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ON A PROBLEM OF SATURATION OF CERTAIN REDUCED DIRECT PRODUCTS

BY

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1. Introduction. Let $\mathfrak{m}$ be a cardinal. A relational structure is called (*atomic*) $\mathfrak{m}$ *compact* if any system of atomic formulas in which $\mathfrak{m}$ variables and parameters are allowed and such that all its finite subsystems are solvable is solvable. This concept was introduced by Mycielski in a seminal paper [M], which initiated an extensive theory generalizing Kaplansky's theory of algebraically compact abelian groups. An interested reader may consult Wenzel's survey [W] for further information.

Unfortunately, Theorem 1 of [M] is false. Although the subsequent development of the theory based on the definitions, problems and examples of [M] does not depend on Theorem 1, we feel that once an error is found, it should be explained. In this paper we give a counterexample to Theorem 1 of [M] and see to what extent its content can be preserved. A flaw in the proof of Theorem 1 of [M] was spotted by the first author, counterexamples were provided by the second author, and positive results were proved by the first author. We would like to thank Jan Mycielski and A. J. Douglas for their helpful comments.

We assume throughout the paper that $\mathfrak{A}_u$ $(u \in U)$ is a system of similar relational structures and that $\mathbf{I}$ is an $\aleph_\alpha$ additive ideal of subsets of U such that U is the union of $\aleph_\alpha$ members of $\mathbf{I}$. Since we are interested in the claim that the reduced direct product $\prod_{u\in U}\mathfrak{A}_u/\mathbf{I}$ is $\aleph_\alpha$ compact, which is trivially true if $\aleph_\alpha$ is singular (because of $U \in \mathbf{I}$), we assume that $\aleph_\alpha$ is regular. If $c \in \prod_{u\in U}\mathfrak{A}_u$, let $\mathbf{c}$ be the element of $\prod_{u\in U}\mathfrak{A}_u/\mathbf{I}$ that corresponds to c.

2. Counterexample. Theorem 1 of [M] says that each system of $\aleph_\alpha$ atomic formulas in $\aleph_\alpha$ variables and with parameters in $\prod_{u\in U}\mathfrak{A}_u/\mathbf{I}$ such that all its subsystems having less than $\aleph_\alpha$ formulas are solvable is solvable. Mycielski's proof is correct for $\alpha = 0$. For $\alpha > 0$ we have the following counterexample.

Let $U = \omega_\alpha \setminus \omega_0$ and let $\mathbf{I} = [U]^{<\aleph_\alpha}$. For $u \in U$ let $\mathfrak{A}_u = \langle u, \neq\rangle$, i.e.

the domain of $\mathfrak{A}_u$ is the ordinal u and the relation is the inequality relation between elements of u. Define $c_\beta : U \to \omega_\alpha$ $(\beta < \omega_\alpha)$ by

$$c_\beta(u) = \begin{cases} \beta & \text{if } u > \beta, \\ 0 & \text{if } u \le \beta. \end{cases}$$

Consider the system $\mathbf{x} \ne \mathbf{c}_\beta$ $(\beta < \omega_\alpha)$. Clearly its subsystems having less than $\aleph_\alpha$ formulas are solvable. The whole system, however, is not solvable: Each $c \in \prod_{u\in U} \mathfrak{A}_u$ is a regressive function, so, by Fodor's theorem, there is $\beta < \omega_\alpha$ such that $\{u : c(u) = \beta\}$ is stationary. Thus $\mathbf{c}$ does not solve the formula $\mathbf{x} \ne \mathbf{c}_\beta$.

The error in Mycielski's proof is due to overlooking the fact that $\bigcup_{\zeta<\eta} B^\eta_\zeta \setminus \bigcup_{\vartheta<\eta} \bigcup_{\zeta<\vartheta} B^\vartheta_\zeta$ may be nonempty for limit η, and if this is the case, we may not be able to find the sequence A_ξ $(\xi < \aleph_\alpha)$ (notation of [M], p. 5).

3. What can be saved

THEOREM 1. *Suppose that* $\{u : \mathfrak{A}_u$ *is not* $\aleph_\beta$ *compact for some* $\beta < \alpha\} \in \mathbf{I}$. *Then* $\prod_{u\in U} \mathfrak{A}_u/\mathbf{I}$ *is* $\aleph_\alpha$ *compact.*

Proof. Without loss of generality we can assume that all $\mathfrak{A}_u$'s are $\aleph_\beta$ compact for all $\beta < \alpha$. Now we repeat the proof of Theorem 1 of [M] with the following changes:

(1) "$\eta < \omega_\alpha$" is replaced by "$\eta \in [\omega_\alpha]^{<\aleph_0}$" in $5_{12,13}$.

