μ is non-atomic and separable (i.e. μ is metrically isomorphic to the Lebesgue measure λ on the unit interval $[0, 1]$). Let $\mathbf{H}$ be a non-zero separable Banach space with its Borel structure denoted by $\mathcal{B}(\mathbf{H})$. We denote by $\Phi(\mathbf{E}, \mathbf{H})$ the vector space of all measurable mappings from $\mathbf{E}$ into $\mathbf{H}$, factorized according to the usual equivalence relation. Then it is well known that the metric

$$\varrho(\phi, \psi) = \int\limits_{\mathbf{E}} \frac{|\phi - \psi|}{1 + |\phi - \psi|}\, d\mu \qquad (\phi, \psi \in \Phi(\mathbf{E}, \mathbf{H}))$$

gives the topology of convergence in measure in $\Phi(\mathbf{E}, \mathbf{H})$. In this way $\Phi(\mathbf{E}, \mathbf{H})$ is a Polish topological vector space.

By a *generalized integral* in $\Phi(\mathbf{E}, \mathbf{H})$ we mean any functional $g : \mathcal{L} \to \mathbf{H}$ such that

1) $\mathcal{L}$ is a vector subspace of $\Phi(\mathbf{E}, \mathbf{H})$ containing all measurable step functions from $\mathbf{E}$ into $\mathbf{H}$;

2) g is linear (or at least additive);

3) for every μ-measurable set $Y \subseteq \mathbf{E}$ and every $h \in \mathbf{H}$, $g(\chi_Y \cdot h) = \mu(Y) \cdot h$, where χ_Y is the characteristic function (the indicator) of Y.

We require as little as possible in the definition of generalized integral, and this notion fully fits in with various different definitions of integrals (for instance, the most often used definition of Bochner integral). Now we formulate a result concerning the properties of generalized integrals.

THEOREM 1.7. *Let T_1 be the theory given in* (ZF) & (*axiom of dependent choice*) & (*every subset of the real line has the Baire property*). *Then in T_1 every generalized integral has the domain of the first category in* $\Phi(\mathbf{E}, \mathbf{H})$.

A proof of the theorem above can be found in [20] (only the case $\mathbf{H} = \mathbb{R}$ is considered there, yet the proof carries over to any separable non-zero Banach space $\mathbf{H}$).

It is useful to compare Theorem 1.7 with the next result, essentially due to A. N. Kolmogorov (see [23]).

THEOREM 1.8. *Let* $(\mathbf{E}, \mathcal{S}, \mu)$ *be the unit interval with Lebesgue measure and let* $\mathbf{H}$ *be the real line. Let* T_2 *be the theory given by* (ZF) & (*axiom of dependent choice*) & (*every subset of the real line is Lebesgue measurable*). *Then in* T_2 *the domain of each generalized integral is a proper subspace of* $\Phi(\mathbf{E}, \mathbf{H})$.

For several reasons the result in Theorem 1.7 is preferable to that in Theorem 1.8. First, Theorem 1.7 provides insight into the topological structure of the domain of an arbitrary generalized integral (and, what is more, taking values in an arbitrary separable non-zero Banach space $\mathbf{H}$). Furthermore, in providing the relative consistency of each of the theories T_1 and T_2 different techniques are employed. Namely, to establish the relative consistency of T_1 it suffices to have the consistency of ZF only (or, which is equivalent, the consistency of ZFC). On the other hand, to get the relative consistency of T_2 it is necessary to require that a stronger theory (ZFC) & (there exists an uncountable strongly inaccessible cardinal number) be consistent.

A more detailed discussion of the above fact can be found in a paper of S. Shelah [29], where it is shown that great cardinal numbers are necessary to justify the relative consistency of T_2.

Going back to Theorem 1.7, we can see that the domain of any effectively defined generalized integral can only be a set of the first category in the topological vector space $\Phi(\mathbf{E}, \mathbf{H})$. In particular, we find that if g is an effectively defined generalized integral then $\operatorname{dom} g \neq \Phi(\mathbf{E}, \mathbf{H})$ (by "effectively" we mean that the Axiom of Choice is not made use of).

The following problem is naturally related to the remarks above.

PROBLEM 1.3. Can we prove any analogue of Theorem 1.7 in terms of measure theory? In other words, does there exist, in $\Phi(\mathbf{E}, \mathbf{H})$, a non-degenerate σ-finite diffuse Borel measure μ such that $\mu^*(\operatorname{dom} g) = 0$ for any effectively defined generalized integral g?

In investigating the last problem it is useful to have the following example in mind.

EXAMPLE 1.5. Let $\mathbf{K}_1$ be the unit circle in the Euclidean plane $\mathbb{R}^2$, considered as a compact abelian group, and let $\mathbf{K}_1^\omega = \mathbf{K}_1 \times \mathbf{K}_1 \times \ldots$; let μ be the Haar measure in $\mathbf{K}_1^\omega$. Analogously to the notion of generalized integral, for the space $\mathbf{K}_1^\omega$ one can introduce the notion of generalized limit (see [16]). Then it turns out that the following statements hold:

1) *In the theory* T_1 *the domain of any generalized limit is of the first category in the topological group* $\mathbf{K}_1^\omega$;

2) *In the theory T_2 the domain of any generalized limit is a set of μ-measure zero in $\mathbf{K}_1^\omega$.*

2. Quasi-invariant and invariant measures. In the present section we shall touch upon some questions connected with various properties of non-zero σ-finite quasi-invariant (invariant) Borel measure defined in topological vector spaces.

Let $(\mathbf{E}, \mathcal{S}, \mu)$ be a measure space and Γ a group of transformations of the basic set $\mathbf{E}$. The measure μ is said to be *quasi-invariant* under Γ (shortly: *Γ-quasi-invariant*) if the following conditions are satisfied:

1) the σ-algebra $\mathcal{S}$ is invariant under Γ;

2) $(\forall h \in \Gamma)\,(\forall X \in \mathcal{S})\,(\mu(h(X)) = 0 \Leftrightarrow \mu(X) = 0)$.

If condition 1) is satisfied together with

3) $(\forall h \in \Gamma)\,(\forall X \in \mathcal{S})\,(\mu(h(X)) = \mu(X))$,

then we say that μ is *invariant under Γ* (or, shortly, that μ is *Γ-invariant*). It is obvious that every Γ-invariant measure is Γ-quasi-invariant and that the converse is in general false.

In Euclidean space $\mathbb{R}^n$, the standard Lebesgue measure λ_n is invariant with respect to the group of all isometries. Note that λ_n can be considered both on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ and on the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^n)$ of $\mathbb{R}^n$. Denote by $\overline{\mathcal{B}}(\mathbb{R}^n)$ the class of all subsets of $\mathbb{R}^n$ with the Baire property. It is evident that $\mathcal{B}(\mathbb{R}^n) \subseteq \overline{\mathcal{B}}(\mathbb{R}^n)$ and that $\overline{\mathcal{B}}(\mathbb{R}^n)$ is an σ-algebra. The following question arises: is it possible to define on $\overline{\mathcal{B}}(\mathbb{R}^n)$ a non-degenerate σ-finite measure which is invariant (or at least quasi-invariant) under a fairly large group of isometric transformations of $\mathbb{R}^n$? It turns out that the answer to this question is negative. Precisely, the following result holds.

THEOREM 2.1. *Let Γ be a group of motions of Euclidean space $\mathbb{R}^n$. Then the following conditions are equivalent*:

1) *for every $x \in \mathbb{R}^n$ the orbit $\Gamma(x)$ is uncountable*;

2) *for every non-zero σ-finite Γ-quasi-invariant measure μ in $\mathbb{R}^n$ there exists a μ-non-measurable set in the σ-algebra $\overline{\mathcal{B}}(\mathbb{R}^n)$.*

In particular, *if condition* 1) *is satisfied then there exists no non-degenerate σ-finite Γ-quasi-invariant measure on $\overline{\mathcal{B}}(\mathbb{R}^n)$.*

The proof of Theorem 2.1 uses induction on n together with Ulam's theorem stating that the first uncountable cardinal ω_1 is not measurable in the wide sense.

So we see that in the finite-dimensional Euclidean space $\mathbb{R}^n$ there does exist a non-zero σ-finite measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ which is invariant under the group of all motions of the space, while no such measure exists on the σ-algebra $\overline{\mathcal{B}}(\mathbb{R}^n)$. If we turn to the infinite-dimensional complete separable metrizable locally convex topological vector space $\mathbb{R}^\omega$ consisting of all real sequences, then the situation changes. In fact, the following result holds.

THEOREM 2.2. *Let* $\mathbf{E}$ *be an arbitrary infinite-dimensional Polish topological vector space. Then there is no non-degenerate σ-finite Borel measure in* $\mathbf{E}$, *quasi-invariant under the group of all translations of the space.*

A proof can be found, for example, in [20], [34] or [32]. Note that since every σ-finite Borel measure in a Polish space is a Radon measure, Theorem 2.2 follows from the result below, which is due to Xia Dao-Xing.

THEOREM 2.3. *Let* $\mathbf{E}$ *be a topological metrizable group admitting a complete metric. If there is in* $\mathbf{E}$ *a non-zero σ-finite Radon measure quasi-invariant under the group of all left translations of* $\mathbf{E}$, *then* E *is locally compact.*

A very simple proof of Theorem 2.3, based on arguments connected with the category of the space $\mathbf{E}$, can be found in [34]. In particular, it follows from the above theorem that if $\mathbf{E}$ is an infinite-dimensional complete metrizable topological vector space then it is impossible to define in $\mathbf{E}$ a non-degenerate σ-finite Radon measure quasi-invariant under the group of all translations of the space. It is worthwhile to compare Theorem 2.3 with the result below, following quite easily from Theorem 1.1.

THEOREM 2.4. *Let* $\mathbf{E}$ *be a metrizable topological group*, *viewed as the group of left translations of itself. If the topological weight of* $\mathbf{E}$ *is not measurable in the wide sense and if there is at least one non-zero σ-finite Borel measure in* $\mathbf{E}$, *quasi-invariant under some dense subgroup* $\Gamma \subseteq \mathbf{E}$, *then* $\mathbf{E}$ *is separable.*

From the above we see that for an infinite-dimensional topological vector space $\mathbf{E}$ it is in general impossible to define a non-degenerate σ-finite Borel measure, quasi-invariant under the whole additive group of the space. Under these circumstances it is natural to relax the requirement of quasi-invariance of the measure. In particular, one can ask about the existence in $\mathbf{E}$ of a non-zero σ-finite Borel measure, quasi-invariant under some fairly large group Γ of translations of $\mathbf{E}$. By "large" we can mean, for instance, that Γ is dense in $\mathbf{E}$. In this respect the following open problem appears to be interesting.

PROBLEM 2.1. Give a characterization (in purely algebraic and topological terms) of all topological vector spaces $\mathbf{E}$ satisfying the condition: there exists in $\mathbf{E}$ at least one non-degenerate σ-finite Borel measure quasi-invariant under some dense vector subspace of $\mathbf{E}$.

Furthermore, the following statement holds even without any additional set-theoretic assumptions.

THEOREM 2.5. *Let* $\mathbf{E}$ *be a non-separable normed vector space and let* Γ *be a dense subgroup of the additive group* $\mathbf{E}$. *Then there exists no non-zero* Γ*-quasi-invariant* σ*-finite Borel measure in* $\mathbf{E}$.

It was shown in [20] that the reason for that fact lies in the Γ-absolute negligibility of every ball (or, which is the same, of every bounded set) in a non-separable normed vector space.

However, for infinite-dimensional separable Banach spaces it turns out that the last question can be answered in the affirmative. More precisely, we have the following

THEOREM 2.6. *Let* $\mathbf{E}$ *be an infinite-dimensional separable Banach space. Then there exists in* $\mathbf{E}$ *a probability Borel measure quasi-invariant under some dense vector subspace of* $\mathbf{E}$.

To prove Theorem 2.6 we need the following simple

LEMMA 2.1. *Denote by* l^1 *the separable Banach space of all absolutely summable real sequences and suppose that* $\mathbf{E}$ *is a separable Banach space. Then there exists a continuous linear mapping of* l^1 *onto* $\mathbf{E}$.