(2) "$\xi < \eta$" is replaced by "$\xi \in \eta$" in 5_{13} and by "$\xi \le \bigcup \eta$" in 5_4.

(3) "$\zeta < \eta$" is replaced by "$\zeta \in \eta$" in 5_7.

(4) the line 5_2 is replaced by "$\{a_s(u)\}_{s\in S}$ is a solution of the system R^u_ξ $(\xi < \xi(u))$". ■

As an immediate corollary we get the following theorem.

THEOREM 2. *Suppose that* $\aleph_\alpha$ *is not weakly inaccessible and that* $\{u : \mathfrak{A}_u$ *is not* $\aleph_\beta$ *compact*$\} \in \mathbf{I}$ *for every* $\beta < \alpha$. *Then* $\prod_{u\in U} \mathfrak{A}_u/\mathbf{I}$ *is* $\aleph_\alpha$ *compact.*

Without the requirement that $\aleph_\alpha$ is not weakly inaccessible this is Theorem 2 of [M]. However, in our counterexample every structure $\mathfrak{A}_u$ is $|u|$ compact, which shows that this requirement is necessary.

One may be tempted to save Theorem 1 of [M] by strengthening its assumptions. So, we can require that $\prod_{u\in U} \mathfrak{A}_u/\mathbf{I}$ is $\aleph_\beta$ compact for all $\beta < \alpha$. It is easily seen, however, that the reduced direct product in our counterexample has this property.

Another reasonable possibility is to replace "product" by "power", i.e. assume that the structures $\mathfrak{A}_u$ are all equal. This also fails because of the following counterexample: Let $U = \omega_\alpha$, $\mathbf{I} = [U]^{<\aleph_\alpha}$. Let $c_\beta : U \to \omega_\alpha$ $(\beta < \omega_\alpha)$ be a function constantly equal to β, and let $d : U \to \omega_\alpha$ be the diagonal function, i.e. $d(u) = u$ ($u \in U$). Consider the system $\mathbf{x} < \mathbf{d}$, and $\mathbf{c}_\beta < \mathbf{x}$ ($\beta < \omega_\alpha$) in the reduced direct power $\langle \omega_\alpha, < \rangle^U/\mathbf{I}$. Clearly every subsystem having less than $\aleph_\alpha$ formulas is solvable, and the whole system is not solvable, again due to Fodor's theorem.

We have, however, the following.

THEOREM 3. *If $\mathfrak{A}$ is a relational structure such that the reduced direct power $\mathfrak{A}^U/\mathbf{I}$ is $\aleph_\beta$ compact for all $\beta < \alpha$, then $\mathfrak{A}^U/\mathbf{I}$ is $\aleph_\alpha$ compact.*

Proof. If $U \in \mathbf{I}$, there is nothing to prove; so assume that $U \notin \mathbf{I}$. By Theorem 1, it is enough to show that $\mathfrak{A}$ is $\aleph_\beta$ compact for all $\beta < \alpha$. Let $\beta < \alpha$ and let R_ξ ($\xi < \omega_\beta$) be a finitely solvable system of atomic formulas. Let $\{x_s\}_{s\in S}$ be the set of all variables and let C be the set of all constants appearing in this system. For each $c \in C$ let $\mathbf{c} \in \mathfrak{A}^U/\mathbf{I}$ correspond to the function constantly equal to c, and let P_ξ ($\xi < \omega_\beta$) be obtained from R_ξ ($\xi < \omega_\beta$) by replacing each constant c by $\mathbf{c}$. Clearly the system P_ξ ($\xi < \omega_\beta$) is finitely solvable in $\mathfrak{A}^U/\mathbf{I}$, so, by $\aleph_\beta$ compactness, it has a solution $\{\mathbf{a}_s\}_{s\in S}$. Thus, for each $\xi < \omega_\beta$,

$$V_\xi = \{u \in U : \{a_s(u)\}_{s\in S} \text{ does not solve } R_\xi\} \in \mathbf{I}\,,$$

so

$$V = \bigcup_{\xi<\omega_\beta} V_\xi \in \mathbf{I}\,.$$

Let $u \in U \setminus V$. Then $\{a_s(u)\}_{s\in S}$ solves the system R_ξ ($\xi < \omega_\beta$). ■

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