In other words, in the case of separable Banach spaces the space l^1 plays a role similar to the role of 2^ω (Cantor set) in the class of all non-empty compact metric spaces (because every non-empty compact metric space is the image of the Cantor set under a continuous mapping).

Once we have Lemma 2.1, Theorem 2.6 can be proved quite easily when we use the fact that in l^1 there exists a probability Borel measure, quasi-invariant under a dense vector subspace.

EXAMPLE 2.1. Let μ be an arbitrary probability Borel measure in the real line $\mathbb{R}$, quasi-invariant under the group of all translations of $\mathbb{R}$. By well known Fubini theorem it is not hard to show that μ has the form $\mu(x) = \int_X \phi\, d\lambda$, $X \in \mathcal{B}(\mathbb{R})$, where ϕ is a strictly positive real function, integrable with respect to Lebesgue measure λ and such that $\int_{\mathbb{R}} \phi\, d\lambda = \mu(\mathbb{R}) = 1$.

EXAMPLE 2.2. Consider an infinite countable family $(\mu_i)_{i\in I}$ of probability Borel measures on the real line $\mathbb{R}$ such that every μ_i is $\mathbb{R}$-quasi-invariant. We can identify the set I with ω. Then the product $\mu = \prod_{i\in I} \mu_i$ is a probability Borel measure in the topological vector space $\mathbb{R}^\omega$. Denoting by $\mathbb{R}^{(\omega)}$ the vector subspace of R^ω consisting of all finite real sequences we see that μ is $\mathbb{R}^{(\omega)}$-quasi-invariant. It is also evident that $\mathbb{R}^{(\omega)}$ is dense in $\mathbb{R}^\omega$. It can be proved that $\mathbb{R}^{(\omega)}$ is maximal (in the sense of inclusion) among all groups $\Gamma \subseteq \mathbb{R}^\omega$ with the following property: for every infinite countable collection

$(\mu_i)_{i\in I}$ of probability Borel $\mathbb{R}$-quasi-invariant measures on $\mathbb{R}$, the product of μ_i's is Γ-quasi-invariant on $\mathbb{R}^\omega$ (see [17]).

The space $\mathbb{R}^\omega$ is also interesting from the point of view of invariant measures. The following statement holds.

THEOREM 2.7. *There exists in* $\mathbb{R}^\omega$ *a non-degenerate* σ*-finite Borel* $\mathbb{R}^{(\omega)}$*-invariant measure* μ; *moreover*, *given a real number* p, $1 \le p < \infty$, *the measure* μ *can be chosen so that it is concentrated on the space* $l^p \subseteq \mathbb{R}^\omega$.

A proof of the above theorem is essentially provided in [18] (for $p = 2$, yet the argument remains valid for any real $p \ge 1$). Starting from the existence of the measure indicated in Theorem 2.7 it is not difficult to construct:

1) a non-zero σ-finite Borel measure in $C([0,1])$, invariant under the dense vector subspace consisting of all polynomials on $[0,1]$;

2) a non-zero σ-finite Borel measure in the space $\mathbb{R}^{[0,1]}$ of all real functions on $[0,1]$, invariant under the subspace of all polynomials on $[0,1]$ (we note that in the present case the space $\mathbb{R}^{[0,1]}$ is equipped with the Tikhonov topology and that the space of all polynomials is dense in $\mathbb{R}^{[0,1]}$).

In general, the following problem remains open.

PROBLEM 2.2. Give a characterization (in purely algebraic and topological terms) of all topological vector spaces $\mathbf{E}$ with the property: there is at least one non-degenerate σ-finite Borel measure in $\mathbf{E}$, invariant under some dense vector subspace of $\mathbf{E}$.

It may be useful to compare Problem 2.2 with the analogous Problem 2.1 as well as with the example below.

EXAMPLE 2.3. Let α be an arbitrary infinite cardinal number. Consider the topological vector space $\mathbb{R}^\alpha$ (equipped with the Tikhonov product topology and the product vector structure). Denote by $\mathbb{R}^{(\alpha)}$ the subspace of $\mathbb{R}^\alpha$ defined as the collection of all finite α-sequences of real numbers. $\mathbb{R}^{(\alpha)}$ is obviously dense in $\mathbb{R}^\alpha$. It turns out that there exists in $\mathbb{R}^\alpha$ a probability Borel $\mathbb{R}^{(\alpha)}$-quasi-invariant measure ν. A proof of this fact can be found in [19]. We wish to indicate that in that proof a certain imbedding f of $\mathbb{R}^\alpha$ into the compact abelian group $\mathbf{K}_1^\alpha$ is employed; here $\mathbf{K}_1$ stands for the unit circle in $\mathbb{R}^2$. The imbedding f has the property that $f(\mathbb{R}^\alpha)$ is μ-massive in $\mathbf{K}_1^\alpha$ where μ stands for the ordinary Haar measure on this group. In fact one can assert that the measure ν is the restriction of μ to the μ-massive set $\mathbb{R}^\alpha$ (via the identification of $\mathbb{R}^\alpha$ with $f(\mathbb{R}^\alpha)$). In this connection we note that for $\alpha > \omega$ it is not known whether there exists in $\mathbb{R}^\alpha$ a non-zero σ-finite Borel measure, invariant under the group $\mathbb{R}^{(\alpha)}$.

Deliberately we do not touch upon any problems connected with various properties of Haar measures on locally compact topological groups, since the appropriate theory is well-developed and presented in detail in many works (see, e.g., [11], [4], Chap. 7, 8, or [12]).

3. Borel measures in topological spaces. In this section we shall consider miscellaneous Borel measures in Hausdorff topological spaces. In many cases the Hausdorff separation condition is unnecessary and is introduced exclusively for the sake of convenience. Besides, we shall be primarily interested in diffuse probability (or at least σ-finite) Borel non-zero measures.

Assuming the Continuum Hypothesis and applying transfinite induction, N. N. Lusin constructed in 1914 (see [26]) a subset X of the real line $\mathbb{R}$ with the following properties:

1) X is uncountable:

2) X is dense in $\mathbb{R}$:

3) for any subset Z of the first category in $\mathbb{R}$ the intersection $Z \cap X$ is at most countable.

From those properties it follows that each uncountable subset of X fails to have the Baire property in $\mathbb{R}$. Besides, the properties above imply that if X is considered as a topological space (with the topology induced from $\mathbb{R}$) then every subset of the first category in X is at most countable. Let us remark at this point that every diffuse σ-finite Borel measure in a separable metric space is concentrated on a set of the first category (a similar fact is true, in view of Theorem 1.1, for every metric space whose topological weight is not measurable in the wide sense). In this way, using the property of X indicated above we immediately get the following result.

THEOREM 3.1. *In the space X every diffuse σ-finite Borel measure is identically zero.*

Let us note, however, that without additional set-theoretic assumptions it is impossible to show the existence of a set $X \subseteq \mathbb{R}$ with properties 1) and 3), since it is easy to see that the conjunction of Martin's Axiom and the negation of the Continuum Hypothesis implies the absence of such a set on the real line. On the other hand, applying standard methods of the classical theory of analytic sets one can prove the existence (in ZFC) of an uncountable set $X \subseteq \mathbb{R}$ satisfying the statement of Theorem 3.1.

A topological space $\mathbf{E}$ is usually called a Lusin space if it does not contain isolated points and if every subset of the first category in $\mathbf{E}$ is at most countable. Consequently, in any Lusin space the class of all sets of the first category coincides with the collection of all subsets which are at most countable. According to the result formulated above, under some set-theoretic assumptions there exist metrizable separable Lusin spaces of cardinality continuum.

Motivated by Theorem 3.1, by a *Lusin space* we shall understand any Hausdorff topological space with no non-zero diffuse σ-finite Borel measure. The following problem is open.

PROBLEM 3.1. Give a characterization of Lusin spaces in the class of all Hausdorff topological spaces.

Clearly, this problem is equivalent to the problem of description of all Hausdorff topological spaces with at least one diffuse probability Borel measure. It could be useful to compare the last problem with a well-known result of Kelley on Boolean algebras with measure (see [14] and [31]).

In 1924, assuming the Continuum Hypothesis and using transfinite induction, W. Sierpiński constructed a subset Y of the real line $\mathbb{R}$ such that:

1) Y is uncountable;
2) Y is dense in $\mathbb{R}$;
3) for every subset $Z \subseteq \mathbb{R}$ of Lebesgue measure zero the intersection $Z \cap Y$ is at most countable.

It follows immediately from the properties above that no uncountable subset of Y is Lebesgue measurable. In general, the Lusin set X considered earlier and the Sierpiński set Y are quite similar and have dual properties (for more details see [26], where a proof of the Sierpiński-Erdős duality principle is given). Various applications of the Lusin and Sierpiński sets can be found in [24] and [26].

We are now going to consider some problems concerning Radon measures and radon topological spaces. Recall that a Hausdorff topological space $\mathbf{E}$ is called a *Radon space* if every σ-finite Borel measure in $\mathbf{E}$ is a Radon measure. In the definition of Radon space a weaker condition is usually required: every finite (or even probability) measure in $\mathbf{E}$ is a Radon measure. It is not difficult to notice, however, that both definitions are in fact equivalent. Classical examples of radon spaces are Polish topological spaces, analytic subsets of Polish spaces, and their complements.

Theorem 2.7 formulated in the previous section ensures the existence, in $\mathbb{R}^\omega$, of a non-degenerate σ-finite Borel measure μ, invariant under the group $\mathbb{R}^{(\omega)}$. It is easy to see that μ has the following property: the μ-measure of each non-empty open set in $\mathbb{R}^\omega$ is infinite. Thus we see that in the Polish space $\mathbb{R}^\omega$ there exist σ-finite $\mathbb{R}^{(\omega)}$-invariant Radon measures which are not locally bounded at any point of the space.

PROBLEM 3.2. Describe all Radon spaces in the class of all Hausdorff topological spaces.

This problem is well known and so far has not been solved in a satisfactory way. It is also interesting to investigate the behavior of Radon spaces under usual set-theoretic operations. For instance, it cannot be claimed in general that the class of Radon spaces is closed under taking continuous images (which satisfy the Hausdorff separation axiom). Indeed, in Section 1 we have noted that under Gödel's constructivity axiom there exists an analytic subset Z of the real line $\mathbb{R}$ and a continuous mapping $f : \mathbb{R} \setminus Z \to \mathbb{R}$ such that $f(\mathbb{R} \setminus Z)$ is not Lebesgue measurable. In particular, $f(\mathbb{R} \setminus Z)$ is not universally measurable in the real line. Consequently, the topological space $f(\mathbb{R} \setminus Z)$ is not a Radon space, since in this case the property of being Radon space is equivalent to universal measurability. More generally, given a Radon space $\mathbf{E}$ and its subspace $\mathbf{E}'$, $\mathbf{E}'$ is a Radon space if and only if it is universally measurable with respect to the class of all σ-finite Borel measures in $\mathbf{E}$ (see [4]).

The question whether the product of Radon spaces is a Radon space appears to be equally interesting. In some special cases the product of a countable family of Radon spaces turns out to be a Radon space (see, e.g., [28]). However, in general even the product of two Radon spaces may fail to be Radon. More precisely, the following result holds (see [35]).

THEOREM 3.2. *Assume the Continuum Hypothesis. Then there exist two Hausdorff topological spaces* $\mathbf{E}$ *and* $\mathbf{F}$ *with the following properties*:

1) $\mathbf{E}$ *and* $\mathbf{F}$ *are compact and every closed subset of both* $\mathbf{E}$ *and* $\mathbf{F}$ *is a* G_δ*-set*;

2) *in the topological product* $\mathbf{E} \times \mathbf{F}$ *there exists a Borel subset* Z *of cardinality continuum*, *without uncountable compact subsets and such that* Z *is Borel isomorphic to the unit interval* $[0, 1]$ *with the usual Euclidean topology.*

From condition 1) of the last theorem it follows directly that the spaces $\mathbf{E}$ and $\mathbf{F}$ are both Radon. On the other hand, by means of the set Z in condition 2) it is possible to define a probability Borel measure in $\mathbf{E} \times \mathbf{F}$ which is not a Radon measure. Therefore the product $\mathbf{E} \times \mathbf{F}$ is not a Radon space.

A similar result can be obtained by the use of the classical "two arrows" space of P. S. Aleksandrov, which is a compact Radon space. If we assume that $\mathbf{c}$ is measurable in the wide sense then it is easy to show that the product of two copies of the "two arrows" space is not Radon (the fact first observed by D. Fremlin). As far as topological sums of Radon spaces are concerned, it is easy to see that the following result holds.

THEOREM 3.3. *Suppose that* $(\mathbf{E}_i)_{i \in I}$ *is a family of Radon spaces and let* $\mathbf{E}$ *be the topological sum of that family. If the cardinality of* I *is not measurable in the wide sense then* $\mathbf{E}$ *is a Radon space.*

We shall now consider the so-called Prohorov topological spaces. First we recall some definitions.

Let $\mathbf{E}$ be a completely regular topological space and $C_b(\mathbf{E})$ the Banach space of all bounded continuous real functions on $\mathbf{E}$, equipped with the norm of uniform convergence. We write $M_b(\mathbf{E})$ for the vector space of all finite signed Radon measures on $\mathbf{E}$. The bilinear form $(\phi, \mu) \to \int \phi \, d\mu$ defined on the product $C_b(\mathbf{E}) \times M_b(\mathbf{E})$ gives rise to a separated duality between $C_b(\mathbf{E})$ and $M_b(\mathbf{E})$. The weak topology in $M_b(\mathbf{E})$ associated with this duality is called the *topology of tight convergence* (or the *tight topology*) in $M_b(\mathbf{E})$. It is evident that the tight topology in $M_b(\mathbf{E})$ is completely regular. Many properties of the original space $\mathbf{E}$ carry over to the completely regular space $M_b^+(\mathbf{E})$ consisting of all finite positive Radon measures on $\mathbf{E}$. For instance, if $\mathbf{E}$ is Polish then so is $M_b^+(\mathbf{E})$ with its tight topology; if $\mathbf{E}$ is a separable metric space then $M_b^+(\mathbf{E})$ is a separable metrizable space; finally, if $\mathbf{E}$ is an analytic (i.e. Suslin) topological space then so is $M_b^+(\mathbf{E})$ (see [4], Chap. 9).

Let now $\mathbf{E}$ be a Hausdorff topological space and suppose that M is a subset of $M_b^+(\mathbf{E})$. We say that M has the *Prohorov property* if

1) $\sup_{\mu \in M} |\mu|(\mathbf{E}) < \infty$ where $|\mu|$ denotes the total variation of μ;

2) for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \mathbf{E}$ such that $|\mu|(\mathbf{E} \setminus K_\varepsilon) < \varepsilon$ for any measure $\mu \in M$.

The Prohorov property provides a criterion for relative compactness in the tight topology. Precisely, we have the following important result, due to Prohorov.

THEOREM 3.4. *Let $\mathbf{E}$ be a completely regular topological space and let M be a subset of $M_b(\mathbf{E})$ with the Prohorov property. Then M is relatively compact in the tight topology of $M_b(\mathbf{E})$.*

In some special cases (important in applications, however) the converse is also true:

THEOREM 3.5. *Let $\mathbf{E}$ be a topological space. Suppose that $\mathbf{E}$ is either locally compact or Polish. Let $M \subseteq M_b^+(\mathbf{E})$ be a relatively compact set in the tight topology of $M_b^+(\mathbf{E})$. Then M has the Prohorov property.*

The theorem just stated is essentially due to Prohorov. The last two results provide a natural motivation to introduce the notion of Prohorov space. Accordingly, we call a completely regular Radon space $\mathbf{E}$ a *Prohorov space* if every relatively compact (in the tight topology) family of finite Borel measures in $\mathbf{E}$ has the Prohorov property. The following problem remains open.

PROBLEM 3.3. Give a characterization of Prohorov spaces in the class of all completely regular topological spaces.

In [27] it was shown that any metrizable Prohorov space is of the second category in itself. Hence it follows in particular that the set $\mathbb{Q}$ of rationals with the usual Euclidean topology is not a Prohorov space (even though it is Radon and σ-compact). It is also not difficult to see that the class of Prohorov spaces is closed under taking finite topological sums and that the property of being a Prohorov space is inherited by closed subsets.

A large number of interesting examples of Borel measures in various topological spaces, possessing (or not) miscellaneous regularity properties, can be found in [10].

4. Cylindrical and Borel σ-algebras. Let $\mathbf{E}$ denote, as usual, a given basic set and let Φ be a family of mappings from $\mathbf{E}$ into a measurable space. The smallest (with respect to inclusion) σ-algebra $\mathcal{S}$ of subsets of $\mathbf{E}$ such that all mappings in Φ are measurable with respect to $\mathcal{S}$ is called the *cylindrical σ-algebra induced by Φ* (shortly: *Φ-cylindrical σ-algebra*), and denoted by $\mathcal{B}(\mathbf{E}, \Phi)$. Cylindrical σ-algebras play an important role in various questions from measure theory and random processes theory (to mention just one let us recall the well known theorem of Kolmogorov concerning families of finite-dimensional distributions satisfying the consistency conditions).

EXAMPLE 4.1. Let $\mathbf{E}$ be a topological space and let Φ be the family of all continuous real functions defined on $\mathbf{E}$. We regard $\mathbb{R}$ as a measurable space with the usual Borel σ-algebra $\mathcal{B}(\mathbb{R})$. In this case the Φ-cylindrical σ-algebra $\mathcal{B}(\mathbf{E}, \Phi)$ is called the *Baire σ-algebra* and denoted by $\mathcal{B}_0(\mathbf{E})$. It is clear that $\mathcal{B}_0(\mathbf{E}) \subseteq \mathcal{B}(\mathbf{E})$ where $\mathcal{B}(\mathbf{E})$ stands for the Borel σ-algebra of $\mathbf{E}$. The situation where the last inclusion becomes equality presents a considerable interest for topological measure theory.

The following well-known definition will be needed later on. Let $\mathbf{E}$ be a set and let Φ be a family of mappings defined on $\mathbf{E}$. We say that Φ *separates points of* $\mathbf{E}$ if, for any two distinct points $x, y \in \mathbf{E}$, there is $\phi \in \Phi$ such that $\phi(x) \neq \phi(y)$.

Further, let $\mathcal{T}$ be a family of subsets of $\mathbf{E}$. Then $\mathcal{T}$ is said to *separate points of* $\mathbf{E}$ if this is true for the family of all characteristic functions of sets in $\mathcal{T}$.

THEOREM 4.1. *Let $\mathbf{E}$ be a Polish space. Then every σ-algebra generated by any countable family of Borel sets separating points of $\mathbf{E}$ coincides with the Borel σ-algebra of $\mathbf{E}$.*

The proof can be derived without much difficulty from the following well-known fact from descriptive set theory: the image of a Borel set under an injective Borel mapping from a Polish space into a separable metric space is again a Borel set.

The next result follows directly from Theorem 4.1.

THEOREM 4.2. *Let $\mathbf{E}$ be a Polish space. Suppose that Φ is an arbitrary countable family of real Borel functions on $\mathbf{E}$, separating the points of $\mathbf{E}$.*

Then the Φ-cylindrical σ-algebra of $\mathbf{E}$ *is equal to the Borel σ-algebra of this space.*

Using the last theorem it is not difficult to obtain the following result (see, e.g., [34]).

THEOREM 4.3. *Let* $\mathbf{E}$ *be a Polish space and let Φ be a family of continuous real functions on* $\mathbf{E}$. *Then the following two conditions are equivalent*:

1) *Φ separates points of* $\mathbf{E}$;

2) *the Φ-cylindrical σ-algebra $\mathcal{B}(\mathbf{E},\Phi)$ and the Borel σ-algebra $\mathcal{B}(\mathbf{E})$ are identical.*

A special case of Theorem 4.3 is the next result, having various applications in measure theory and probability theory.

THEOREM 4.4. *Let* $\mathbf{E}$ *be a complete separable metrizable topological vector space, let* $\mathbf{E}^*$ *be its topological dual space and let Φ be a subset of* $\mathbf{E}^*$. *The following two conditions are equivalent*:

1) *Φ separates points of* $\mathbf{E}$;

2) *the Φ-cylindrical σ-algebra $\mathcal{B}(\mathbf{E},\Phi)$ coincides with the Borel σ-algebra $\mathcal{B}(\mathbf{E})$.*

We remark that some examples in [6] indicate that each of the following assumptions imposed on $\mathbf{E}$: completeness, separability, metrizability, is essential for the validity of Theorem 4.4. At the same time, one can find non-Polish topological vector spaces for which the equivalence of 1) and 2) still holds.

EXAMPLE 4.2. Let $\mathbb{R}^{(\omega)}$ denote the vector space of all finite real sequences. We introduce a norm in $\mathbb{R}^{(\omega)}$ by putting $\|x\| = \sum_{n\in\mathbb{N}} |x_n|$ for $x \in (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{(\omega)}$ (note that the definition is correct, since the number of non-zero terms in the sum above is finite). It is easy to see that $\mathbb{R}^{(\omega)}$ with $\|\cdot\|$ is a separable normed vector space. Now, let Φ be an arbitrary family of continuous linear forms on $\mathbb{R}^{(\omega)}$, separating points of $\mathbb{R}^{(\omega)}$. It is easy to see that $\mathcal{B}(\mathbb{R}^{(\omega)},\Phi) = \mathcal{B}(\mathbb{R}^{(\omega)})$. On the other hand, it is quite obvious that $\mathbb{R}^{(\omega)}$ is not complete, in fact it is of the first category in itself.

In relation to the example above it is natural to pose the following

PROBLEM 4.1. Characterize all topological vector spaces $\mathbf{E}$ such that $\mathcal{B}(\mathbf{E},\Phi) = \mathcal{B}(\mathbf{E})$ for every family $\Phi \subseteq \mathbf{E}^*$ separating points of $\mathbf{E}$.

In a sense, the next problem is related to Problem 4.1.

PROBLEM 4.2. Give a characterization of all topological vector spaces $\mathbf{E}$ such that $\mathcal{B}(\mathbf{E},\Phi_1) = \mathcal{B}(\mathbf{E},\Phi_2)$ for any two families $\Phi_1,\Phi_2 \subseteq \mathbf{E}^*$ separating points of $\mathbf{E}$.

An interesting result concerning cylindrical and Borel σ-algebras was obtained in [9]. Let $\mathbf{E}$ be a Banach space and let $\mathcal{B}(\mathbf{E},\mathbf{E}^*)$ stand for the cylindrical σ-algebra in $\mathbf{E}$ induced by the collection of all continuous linear forms on $\mathbf{E}$. Suppose that $\mathcal{B}(\mathbf{E},\mathbf{E}^*)=\mathcal{B}(\mathbf{E})$. Does this imply that $\mathbf{E}$ is separable? The answer turns out to be negative. Let Y be a set and denote by $l^1(Y)$ the Banach space of all absolutely summable real functions on Y. In other words,

$$f \in l^1(Y) \quad \text{if and only if} \quad \sum_{y\in Y}|f(y)|<\infty.$$

Next, let $\mathcal{P}(Y)$ denote the σ-algebra of all subsets of Y and let $\mathcal{P}(Y)\otimes\mathcal{P}(Y)$ stand for the σ-algebra in $Y\times Y$ defined as the product of two copies of $\mathcal{P}(Y)$. Finally, let $\mathcal{P}(Y\times Y)$ denote the σ-algebra of all subsets of $Y\times Y$. The following result holds (see [9]).

THEOREM 4.5. *If, for an infinite set Y, the equality $\mathcal{P}(Y)\otimes\mathcal{P}(Y)=\mathcal{P}(Y\times Y)$ is satisfied, then the Borel σ-algebra of the Banach space $l^1(Y)$ coincides with the cylindrical σ-algebra induced by the family of all continuous linear forms on this space.*

Taking for Y the first uncountable cardinal ω_1 we have, by a classical result of Sierpiński, $\mathcal{P}(\omega_1)\otimes\mathcal{P}(\omega_1)=\mathcal{P}(\omega_1\times\omega_1)$. Thus we see that although the Banach space $l^1(\omega_1)$ is non-separable, its Borel σ-algebra coincides with the cylindrical σ-algebra induced by all continuous linear forms on that space.

Let $\mathbf{E}$ be a Hausdorff topological space and let Φ be a family of continuous real functions on $\mathbf{E}$. Given a probability measure on the Φ-cylindrical σ-algebra in $\mathbf{E}$, the question arises whether that measure can be extended to a Borel measure on $\mathbf{E}$. One can at the same time demand that the extension have certain additional properties (such as being a Radon measure). Some aspects of this problem are considered in [34]. Clearly, the above question is a special case of the following more general problem.

Let $(\mathbf{E},\mathcal{S},\mu)$ be a σ-finite measure space and let $\overline{\mathcal{S}}$ be an arbitrary σ-algebra of subsets of $\mathbf{E}$, containing $\mathcal{S}$. The problem is to find necessary and sufficient conditions for the existence of the extension of μ to $\overline{\mathcal{S}}$. These conditions, as is naturally understood, should be formulated in measure-theoretic terms.

EXAMPLE 4.3. Consider the unit interval $[0,1]$ with the σ-algebra $\mathcal{S}$ consisting of all Borel sets of the first category in $[0,1]$ and their complements. We define a probability two-valued measure μ by putting, for $X\in\mathcal{S}$,

$$\mu(X)=\begin{cases}0 & \text{if } X \text{ is of the first category},\\ 1 & \text{if } [0,1]\setminus X \text{ is of the first category}.\end{cases}$$

It is clear that μ is a separable measure. Moreover, the metric space associated with μ consists of just two points. It is easy to see that the measure μ cannot be extended to a probability Borel measure on $[0,1]$. Indeed, every such extension would have to be concentrated on a first category set, which contradicts the definition of the original measure μ.

A much more complicated example in [8] exhibits a two-valued probability measure, defined on a certain σ-algebra of Borel subsets of $[0,1]^\omega$, with no extension to a probability Borel measure. That example was used in [8] to conclude that Problem 6 in Marczewski's paper [33] has a negative solution. We remark, however, that, as seen from the context, the measures considered in [33] are assumed to be non-atomic. Therefore the formulation of Problem 6 in that paper should read: is it always possible to extend a non-atomic separable probability measure from the σ-algebra of Borel subsets of a Polish space $\mathbf{E}$ to the whole Borel σ-algebra of $\mathbf{E}$? Recall that every non-atomic separable probability measure is metrically isomorphic to the classical Lebesgue measure on $[0,1]$. Therefore Marczewski's Problem 6 is in fact interesting for such measures only (Example 4.3 shows that without non-atomicity condition the problem is trivial). On the other hand, using Example 4.3 one can provide a solution of Marczewski's problem even for non-atomic measures. Indeed, let λ be Lebesgue measure on the Borel σ-algebra of the interval $[0,1]$. Put $\nu = \mu \times \lambda$, where μ is the measure defined in Example 4.3. Then it is not hard to show that ν is a non-atomic separable probability measure on the unit square $[0,1]^2$ and that it cannot be extended to a Borel measure on the square.

It is worth noticing that in the paper [21] a relation is established between Marczewski's problem mentioned above and the existence of Lebesgue nonmeasurable subsets of $[0,1]$. Moreover, in [21] it is shown that Gödel's constructivity axiom implies the existence of a coanalytic topological space $\mathbf{E}$, a countably generated σ-algebra $\mathcal{S} \subseteq \mathcal{B}(\mathbf{E})$ and a non-atomic probability measure μ such that μ is defined on $\mathcal{S}$ and has no extension to a measure on $\mathcal{B}(\mathbf{E})$. Note also that, on the other hand, for every countably generated σ-algebra $\mathcal{S} \subseteq \mathcal{B}(\mathbf{E})$ and every non-atomic probability measure μ defined on $\mathcal{S}$ there exists an extension of μ to a measure on $\mathcal{B}(\mathbf{E})$ (see [8]).

5. Miscellaneous families of probability measures. Let again $\mathbf{E}$ be a basic set and let $\mathcal{S}$ be a σ-algebra of its subsets. In the present section we shall consider various families of measures defined on $\mathcal{S}$ and single out some classes of such families. For the sake of simplicity we restrict ourselves to probability measures though the notions introduced below make sense for σ-finite measures, too.

Let $(\mu_i)_{i\in I}$ be a family of probability measures defined on $\mathcal{S}$. Such a family is called *orthogonal* if for any two distinct $i, j \in I$ the measures μ_i and μ_j are mutually singular, i.e., if there are disjoint subsets $X, Y \in \mathcal{S}$ such that $\mu_i(X) = \mu_j(Y) = 1$.

It is clear that the sets X and Y may depend on the pair (i, j). Removing this dependence we arrive at the concept of weakly separating family of probability measures. A family $(\mu_i)_{i\in I}$ of probability measures on $\mathcal{S}$ is called *weakly separating* if there exists a collection $(X_i)_{i\in I} \subseteq \mathcal{S}$ such that $\mu_i(X_j) = \delta_{ij}$ for every pair $(i, j) \in I \times I$ (δ_{ij} denotes the Kronecker symbol). It is quite obvious that every weakly separating family of measures is orthogonal. A family $(\mu_i)_{i\in I}$ of measures on $\mathcal{S}$ is said to be *strongly separating* if there exists a collection $(X_i)_{i\in I} \subseteq \mathcal{S}$ such that

$$(\forall i, j \in I)(i \neq j \Rightarrow X_i \cap X_j = \emptyset), \qquad (\forall i \in I)(\mu_i(X_i) = 1).$$

Let us note that under the Continuum Hypothesis every weakly separating family $(\mu_i)_{i\in I}$ with $\mathrm{card}(I) \leq \mathbf{c}$ is strongly separating.

Finally, we introduce the notion of a family of probability measures admitting a statistical estimation. For that purpose we assume in addition that the set I of indices is endowed with a measurable structure, i.e., that a σ-algebra of subsets of I is given. This σ-algebra is assumed to contain all singletons. Now we say that a family $(\mu_i)_{i\in I}$ of probability measures on $\mathcal{S}$ *admits a statistical estimation* if there exists a measurable mapping $f : \mathbf{E} \to I$ with the property $(\forall i \in I)(\mu_i(\{x \in \mathbf{E} : f(x) = i\}) = 1)$. The mapping f is then called a *statistical estimation* for the family $(\mu_i)_{i\in I}$. It is clear that every family of measures admitting a statistical estimation must be strongly separating. As we shall see below, the converse is in general false.

The classes of families of probability measures just presented were first introduced in [13], originating from various problems in statistics of random processes. At the beginning we shall be concerned with various set-theoretic characteristics of the families of measures having the properties mentioned above. In the sequel the cardinality of $\mathbf{E}$ is assumed to be $\mathbf{c}$ (continuum), though the results below carry over with easy modifications to the case of an arbitrary uncountable basic set $\mathbf{E}$.

Theorem 5.1. *There exist a σ-algebra $\mathcal{S}$ in the unit interval* $[0, 1]$ *and a family $(\mu_i)_{i\in I}$ of probability measures on $\mathcal{S}$ such that*

1) *every measure μ_i $(i \in I)$ is an extension of Lebesgue measure on* $[0, 1]$;
2) *the family $(\mu_i)_{i\in I}$ is orthogonal*;
3) $\mathrm{card}(I) = 2^{2^{\mathbf{c}}}$.

In other words, Theorem 5.1 shows that on $[0, 1]$, there exists an orthogonal family of measures $(\mu_i)_{i\in I}$ having the greatest possible cardinality, equal to the cardinality of the injective family of all measures on [0,1]. We note at this point that every measure μ_i $(i \in I)$ can be required to have additional properties. In particular, we can demand that every μ_i satisfies the Marczewski axiom, i.e., that any μ_i-measurable set can be represented as $(X \cup X_1) \setminus X_2$, where X is a Lebesgue measurable subset of $[0,1]$ and $\mu_i^*(X_1) = \mu_1^*(X_2) = 0$ (see [15]).

The proof of Theorem 5.1 is based on the method of independent families of sets, first employed by Marczewski.

The following result is a direct consequence of Theorem 5.1.

THEOREM 5.2. *There exist a σ-algebra $\mathcal{S}$ of subsets of* $[0,1]$ *and a family of probability measures* $(\mu_i)_{i\in I}$ *on $\mathcal{S}$ such that*

1) *every* μ_i $(i \in I)$ *is an extension of Lebesgue measure on* $[0,1]$ *and satisfies the Marczewski axiom*;
2) *the family* $(\mu_i)_{i\in I}$ *is weakly separating*;
3) $\mathrm{card}(I) = 2^{\mathbf{c}}$.

In other words, Theorem 5.2 indicates that on $[0,1]$ there exists a weakly separating family of probability measures $(\mu_i)_{i\in I}$ having the greatest possible cardinality, equal to the cardinality of the class of all subsets of $[0,1]$.

Further, it is easy to see that if $(\mu_i)_{i\in I}$ is a strongly separating family of probability measures on a σ-algebra of subsets of $[0,1]$, then the cardinality of the set I of indices does not exceed continuum. Without much trouble one can construct a strongly separating family of Borel probability measures on $[0,1]$ such that each measure from this family is isomorphic to Lebesgue measure and such that the cardinality of the family is $\mathbf{c}$.

Among families of probability measures defined on a common σ-algebra of sets, one can distinguish so-called statistical structures (see [25]). Let $\mathbf{E}$ be a basic set and let $\mathcal{S}$ be a σ-algebra of subsets of $\mathbf{E}$. Let $(\mu_i)_{i\in I}$ be a family of probability measures on $\mathcal{S}$. We assume that a σ-algebra $\mathcal{T}$ of subsets of I is given, providing thus a measurable structure in I. The family $(\mu_i)_{i\in I}$ is called a *statistical structure* if for every $X \in \mathcal{S}$ the mapping $g_X : (I, \mathcal{T}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by $g_X(i) = \mu_i(X)$ $(i \in I)$ is measurable with respect to the σ-algebras $\mathcal{T}$ and $\mathcal{B}(\mathbb{R})$. It is evident that statistical structures can be orthogonal, weakly separating, strongly separating and, besides, they can admit a statistical estimation. The last property is possible only in the case where $\mathcal{T}$ contains all singletons, which is usually assumed in practice.

It is an interesting question whether for a given statistical structure there exists at least one statistical estimation. It is clear that a necessary condition is the strong separation of the statistical structure in question. However, this condition is not sufficient, which can be seen from the following result.

THEOREM 5.3. *Put* $\mathbf{E} = [0,1] \times \omega_1 \times \omega_1$ *and let I be a set of indices having the first uncountable cardinality* ω_1. *Then it can be effectively shown* (*i.e.*, *without the Axiom of Choice*) *that there exist a* σ-*algebra* $\mathcal{S}$ *in* $\mathbf{E}$, *a family of probability measures* $(\mu_i)_{i\in I}$ *and a* σ-*algebra* $\mathcal{T}$ *in* I *such that*

1) *all measures* μ_i *are defined on* $\mathcal{S}$,
2) *each* μ_i *is metrically isomorphic to Lebesgue measure on* $[0,1]$;
3) $(\mu_i)_{i\in I}$ *is a strongly separating statistical structure*;
4) $\mathcal{T}$ *contains all singletons*;
5) *there is no statistical estimation for* $(\mu_i)_{i\in I}$.

For a proof see [22]. We remark that the strong separation of this statistical structure $(\mu_i)_{i\in I}$ is shown effectively, too.

In [13] a construction is given, however with the help of the Continuum Hypothesis, of another strongly separating statistical structure $(\mu_i)_{i\in I}$ with the following properties:

1) all measures μ_i are defined on the Borel σ-algebra of the unit square $[0,1]^2$;
2) each μ_i is isomorphic to Lebesgue measure on $[0,1]$;
3) I is an uncountable Polish space, equipped with its Borel σ-algebra;
4) $(\mu_i)_{i\in I}$ does not admit any statistical estimation.

Note, finally, that it is not known whether one can effectively construct a strongly separating statistical structure having the above four properties.

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Reçu par la Rédaction le 22.9.1988;
en version modifiée le 19.10.1990